A three-dimensional theory of coastal currents and upwelling over a continental shelf

By R. B. HILL and J. A. JOHNSON, *School of Mathematics and Physics, University of East Anglia, Norwich, England*

(Manuscript received April 10; revised version August 13, 1974)

ABSTRACT

The steady circulation of a homogeneous ocean is investigated in the region where the bottom rises to meet the surface at the coast as over a continental shelf. In this region the bottom Ekman layer becomes important and carries the coastal upwelling from the shelf interior region to the surface Ekman layer. A strong longshore current is produced in the shelf boundary layer and for certain distributions of wind stress an offshore deep counter-current is formed. At a given latitude the longshore flow near the coast depends on the local wind, whereas the offshore counter-current and the net northwards transport in the shelf layer are a function of the latitudinal distribution of wind stress.

1. Introduction

Theories of ocean circulation with variable Coriolis parameter that take account of bottom topography generally close the ocean basin by a shallow vertical coast rather than allow the depth to tend to zero as the coast is approached. Examples include the transport theories of Welander (1968), Schulman & Niiler (1970), and the three-dimensional theory of Johnson et al. (1971). The reason for this inclusion of shallow vertical coasts is that all these theories break down as the depth of the ocean approaches zero. A theory in which account is taken of the shallowing shelf region is given by Birchfield (1973). This latter theory is based on an f-plane and is more appropriate to a lake or small sea where variations in Coriolis parameter f are unimportant.

In this paper the bottom of the ocean is allowed to rise slowly, as over the continental shelf, until it reaches the surface at the coast. Away from the coastal region where the depth is small, the interior solution given by Johnson et al. (1971) is used. In the neighbourhood of the coast a boundary layer correction has to be applied to deal with the singularity in the interior solution. In this region the lower Ekman layer becomes important and a significant longshore flow occurs.

Observations made over the continental shelf off Oregon are discussed by Smith et al. (1971). A typical width of the shelf is $l = 30$ km and at this distance from the coast the depth is about $d = 200$ m. This produces a typical gradient of $d/l = 2/3 \times 10^{-2}$. In the model developed below the variables are scaled by typical values for an ocean basin. The horizontal scale is $L = 2 \times 10^3$ km and the vertical scale is $D = 4 \times 10^3$ m. The scaled gradient of the continental shelf in terms of the dimensionless scaled variables is

$$\frac{d/D}{l/L} = \frac{0.5 \times 10^{-1}}{1.5 \times 10^{-2}} = 3.33,$$

an 0(1) quantity. Hence in terms of dimensionless variables the width and depth of the shelf boundary layer have the same order of magnitude. The aspect ratio of the ocean, $\delta = D/L$ is 2×10^{-3}.

In the model of Birchfield (1973) the depth of the boundary layer is considered to be greater than the width so that vertical scaling is not introduced into the analysis. A theory for a slope region in which the scaled gradient is large is presented by Killworth (1973). He shows how the surface Ekman layer transport is transferred into the interior of the ocean when the lower Ekman layer overlaps with the upper Ekman

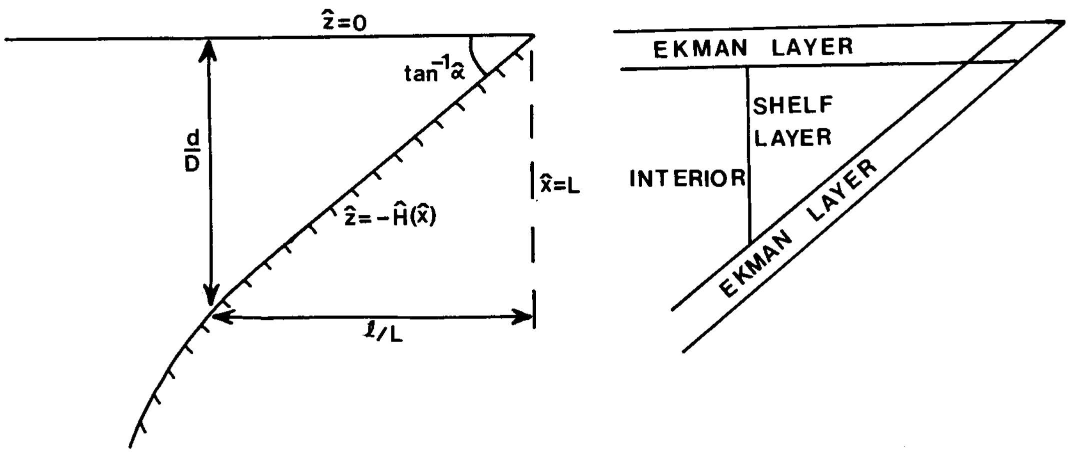

Fig. 1. Notation and position of boundary layers.

layer at the top of the slope. The theory presented below will be concerned only with shelf slopes that have an $O(1)$ gradient.

The notation to be used is illustrated in Fig. 1. The coordinate axes are chosen so that $\hat{x}$ is eastward, $\hat{y}$ northward and $\hat{z}$ vertically upward. The surface of the ocean is at $\hat{z} = 0$. The bottom of the ocean is $\hat{z} = -\hat{H}(\hat{x})$ and near the eastern boundary $\hat{H} = \hat{\alpha}(L - \hat{x}) + O(L - \hat{x})^2$ where $\tan^{-1} \hat{\alpha}$ is the angle that the bottom makes with the surface at the coast. The gradient of the shelf is $\hat{H}_x(L) = -\hat{\alpha}$.

The solution is illustrated in section 7 for a particular wind stress. The distribution of velocity across the shelf is determined and the total transport calculated. A discussion of the solution is given in section 9.

2. Equations of motion and boundary conditions

Let the dimensionless variables be defined by

$$x,\, y = (\hat{x}, \hat{y})L;\; z = \hat{z}/D;\; u, v = (\hat{u}, \hat{v})/U;\; w = \hat{w}/(\delta U);$$

$$H = \hat{H}/D;\; \alpha = \hat{\alpha}/\delta;\; f = \hat{f}/f_0;\; \beta = \hat{\beta}L/f_0$$

where U is a horizontal velocity scale, $\hat{f} = f_0 + \hat{\beta}\hat{y}$ is the variable Coriolis parameter and $\hat{u}, \hat{v}, \hat{w}$ are the dimensional velocity components. The dimensionless equations of motion for steady linear barotropic flow are

$$\left.\begin{aligned} -fv &= -p_x + E_H(u_{xx} + u_{yy}) + Eu_{zz}, \\ fu &= -p_y + E_H(v_{xx} + v_{yy}) + Ev_{zz}, \\ O &= -p_z + \delta^2 E_H(w_{xx} + w_{yy}) + \delta^2 E w_{zz}, \\ & u_x + v_y + w_z = 0, \end{aligned}\right\} \quad (1)$$

where p is the reduced pressure, $f = 1 + \beta y$ and the Ekman numbers are defined as

$$E_H = \nu_H/(f_0 L^2),\; E = \nu/(f_0 D^2)$$

where ν_H, ν are horizontal and vertical coefficients of eddy viscosity respectively. For convenience of presentation it will be assumed that $E_H = E$. In the theory of the shelf boundary layer it will be assumed that the weak restriction $\delta < E^{1/4}$ holds.

The wind stress condition at the surface requires

$$u_z = E^{-\frac{1}{2}}\tau^x,\; v_z = E^{-\frac{1}{2}}\tau^y,\; w = 0;\; z = 0$$

On the sloping bottom,

$$u = v = w = 0,\; z = -H(x).$$

To find a solution that is valid as the depth of the ocean tends to zero as $x \to 1$, the ocean is divided up into the regions illustrated in Fig. 1. Away from $x = 1$ there is an interior region where frictional effects are not important, with Ekman layers at the surface and at the bottom. The lower layer here is weak and does not carry any significant transport. Near the coast there is a shelf boundary layer. Here the northward velocity is larger and the lower Ekman layer becomes important and carries the coastal upwelling. The interior solution is discussed briefly in the next section. The remainder of the paper is devoted to a discussion of the solution in the coastal layer and its associated Ekman layers.

3. The interior solution

As the scaled gradient of the shelf is $O(1)$, the solution of Johnson et al. (1971) for the flow over bottom topography may be used in the interior region away from the coast. For this solution the velocity components are

$$u_I = \frac{E^{1/2}\beta}{fH}\left\{\int^x \frac{\partial}{\partial\eta}\left(\frac{f^2}{\beta H}\mathbf{k}\cdot\operatorname{curl}\frac{\tau}{f}\right)dx' - c(\eta^{-1})\right\} \tag{2}$$

$$v_I = \frac{E^{1/2}H_x}{H^2}\left\{\int^x \frac{\partial\eta}{\partial}\left(\frac{f^2}{\beta H}\mathbf{k}\cdot\operatorname{curl}\frac{\tau}{f}\right)dx' - c(\eta^{-1})\right\}$$

$$+E^{1/2}\frac{f}{\beta H}\mathbf{k}\cdot\operatorname{curl}\frac{\tau}{f}, \tag{3}$$

$$w_I = E^{1/2}\mathbf{k}\cdot\operatorname{curl}\frac{\tau}{f} + \frac{\beta z}{f}v_I$$

where $\eta = -f/H$ is a characteristic coordinate and $C(\eta^{-1})$ is an arbitrary function to be determined by matching with the shelf layer solution.

As $x \to 1$ and $H(x)$ becomes small, the behaviour of these interior solutions, to lowest order, is described by

$$u_I \sim O\left(\frac{E^{1/2}}{H}\right), \quad v_I \sim O\left(\frac{E^{1/2}H_x}{H^2}\right) \tag{4}$$

Hence u_I and v_I are singular as $H \to 0$ and to avoid this singularity in the solution a boundary layer must be inserted near the coast.

The general character of the interior solutions and its attendant Ekman layers does not change much so long as v_I remains less than $O(1)$. The upper Ekman layer transmits the wind stress to the interior and carries the $O(E^{\frac{1}{2}})$ Ekman transport. The lower Ekman layer is weak and merely serves to satisfy the bottom boundary condition. It carries only $O(E)$ transport and so does not take part in the lowest order circulation.

When the northwards velocity between the Ekman layers becomes $O(1)$ a distinct change takes place. An $O(1)$ eastward flow is induced in the lower Ekman layer due to the Ekman

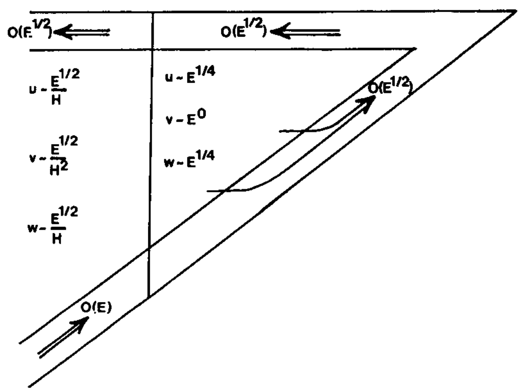

Fig. 2. Magnitude of the velocity components in the shelf layer and interior regions. The open arrows represent the transport in the Ekman layers.

spiral and the lower layer can now carry an $O(E^{\frac{1}{2}})$ transport and take part in the basic circulation. The lower Ekman layer is able to carry the upwelling water to the surface layer which carries it offshore. It may be seen from (4) that for the case in which the scaled gradient H_x is $O(1)$, the northward velocity v_I will become $O(1)$ when $H = O(E^{1/4})$. This occurs at a distance $O(E^{1/4})$ from the coast and from (4),

$$u_I = O(E^{1/4}), \quad v_I = O(E^0)$$

In this coastal layer of depth $O(E^{1/4})$ and width $O(E^{1/4})$ the northward transport will be $O(E^{1/2})$ and so forms part of the basic circulation. Fig. 2 is a summary of the magnitudes of the velocities of the various layers and of the transport in the Ekman layer.

4. The $E^{1/4}$ eastern shelf layer

The order of magnitude discussion presented in the preceding section is used to determine a formal expansion for the velocity field in a shelf layer of thickness $E^{1/4}$ near the *eastern* boundary of the ocean. The theory has to be modified considerably for use on the western side of the ocean as the interior solution (2) and (3) has a different asymptotic form as $x \to 0$ because of the asymmetry of the westward intensification of the interior circulation.

In the eastern shelf layer, let

$$X = \frac{x-1}{E^{1/4}}, \quad \varrho = \frac{z}{\alpha(x-1)} = \frac{z}{E^{1/4}\alpha X} \tag{5}$$

so that the upper surface of the layer is $\varrho = 0$ and the lower surface is $\varrho = 1$, and expand the velocity components and pressure so that

$$u = E^{1/4}U_0 + E^{1/2}U_1 + \dots,$$

$$v = V_0 + E^{1/4}V_1 + \dots,$$

$$w = E^{1/4}W_0 + E^{1/2}W_1 + \dots,$$

$$p = E^{1/4}P_0 + E^{1/2}P_1 + \dots$$

Substituting these expansions into (1) and observing from (1c) that the pressure does not vary with depth to $O(E^{3/4})$, it follows that

$$U_{0\varrho} = U_{1\varrho} = 0, \quad V_{0\varrho} = V_{1\varrho} = 0$$

Therefore the equations of motion reduce to

$$fV_0 = P_{0X}, \quad fV_1 = P_{1X} \tag{6}$$

$$fU_0 = -P_{0y}, \quad fU_1 = -P_{1y} + V_{0XX} \tag{7}$$

$$P_{0\varrho} = 0, \quad P_{1\varrho} = 0 \tag{8}$$

$$U_{0X} + V_{0y} + \frac{1}{\alpha X} W_{0\varrho} = 0, \quad U_{1X} + V_{1y} + \frac{1}{\alpha X} W_{1\varrho} = 0 \tag{9}$$

The derivative of the continuity equations (9) with respect to ϱ gives

$$W_{0\varrho\varrho} = 0, \quad W_{1\varrho\varrho} = 0$$

and hence

$$W_0 = A_0 + B_0\varrho, \quad W_1 = A_1 + B_1\varrho \tag{10}$$

where A_0, A_1, B_0, B_1 are unknown functions of x and y. The Ekman suction conditions at $\varrho = 0$ determine A_0 and A_1. In the upper Ekman layer the flow produced by the wind stress has the well-known solution for the vertical velocity, $w_E = E^{1/2}\{\mathbf{k}\cdot\text{curl}\,(\boldsymbol{\tau}/f) + \text{exponential terms}\} + O(E)$ Asymptotic matching with the shelf layer as $\varrho \to 0$ implies that

$$A_0 = 0, \quad A_1 = \mathbf{k}\cdot\text{curl}\,(\boldsymbol{\tau}/f) \tag{11}$$

If the pressure is eliminated from (6a) and (7a), and the result is substituted into (9a), then

$$fW_{0\varrho} = \alpha X\beta V_0 = -f\alpha X(U_{0X} + V_{0y}) \tag{12}$$

Similarly to the next order

$$fW_{1\varrho} = \alpha X\beta V_1 - \alpha X V_{0XXX} \tag{13}$$

Equations (10)–(13) show that

$$fW_0 = \alpha X\beta\varrho V_0$$

$$fW_1 = f\mathbf{k}\cdot\text{curl}\,(\boldsymbol{\tau}/f) + \alpha X\beta\varrho V_1 - \alpha X\varrho V_{0XXX} \tag{14}$$

The following analysis of the lower Ekman layer will give a second equation connecting W_0 and V_0 from which the differential equation for V_0 may be obtained.

5. The Ekman layer beneath the shelf layer

Let the vertical and horizontal coordinates be scaled to

$$\bar\eta = \frac{z - \alpha(x-1)}{E^{1/2}}, \quad X = \frac{x-1}{E^{1/4}}$$

so that the bottom is $\bar\eta = 0$. In this layer the $O(1)$ northward velocity in the shelf layer must decay to zero to satisfy the boundary condition. The Ekman spiral and the sloping boundary require that all the velocity components are $O(1)$. Hence let

$$(\bar u, \bar v, \bar w) = (\bar u_0, \bar v_0, \bar w_0) + E^{1/4}(\bar u_1, \bar v_1, \bar w_1) + \dots,$$

$$p = \bar p_0 + E^{1/4}\bar p_1 + \dots$$

and substitute into (1). After a little manipulation it may be shown that the Ekman layer equations are modified to

$$(1+\alpha^2)^2 \bar v_{0\bar\eta\bar\eta\bar\eta\bar\eta} + f^2 \bar v_{0\bar\eta} = 0 \tag{15}$$

$$(1+\alpha^2)^2 \bar u_{0\bar\eta\bar\eta\bar\eta\bar\eta} + f^2 \bar u_{0\bar\eta} = 0 \tag{16}$$

The solutions of (15) and (16) that match the shelf layer as $\bar\eta \to \infty$ and satisfy the boundary condition $\bar u_0 = \bar v_0 = 0$ on $\bar\eta = 0$ are

$$\left.\begin{aligned} \bar v_0 &= V_0(X, y)[1 - \tfrac{1}{2}\exp\{-(1+i)F\bar\eta\} \\ &\qquad - \tfrac{1}{2}\exp\{-(1-i)F\bar\eta\}], \\ \bar u_0 &= iV_0(X, y)[-\tfrac{1}{2}\exp\{-(1+i)F\bar\eta\} \\ &\qquad + \tfrac{1}{2}\exp\{-(1-i)F\bar\eta\}] \end{aligned}\right\} \tag{17}$$

where $F^2 = \tfrac{1}{2}f/(1+\alpha^2)$. A similar analysis to $O(E^{1/4})$ reveals that

$$\bar u_1 = U_0(X, y) + \text{exponential terms}$$

The continuity equation to zeroth and first order is

$$-\alpha\bar{u}_{0\bar{\eta}} + \bar{w}_{0\bar{\eta}} = 0, \quad \bar{u}_{0X} - \alpha\bar{u}_{1\bar{\eta}} + \bar{w}_{1\bar{\eta}} = 0$$

Integration of these two equations with respect to $\bar{\eta}$ and use of the boundary conditions $\bar{u}_0 = \bar{u}_1 = \bar{w}_0 = \bar{w}_1 = 0$ at $\bar{\eta} = 0$ leads to

$$\bar{w}_0 = \alpha\bar{u}_0,$$

$$\bar{w}_1 = \frac{V_{0X}}{2F} + \alpha U_0 + \text{exponential terms} \tag{18}$$

6. Completion of the eastern shelf layer solution

Matching (14) as $\varrho \to 1$ with (18) as $\bar{\eta} \to \infty$ gives the equation

$$\frac{\alpha X\beta}{f} V_0 = \frac{V_{0X}}{2F} + \alpha U_0 \tag{19}$$

Equation (12) may be used to eliminate U_0 and leave the differential equation for V_0 as

$$\frac{1}{2\alpha F} V_{0XX} - \frac{\beta X}{f} V_{0X} - V_{0y} - \frac{2\beta}{f} V_0 = 0 \tag{20}$$

This equation may be simplified by transforming the coordinates to $\xi = X/f$ and

$$Y = \int_0^y \frac{dy'}{2\alpha f^2 F} = \frac{2^{1/2}(1+\alpha^2)^{1/2}}{3\alpha\beta}(1 - f^{-3/2}) \tag{21}$$

The maximum range of y is $(-\beta^{-1}, \infty)$ as the solution breaks down at $y = -\beta^{-1}$ which corresponds to the equator where $f = 0$. The corresponding range of Y is $(-\infty, 2^{\frac{1}{2}}(1+\alpha^2)^{\frac{1}{2}}/3\alpha\beta)$. As a practical upper limit on y is between one and two, Y never approaches its upper limit.

Substitution of (21) changes (20) into

$$\frac{\partial^2}{\partial\xi^2}(f^2 V_0) - \frac{\partial}{\partial Y}(f^2 V_0) = 0 \tag{22}$$

a form of the one-dimensional diffusion equation. Carslaw & Jaeger (1959) give a solution of (22) in the form of the convolution integral,

$$f^2 V_0(\xi, Y) = \int_{-\infty}^{Y} a(t)\,(Y-t)^{-1/2} \exp\left\{\frac{-\xi^2}{4(Y-t)}\right\} dt \tag{23}$$

where $a(t)$ is given in terms of the boundary condition at the coast $\xi = 0$ as

$$a(t) = \frac{1}{\pi}\frac{d}{dt}\int_{-\infty}^{t} \frac{f^2 V_0(0, Y)}{(t-Y)^{1/2}} dY \tag{24}$$

To find $a(t)$ in (24), a condition is required at $\xi = 0$ on the velocity V_0. This is provided by the transport condition which states that the eastward transport at the coast in the lower Ekman layer is equal to the westward transport at the coast in the upper Ekman layer. In other words the offshore Ekman transport driven by the wind stress τ^y is replaced by upwelling from the bottom Ekman layer. The eastward transport in the lower Ekman layer at $X = 0$ (or $x = 1$) may be found by integrating (17) across the layer and is

$$E^{1/2}\int_0^\infty \bar{u}_0\Big|_{X=0} d\bar{\eta} = -\frac{E^{1/2} V_0}{2F}\Big|_{X=0}$$

The westward transport in the surface Ekman layer is $-E^{1/2}\tau^y/f$ evaluated at $x = 1$. Hence

$$V_0\Big|_{\xi=0} = \frac{2F}{f}\tau^y\Big|_{x=1} \tag{25}$$

To simplify the presentation the derivation of $a(t)$ in (24) is carried out in Appendix I, where it is shown that (23) becomes

$$V_0(\xi, Y) = \frac{1}{f^2(Y)}\int_{-\infty}^{Y}\left\{\frac{d}{ds}G(s)\right\} \times \operatorname{erfc}\left\{-\frac{\xi}{2(Y-s)^{1/2}}\right\} ds \tag{26}$$

where $G(Y) = 2fF\tau^y(1, Y)$ and the complementary error function is defined by

$$\operatorname{erfc}(T) = \frac{2}{\pi^{1/2}}\int_T^\infty e^{-t^2} dt$$

The solution (26) gives the northward velocity in the shelf layer as a function of $\xi = X/f$ for any given Y (or latitude y).

The total northwards transport in the shelf current at a given latitude is

$$T_N = E^{1/4}\int_{-\infty}^{0}\left[\int_{\alpha(x-1)}^{0} V_0\, dz\right] dX$$

where $X = (x-1)/E^{1/4}$ and the integral is over the triangular cross section of the shelf layer. As V_0 is independent of depth,

$$T_N = -E^{1/2}\int_{-\infty}^{0} \alpha V_0 X\, dX = -E^{1/2}\int_{-\infty}^{0} \alpha f^2 V_0 \xi\, d\xi \tag{27}$$

The last integral is evaluated in appendix II where (26) is substituted in (27). The result is

$$T_N = E^{1/2}\int_{-1/\beta}^{y} \left(\frac{1+\alpha^2}{2}\right)^{1/2} \frac{V_0(0,t)\,dt}{(1+\beta t)^{1/2}} \tag{28}$$

in terms of the original latitude y. Use of (25) and the definition of F leads to

$$T_N = E^{1/2}\int_{-1/\beta}^{y} \frac{\tau^y(1,t)\,dt}{1+\beta t} \tag{29}$$

This result may be justified from physical considerations. In section 8 it will be shown that to lowest order no water passes directly from the shelf layer to the interior. Hence all the water downwelling from the Ekman layer must turn northward or southward in the shelf layer. The net input into the shelf layer between the equator at $y = -1/\beta$ and a given latitude y is the integral of the Ekman transport τ^y/f at the coast. This is the integral in (29) which represents the total outflow northwards at latitude y if there is no net transport across the equator in the shelf layer.

As the Ekman layer solution breaks down at the equator these solutions should all start just north of $y = -1/\beta$. To avoid this difficulty the example used in the next section has a wind stress which is zero at the equator and has a turning point at the equator.

7. The shelf current for a particular wind stress

The solution (26) is illustrated for a particular wind stress in Fig. 3. The wind stress chosen at the coast is

$$\tau^y = f\cos\tfrac{1}{6}\pi\,(1+4\beta y) \tag{30}$$

which is northward just north of the equator and southwards further north. This induces coastal downwelling from $\beta y = (-1)+$, just north of the equator, to $\beta y = \frac{1}{2}$ and coastal upwelling between $\beta y = \frac{1}{2}$ and $\beta y = 2$.

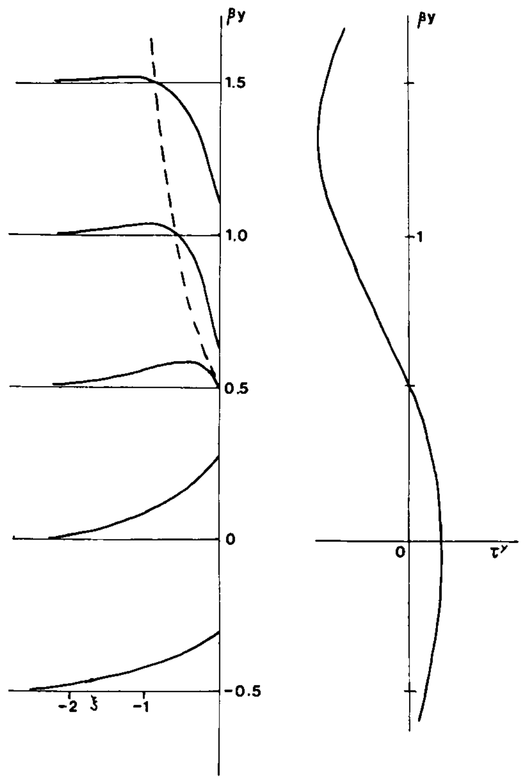

Fig. 3. On the right, the wind stress distribution at the coast, $\tau^y = f\cos\frac{1}{6}\pi(1+4\beta y)$. On the left, the corresponding shelf current and counter-current.

The northward velocity at the inshore edge of the shelf layer satisfies (25) and is therefore northward for $\beta y < \frac{1}{2}$, and southward for $\beta y > \frac{1}{2}$ in the region of coastal upwelling. However further offshore the coastal current is everywhere towards the north. Hence in the region of upwelling there is a counter-current offshore that penetrates to the bottom. Although the velocity distribution just offshore is a function of the local wind given by (25), further offshore the direction of the flow depends on the latitudinal distribution of wind stress used in the integral (26). In this model the presence of a counter-current at a particular latitude depends on the distribution of τ^y between the equator and that latitude.

The total northward transport in the shelf layer is given by substituting (30) into (29) to obtain

$$T_N = E^{1/2}\frac{3}{2\pi\beta}\left\{1+\sin\frac{\pi}{6}(1+4\beta y)\right\}$$

which is never negative. Hence the net transport is northward at all latitudes. The strong southward current near the coast in Fig. 3 for $\beta y > \frac{1}{2}$ does not alter the direction of the net transport as the depth is very small near the coast. The total transport in the shelf current depends on the latitudinal distribution of wind stress and not on the local wind stress.

8. Asymptotic behaviour of the solution

The matching of the shelf layer onto the interior is discussed here. The eastward velocity in the interior is given by (2), which may be simplified for a zonal wind stress of the form

$$\tau^x = \tau^y = -f\int T(f)\,dy, \quad \mathbf{k}\cdot\operatorname{curl}(\boldsymbol{\tau}/f) = T(f),$$

where T is an arbitrary function of f. Then

$$\beta\frac{\partial}{\partial\eta}\left(\frac{f^2}{\beta H}\mathbf{k}\cdot\operatorname{curl}\frac{\boldsymbol{\tau}}{f}\right) = 2H\eta\, T(-\eta H) - H^2\eta^2 T'(-\eta H)$$

where the prime denotes a total derivative. Hence, from (2),

$$u_I = \frac{E^{1/2}}{fH}\int^x \left\{2H\eta T(-\eta H) - H^2\eta^2 T'(-\eta H)\right\}dx' - \frac{E^{1/2}}{fH}\beta C(\eta^{-1}) \qquad (31)$$

The expansion of this integral near $x=1$ is required to facilitate matching with the shelf layer.

The following expansions are valid near $x=1$,

$$H(x) = -\alpha(x-1) + O(x-1)^2$$

as

$$H(1)=0, \quad H'(1) = -\alpha,$$

$$T(-\eta H) = T\Big|_{x=1} + (x-1)\frac{\partial T}{\partial x}\Big|_{x=1} + O(x-1)^2 = T(0) + (x-1)\,\eta\alpha T'(0) + O(x-1)^2$$

as

$$T_x = -\eta H_x T',$$

$$C(\eta^{-1}) = C(0) + (x-1)\frac{\alpha}{f}C'(0) + O(x-1)^2$$

as

$$C_x = -H_x\frac{C'}{f}$$

Substituting in (31), carrying out the integration keeping η constant, and retaining only lowest order terms leads to

$$u_I = E^{1/2}\left[\frac{\beta C(0)}{\alpha f(x-1)} + \dot{O}(x-1)^\circ\right]$$

This may be rewritten in the inner shelf layer variables as

$$u_I = E^{1/4}\frac{\beta C(0)}{\alpha f X} + O(E^{1/2}) \qquad (32)$$

The expression for the eastward flow in the shelf layer may be obtained from (19) and (23). It is clear that the solution decays exponentially for large negative $\xi = X/f$. Therefore matching with (32) gives $C(0)=0$. The determination of other terms in the expansion for $C(\eta^{-1})$ requires higher order solutions. The presence of the shear layer makes the lowest order interior solution well-behaved as the coast is approached. This means that the interior gyre is shielded from the coast by the strong shelf current which takes care of the upwelling and downwelling into the Ekman layer.

Where the wind stress is northward at the coast, it produces downwelling which enters the lower Ekman layer over the shelf. This flow then passes into the shelf layer and moves north or south to provide the source of water for the upwelling being induced by southward winds at some other latitude.

9. Discussion

The wind stress on the surface produces an $O(E^{\frac{1}{2}})$ transport in the Ekman layer just below the surface. If the surface wind has a southward component then there is an offshore component of Ekman transport in the surface boundary layer. This removes water from the region close to the coast which must be replaced by water upwelling from below. This water is drawn up from the bottom Ekman layer on the shelf. To maintain an $O(E^{\frac{1}{4}})$ transport in the bottom layer requires a large $O(E^0)$ longshore flow V_0 in the shelf layer, as the magnitude of the transport in the bottom layer is proportional to $E^{\frac{1}{4}}V_0$ (see section 6). Thus a strong longshore current in the shelf layer is associated with coastal upwelling.

Further away from the coast, beyond the shelf boundary layer the northward velocity in the interior is only $O(E^{\frac{1}{4}})$ and therefore the lower Ekman layer can no longer provide sufficient transport for the coastal upwelling. Consequently the upwelling water must come out of the shelf layer into the lower Ekman layer near the coast. In section 8 it is shown that there is no flow from the interior region into the shelf layer. Therefore the demand for water for upwelling in the lower Ekman layer produces the longshore flow in the shelf layer. The strength of the longshore current will depend on the distribution of upwelling and downwelling along the coast. The solution for a particular wind stress distribution is presented in Fig. 3. In this example an offshore countercurrent occurs where the coastal upwelling extracts less water than is flowing northwards along the layer, produced by the downwelling further south.

Allowing for the fact that this model is steady and homogeneous, how do the results obtained compare with observations. Smith et al. (1971) indicate that during coastal upwelling off Oregon there is onshore flow in a bottom mixed layer corresponding to the lower Ekman layer in this model. Bang (1971) observes a similar situation in the Benguela region. He also states that the longshore currents and counter-currents are intermittant in time and space. However very few actual velocity measurements have been reported. This situation should improve with the current programmes of observation in upwelling regions. When upwelling favourable winds blow for more than four days (the response time for upwelling) it is often observed that a quasi-steady state is reached in the ocean. This state may be related to the steady theory presented here.

The lack of stratification is a serious omission. However its inclusion would significantly increase the complexity of the analytical problem. It is an essential feature in any theory which tries to model the pycnocline mentioned by Smith et al. (1971) and of theories in which the shelf current is allowed to vary with depth to include an undercurrent. In this homogeneous theory the counter-current is offshore of the main coastal current and both penetrate to the bottom.

Appendix I

The analysis presented here gives the derivation of (26). The solution of equation (22) may be expressed as in (23) by

$$f^2(Y)\,V_0(\xi,\,Y) = \int_{-\infty}^{Y} a(t)\,(Y-t)^{-1/2} \times \exp\left\{\frac{-\xi^2}{4(Y-t)}\right\} dt \qquad \text{(A 1)}$$

where $a(t)$ is to be determined from

$$f^2(Y)\,V_0(0,\,Y) = \int_{-\infty}^{Y} a(t)\,(Y-t)^{-1/2} dt$$

The inversion of this integral is given by Whittaker & Watson (1965, p. 229) as

$$a(t) = \frac{1}{\pi}\frac{d}{dt}\int_{-\infty}^{t} \frac{f^2(s)\,V_0(0,s)\,ds}{(t-s)^{1/2}}$$

$$= \frac{1}{\pi}\frac{d}{dt}[-2f^2(s)\,V_0(0,s)\,(t-s)^{1/2}]_{s=-\infty}^{t}$$

$$+\frac{1}{\pi}\frac{d}{dt}\int_{-\infty}^{t} 2(t-s)^{1/2}\frac{d}{ds}\{f^2(s)\,V_0(0,s)\}\,ds$$

In the first term, the lower limit is zero as, from (21), with Y replaced by s,

$$f^2(s)\,(t-s)^{1/2} \approx (-s)^{-4/3}(-s)^{1/2} \quad \text{for} \quad |s| \gg 1$$

Hence

$$a(t) = \frac{1}{\pi}\int_{-\infty}^{t} (t-s)^{-1/2}\frac{d}{ds}\{f^2(s)\,V_0(0,s)\}\,ds$$

and substituting in (A 1) gives, after changing the order of integration,

$$f^2(Y)\,V_0(\xi, Y) = \int_{-\infty}^{Y} \frac{1}{\pi}\frac{d}{ds} \times \{f^2(s)\,V_0(0,s)\}\,I(\xi, Y, s)\,ds \qquad \text{(A 2)}$$

where

$$I(\xi, Y, s) = \int_s^Y (t-s)^{-1/2}(Y-t)^{-1/2} \times \exp\left\{\frac{-\xi^2}{4(Y-t)}\right\} dt.$$

The substitution $z = (Y-s)/(Y-t)$ leads to

$$I(\xi, Y, s) = \int_1^{\infty} \exp\left\{\frac{-\xi^2 z}{4(Y-s)}\right\}\frac{dz}{z(z-1)^{1/2}} = \pi \operatorname{erfc}\left\{\frac{-\xi}{2(Y-s)^{1/2}}\right\} \qquad \text{(A 3)}$$

where the negative square root of the coefficient of $-z$ in the exponential in (A3) has been chosen to select the solution which decays as $\xi \to -\infty$. Substitution for I in (A2) gives

$$f^2(Y)\,V_0(\xi, Y) = \int_{-\infty}^{y} \frac{d}{ds}\{f^2(s)\,V_0(0,s)\} \times \operatorname{erfc}\left\{\frac{-\xi}{2(Y-s)^{1/2}}\right\} ds \qquad \text{(A 4)}$$

The substitution for $V_0(O, s)$ in terms of $\tau^y(1, Y)$ given in (25) leads to the presented solution (26).

Appendix II

To deal with the total northward transport, substitute (A 4) into (27), giving

$$T_N(Y) = -E^{1/2}\int_{-\infty}^{0} \alpha \int_{-\infty}^{Y} \frac{d}{ds}\{f^2(s)\,V_0(0,s)\} \times \operatorname{erfc}\left\{\frac{-\xi}{2(Y-s)^{1/2}}\right\} ds\,\xi\,d\xi \qquad \text{(A 5)}$$

The order of integration may be reversed and the ξ-integral is

$$\int_{-\infty}^{0} \xi \operatorname{erfc}\left\{\frac{-\xi}{2(Y-s)^{1/2}}\right\} d\xi = -4(Y-s)\int_0^{\infty} t \operatorname{erfc} t\,dt$$

$$= -4(Y-s)\left[\tfrac{1}{2}t^2 \operatorname{erfc} t\right]_0^{\infty} - 4(Y-s)\int_0^{\infty} t^2 \pi^{-1/2} e^{-t^2} dt$$

$$= -4(Y-s)\int_0^{\infty} \frac{1}{2} u^{1/2} \pi^{-1/2} e^{-u} du = -(Y-s)$$

Substitute in (A 5) to get

$$T_N(Y) = -E^{1/2}\alpha \int_{-\infty}^{y} (s-Y)\frac{d}{ds}\{f^2(s)\,V_0(0,s)\}\,ds$$

$$= -E^{1/2}\alpha[f^2(s)\,V_0(0,s)\,(s-Y)]_{s=-\infty}^{Y} + E^{1/2}\alpha \int_{-\infty}^{Y} f^2(s)\,V_0(0,s)\,ds$$

$$= E^{1/2}\alpha \int_{-\infty}^{Y} f^2(s)\,V_0(0,s)\,ds \qquad \text{(A 6)}$$

as $f^2(s) \to 0$ as $s \to -\infty$. The substitution

$$1 + \beta t = [1 - 3\alpha\beta s\{2(1+\alpha^2)\}^{-1/2}]^{-2/3}$$

transforms the coordinate Y back into the latitude y and (A 6) becomes

$$T_N(y) = E^{1/2}\int_{-1/\beta}^{y} \left(\frac{1+\alpha^2}{2}\right)^{1/2} \frac{V_0(0,t)\,dt}{(1+\beta t)^{1/2}} \qquad \text{(A 7)}$$

which is (28).

Acknowledgement

R.B.H. was supported by a research studentship from the Natural Environment Research Council.

REFERENCES

Bang, N. D. 1971. The southern Benguela current region in February 1966. Part II. Bathythermography and air–sea interactions. *Deep-Sea Res.* *18*, 209–224.

Birchfield, G. E. 1973. An Ekman model of coastal currents in a lake or shallow sea. *J. Phys. Oceanogr.* *3*, 419–428.

Carslaw, H. S. & Jaeger, J. C. 1959. *Conduction of heat in solids.* Oxford.

Johnson, J. A., Fandry, C. B. & Leslie, L. M. 1971. On the variation of ocean circulation produced by bottom topography. *Tellus 23*, 113–121.

Killworth, P. D. 1973. On the circulation of a homogeneous ocean induced by presence of continental slopes. *J. Phys. Oceanogr. 3*, 3–15.

Schulmann, E. E. & Niiler, P. P. 1970. Topographic effects on the wind-driven ocean circulation. *Geophys. Fluid Dynamics 1*, 439–462.

Smith, R. L., Mooers, C. N. K. & Enfield, D. B. 1971. Mesoscale studies of physical oceanography in two coastal upwelling regions; Oregon and Peru. *Fertility of the Sea 2*, 513–535.

Welander, P. 1968. Wind-driven circulation in one and two layer oceans of variable depth. *Tellus 20*, 1–15.

Whittaker, E. T. & Watson, G. N. 1965. *A course in modern analysis.* Fourth Edition, Cambridge.

ТРЕХМЕРНАЯ ТЕОРИЯ БЕРЕГОВЫХ ТЕЧЕНИЙ И ПОДНЯТИЯ ВОД НАД КОНТИНЕНТАЛЬНЫМ ШЕЛЬФОМ

Циркуляция однородного океана изучается в области, где дно повышается до поверхности у берега, как над континентальным шельфом. В этой области становится важным придонный экмановский слой, в котором происходит поднятие вод вблизи берега из внутренней области над шельфом. В шельфовом пограничном слое образуется сильное вдольбереговое течение и при определенных распределениях напряжения ветра образуется глубокое внебереговое противотечение. На данной долготе вдольбереговое течение вблизи берега зависит от локальных ветров, в то время как внебереговое противотечение и чистый перенос массы к северу в слое над шельфом являются функциями распределения напряжения ветра с широтой.