Discrete solutions of the boundary value problem in physical geodesy

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ABSTRACT

Solutions of the discrete boundary value problem of physical geodesy are discussed. Dirac impulses, Wiener–Hopf predictions and "reflexive predictions" are compared. The dramatic computational gains with "reflexive filtering" are discussed.

1. Formulation of the problem

In classical geodesy the principal problem was considered as a problem where gravity was given all over the physical surface of the earth. This continuous approach was first treated in a more general way by Molodensky (1948) and the general problem is normally called the problem of Molodensky.

For most geodetic applications we only know gravity at discrete points and we define our problem in the following way: A finite number of gravity data is given for a non-spherical surface, and it is required to find such a solution that the boundary values for the gravity data are satisfied in all given points (Bjerhammar, 1963, 1964).

2. Solution

For the solution of the previous problem we introduce an internal sphere.[1] *All boundary values of the physical surface are reduced down to this internal sphere using an integral equation for harmonic functions.* See Bjerhammar (1963), (1968), Reit (1967), Krarup (1969), Moritz (1970), Tscherning (1973) and Lauritzen (1973). All these solutions make use of collocation as well as harmonic reduction of gravity. We call this procedure "translocation" (cf. Bjerhammar, 1969). This new approach was first received with considerable scepticism. For Example Pick (1965) questioned the general stability of the solution. J. J. Levallois gave a "proof" for the impossibility of downward continuation. Moritz (1964) was the first scientist who gave the new procedure full support and the author is grateful for many valuable discussions concerning this subject. Moritz (1970) writes:

"Solution of the Bjerhammar's Problem.

Let the gravity anomalies Δg_1, Δg_2 at n points on the earth's surface, at elevations, $h_1, h_2, ..., h_n$ be known from observation: to determine a gravity field consistent with these observations.

This formulation of the main problem of gravimetric geodesy, due to Bjerhammar, is indeed more realistic than the usual formulation in terms of a boundary-value problem, since we observe at discrete points only. The solution of this problem is not unique. It may therefore be restricted by the additional requirement that the computed gravity field be as smooth as possible. This requirement makes sense because then there will be no spurious irregularities that do not correspond to reality.

In a way, the non-uniqueness of the solution of Bjerhammar's problem corresponds to the non-uniqueness of the problem of interpolation. Thus we may expect that optimum interpolation—in the sense of least standard error—will correspond to an optimum solution of Bjerhammar's problem—in the sense of an optimally smooth gravity field. The remark at the end of section 3 indicates the possibility for such a correspondence to exist, and the solution to be described now will bear this out.

In the solution we shall use another idea of Bjerhammar, analytical continuation of the free air anomaly Δg down to sea level. For the usual case, that the measured gravity anomalies are assumed to be known at every point of the earth's

[1] Here called "geosphere".

surface, this method has been described in (Heiskanen & Moritz, 1967, section 8–10)."

Hotine (1969) wrote:

"Realizing the essential simplicity of the classical approach, Bjerhammar, in one of the most modern methods, uses a spherical reference surface completely embedded in the earth. He then uses the upward continuation integral ...

The method is applicable without modification to gravity reduction or to any function derivable from Poisson's integral ... Once the gravity anomaly or preferably the gravity disturbance, has been found at points in the external field, it is a simple matter to compute actual gravity of such points ... Convergence difficulties caused by external matters do not arise because we are not dealing with an infinite series of spherical harmonics, but are concerned only to find a finite series which fits a finite number of observations ... We have no need to interprete values of $(gA)Q$ on the reference sphere as gravity anomalies. We could consider $(gA)Q$ simply a function defined on the reference sphere, which, when substituted in the integral equation ... correctly reproduces the observed quantities, but we still have to show that such a function can be found by solving the integral equation and is expressible to sufficient accuracy by a practicable number of terms."

In these new early descriptions of the method there was always included a direct computation of the reduced gravity (Δg^*) at the internal sphere. The numerical presentation of the reduced gravity was given with the use of power series or grids having constant gravity inside each surface element. In order to test the validity of the method several test models were analyzed. A numerical model with several thousand tons of mass in one single point outside the internal sphere was properly evaluated without any serious difficulties (Bjerhammar, 1969). The accuracy of the solution was improved with increasing power of the applied polynomial giving a bias smaller than the rounding errors. For this giant mass concentration in one single point outside the sphere, all vertical deflections had errors less than 0.3". The study showed that the final results improved considerably when adding a correction for the residuals at the observation points, using the classical formulas for the computation of the vertical deflections. In the practical application we can consider all external mass to be located in a very large number of points with, for example, one million times smaller mass. For each such point we can make a corresponding gravity reduction and the residual errors will be less than $1.10^{-6''}$ for each point. The individual errors will be positive as well as negative and the total error will have a tendency to cancel. It can easily be proved that the errors cancel exactly when the external mass can be "focussed" into points below the sphere.

Krarup (1969) was the first mathematician who studied this new technique in a more advanced way and he stated (p. 9):

"As far as I can see, the most important point of view introduced in physical geodesy since the appearance of Molodensky's famous articles is Bjerhammar's idea of calculating an approximation of the potential by collocation, at the points where gravity anomalies have been measured, of potentials that are regular down to a certain sphere inside the surface of the earth.

From the classical theory this idea looks very venturesome, because we know that the actual potential of the earth is not regular down to the Bjerhammar sphere. Nor does the evidence Bjerhammar produces in support of his idea seem convincing to me at all. As we shall see later on from the point of the new potential theory, very much can be said in favour of the determination of the potential by interpolation methods as well as the use of potentials that are regular outside a Bjerhammar sphere.

There is a very close connection between the problem of the Bjerhammar sphere and the problem of the convergence of series in spherical harmonics or better the question of approximation of potentials by series in spherical harmonics, and here I believe we have reached the very core of the foundation of physical geodesy."

Krarup's original scepticism resulted finally in the most enthusiastic support. He based his thinking on the famous Runge theorem which he gave a new type of proof (Krarup 1969, 1973). He also gave his own contribution with use of a Hilbert space approach ("least squares collocation").

It is obvious that our discrete approach can always give a solution that fits to all given points. We have an infinite number of harmonics in the potential field of the earth and it will be possible to find an exact fit to a finite number of points. Krarup's concern can hardly refer to the problem we originally defined. His whole discussion is apparently more directed to a different problem, namely the solution of the *continuous* boundary value problem in physical geodesy (Molodensky problem). The Runge theorem is of specific interest for the solutions of this problem and the theorem tells us that a *local* solution on a compact *subset* is possible when using uniform approximation.

For our geodetic problem we have a more interesting theorem by Keldych–Lavrentieff (1937) which tells us that uniform approximation is valid for the geodetic case (global case) when solving the problem for the whole compact set of harmonic potentials of the earth.

The geoid

The *geoid* in the classical approach was the equipotential surface that coincides with mean sea-level and is obtained "if the water is allowed to cross the continents in canals" (physical geoid).

In our approach we have defined the geoid as the *equipotential surface coinciding with the mean sea-level when using an analytical continuation of the gravity field of the earth* (Bjerhammar, 1965). It is interesting to note that already Poincaré used a very similar type of definition. This geoid has a simple mathematical relation to the external gravity field. Molodensky (1948) used the concept of the "quasi-geoid". This geoid is obtained when translating the separation between the equipotential surface of the true earth and a theoretical model earth, from the physical surface, down to the reference ellipsoid. There is no simple relation between this geoid and the external gravity field.

3. A Dirac approach to physical geodesy

The author has in a number of papers presented a solution of the boundary value problem of physical geodesy, using a harmonic reduction of the discrete gravity data down to an internal sphere. This type of collocation has further been studied with the use of Hilbert spaces (Krarup, 1969) and the theory of Wiener prediction and filtering (Moritz, 1970). The author gives here a presentation with Dirac functions (based upon distribution functions) which gives exceptional simplifications of the theoretical presentation and applications. The method is completely discrete and therefore computer oriented. All integrations are avoided.

We use the following relation between gravity anomalies at the surface of the earth Δg and the reduced gravity anomly Δg^* at the internal sphere.

$$\Delta g_j = \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta g^*}{r_{ji}^3} dS \tag{1}$$

Δg_j = gravity anomaly (at the surface of the earth or in space)
Δg^* = reduced gravity anomaly at the "sphere"
r_j = geocentric distance to the actual point
r_0 = radius of the sphere
r_{ji} = distance between the actual point (P_j) at the physical surface and the moving point (P_i) at the sphere
S = surface of the sphere

Impulses

It is known that in many branches of physics we are faced with sources that are nearly instantaneous. In order to avoid the cumbersome study of a detailed functional dependence of various sources, one can often replace them by an idealized source which is exactly instantaneous. Such idealized sources are said to be "impulsive".

Carrier points

In our study we use a distribution of the reduced gravity anomaly, where all information is distributed at impulses on the internal sphere. These "carrier points" carry all the information on the sphere and Δg^* is zero outside these points.

Definition

Carrier points are points on the surface of the internal sphere which cary all gravity information as impulses. All remaining points on the surface of the sphere have zero gravity anomalies.

The set of points on the surface of the internal sphere can be split into three subsets

(1) $\Delta g^* \neq 0$ Carrier points with a Dirac impulse
(2) $\Delta g^* = 0$ Carrier points without a Dirac impulse
(3) $\Delta g^* = 0$ Remaining points at the sphere

The subset (2) is only of formal interest and has no real practical importance.

As we have only information in the carrier points we can rewrite eq. (1) as an exact matrix equation

$$\Delta g_j = \frac{(r_j^2 - r_0^2)\, r_0^2}{r_j} \begin{bmatrix} \frac{1}{r_{11}^3} & \frac{1}{r_{12}^3} & \cdots & \frac{1}{r_{1n}^3} \\ \\ \frac{1}{r_{n1}^3} & \frac{1}{r_{n2}^3} & \cdots & \frac{1}{r_{nn}^3} \end{bmatrix} \begin{bmatrix} \Delta g_1^* \\ \Delta g_2^* \\ \vdots \\ \Delta g_n^* \end{bmatrix} \cdot C$$

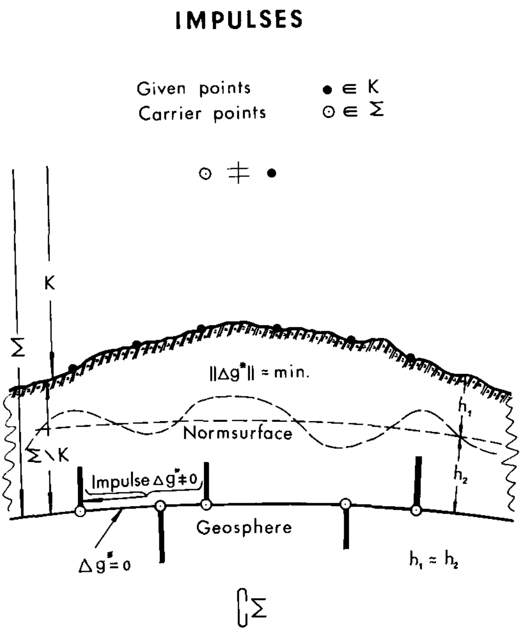

Fig. 1

where

$$C = \frac{1}{4\pi r_0^2}\iint dS = 1$$

or

$$\boxed{\Delta g_j = \boldsymbol{K}\Delta \boldsymbol{g}^*} \tag{2}$$

where

$$K_{ji} = \frac{(r_j^2 - r_0^2)\, r_0^2}{r_j\, r_{ji}^3} = \sum_{n=0}^{\infty} (2n+1)\left(\frac{r_0}{r_j}\right)^{n+2} P_n(\cos\omega_{ji}) \tag{2 a}$$

ω = geocentric angle

We note that Δg^* is different from zero only for the given carrier points. Clearly we are entitled to move all variables outside the integral sign. The size of the impulses has to be computed by solving the linear matrix equation.

If the matrix K is non-singular we obtain the unique solution

$$\Delta \boldsymbol{g}^* = \boldsymbol{K}^{-1}\Delta \boldsymbol{g} \tag{3}$$

Predictions:

$$\Delta g_p = \boldsymbol{K}\Delta \boldsymbol{g}^*$$

$$K_{pi} = \sum_{0}^{\infty} (2n+1)\left(\frac{r_0}{r_p}\right)^{n+2} P_n(\cos\omega_{pi})$$

$$T_p = (\boldsymbol{S}\omega)\,\Delta \boldsymbol{g}^*$$

$$(S\omega)_{pi} = r_p \sum_{n=2}^{\infty} \frac{2n+1}{n-1}\left(\frac{r_0}{r_p}\right)^{n+2} P_n(\cos\omega_{pi})$$

$$\xi_p = \frac{\partial T}{\partial x}\Big/\frac{\partial U}{\partial z} = -\frac{\partial T}{\partial x}\Big/\gamma = -(\boldsymbol{f}_x/\gamma)\,\Delta \boldsymbol{g}^*$$

$$\eta_p = \frac{\partial T}{\partial y}\Big/\frac{\partial U}{\partial z} = -\frac{\partial T}{\partial y}\Big/\gamma = -(\boldsymbol{f}_y/\gamma)\,\Delta \boldsymbol{g}^*$$

$$\begin{bmatrix} f_x \\ f_y \end{bmatrix} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix}\{t^3 \sin\omega_{pi}(2\varrho^{-3} - 8 + 3(\varrho+1)^2/(2\varrho\phi) - 3\ln\phi\}$$

$\varrho^2 = 1 + t^2 - 2t\cos\omega_{pi}$
$\phi = \{(\varrho+1)^2 - t^2\}/4$
α = azimuth
$t = r_0/r_p$ U = theoretical potential
T = disturbance potential

For *filtering we simply* use a smaller number of carrier points than the number of observation points. A least squares solution then gives

$$\Delta \boldsymbol{g}^* = (\boldsymbol{K}^T\boldsymbol{P}\boldsymbol{K})^{-1}\boldsymbol{K}^T\boldsymbol{P}\Delta \boldsymbol{g} \quad \boldsymbol{K}(n\times m)\ n > m$$

Computational gain

We take an example with 1 000 observations of Δg and use a prediction with filtering:

Solution: Wiener–Hopf–Krarup–Moritz.

Number of multiplications:

$$N_1 \approx \frac{n^3}{3} \approx 3\times 10^8$$

Solution: Dirac approach with filtering (50% reduction of unknowns).

Number of multiplications:

$$N_2 \approx \frac{n^3}{24} \approx 4\times 10^7 \quad N_2 = N_1/8$$

Solution: Dirac approach with filtering (75% reduction of unknowns).

Number of multiplications:

$$N_3 \approx \frac{n^3}{192} \approx 5 \times 10^6 \quad N_3 = N_1/64$$

We have now reduced the computational work 64 times (for the solution of the normal equations).

Smoothness: At the topography comparable with solutions according to Wiener–Hopf.

4. Stochastic processes

In the geodetic literature stochastic processes have often been used for the solution of the gravimetric boundary value problem. Krarup (1969), Moritz (1970), Meissl (1971), Lauritzen (1973), Tscherning (1973).

The justification for an application of stochastic processes has not been presented in a convincing way, but we find an interesting discussion in Meissl (1971). He gave four justifications for regarding the gravimetric quantities as stationary stochastic processes.

1. There is "some repetitive pattern" since "the earth's surface may be regarded as consisting of many areas, for example $5° \times 5°$ blocks ... *However, the global random function has still no meaning*".
2. We can choose the selected points at random on the surface of the earth. "*But why shall we choose points at random on the earth's surface?*"
3. "If the gravity anomalies are completely known all over the earth, then there is no need to regard them as a stochastic process."
4. "*The stochastic process-terminology is used merely to make ... the reader more confused.*"

We have here "twisted" the presentation of Meissl somewhat and in fact we only get objections against using the stochastic processes.

It seems tempting to go one step further and try to find an answer to the question if it can be justified using stochastic processes for the solution of the gravimetric boundary value problem.

So far we have seen that the main interest has been devoted to weakly stationary processes. This means that the solutions will include a determination of a covariance function and then also a covariance matrix for the given observations.

Several authors have used the Wiener–Hopf approach in the solution of the boundary value problem. The special solutions of Krarup (1969) and (1973) have been called "least squares collocations" or simply "collocation". There has been some conflicting discussion concerning the ergodicity of the applied stochastic process. We quote:

Lauritzen (1973: p. 80). "This means that even if we knew gravity all over the earth, we would not be able to find the true covariance function. Somehow the problem is not suited for statistical treatment."

Using the theory of stochastic processes we obtain the optimal prediction ($\hat{x}$) for a Wiener–Hopf approach (for a weakly stationary stochastic process)

$$\hat{x} = \boldsymbol{q}\boldsymbol{Q}^{-1}\Delta\boldsymbol{g}$$

where

$\Delta\boldsymbol{g}$ = gravity anomaly with $E\{\Delta\boldsymbol{g}\} = 0$

$\boldsymbol{Q}$ = auto-covariance matrix $E\{\Delta\boldsymbol{g}\,\Delta\boldsymbol{g}^T\}$

$\boldsymbol{q}$ = crosscovariance vector for $x/\Delta g = E\{\boldsymbol{x}\,\Delta\boldsymbol{g}^T\}$

$E\{\}$ = expectation

The elements of the covariance matrix have to be defined by a covariance function

$$Q_{ji} = E\{\Delta g_j \Delta g_i\} \tag{1}$$

Moritz (1970) used the covariance function for Δg

$$Q_{ji} = \sum \sigma_n^2 (2n+1)\,(r_0^2/r_j r_i)^{(n+2)}\,P_n\,(\cos\omega_{ji}) \tag{2}$$

$P_n(\cos\omega_{ji})$ = Legendre polynomial

Krarup (1969) used the covariance function for T

$$Q_{ji} = \sum \sigma_n^2 (2n+1)\,(r_0^2/r_j r_i)^{(n+1)}\,P_n \cos\omega_{ji}) \tag{3}$$

Lauritzen (1973) used the same covariance function with

$$\sigma_n^2 = \frac{A}{(n-2)\,(n-2)\,(2n+1)}$$

$$A = 7.84888;\ r_0/r_j = 0.9945 \tag{4}$$

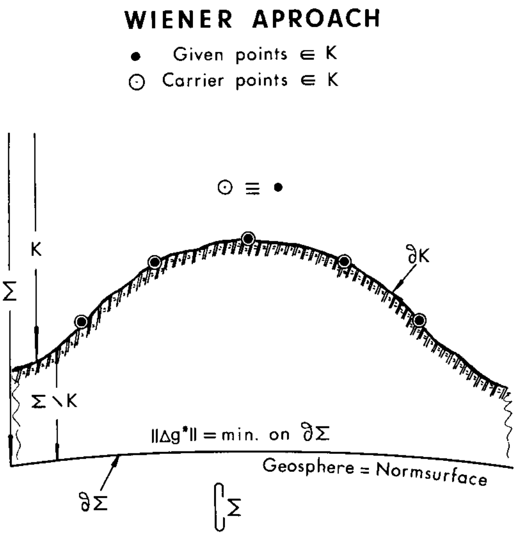

Fig. 2

Lauritzen (1973) proved that the process is not ergodic, which means that "even if we knew gravity all over the earth, we would not be able to find the true covariance function" p. 80. "It means that it is needless work to measure and measure gravity anomalies, deflections of the vertical and so on, in order to determine the covariance function."

5. Reflexive prediction

Abstract. A new prediction problem is defined. For a weakly stationary stochastic process a set of observations is given. It is required to find a set of unknowns (observations) which have the given observations as optimal predictions for a prescribed covariance function. The solution of this problem is very general and includes the Wiener–Hopf approach as a limiting case. An application to the gravimetric boundary value problem in physical geodesy is given. In our study we start with a condensed presentation of the Wiener–Hopf approach for discrete cases.

Suppose that the *expectations* for the "time" (t) and $(t+\tau)$ are identical

$$E\{x(t)\} = E\{x(t+\tau)\} = 0 \tag{1}$$

The covariance function $Q(\tau)$ is defined by

$$Q(\tau) = E\{x(t)x(t+\tau)\} \tag{2}$$

In most applications the following estimator is used

$$\hat{Q}(\tau) = \frac{1}{2(T-\tau)} \int_{T-\tau}^{-T+\tau} x(t)\, x(t+\tau)\, dt \tag{3}$$

This estimator is unbiased with

$$E\{\hat{Q}(\tau)\} = \frac{1}{2(T-\tau)} \int_{T-\tau}^{-T+\tau} E\{x(t)\, x(t+\tau)\}\, dt = Q(\tau) \tag{4}$$

If this estimator converges to the true value when $T \to \infty$ then the process is *ergodic*. For a *non-ergodic* stochastic process we have no simple means of determining the covariance function. Our following study is mainly devoted to problems when the correct covariance function is not a priori given. Instead we want to find "observations" which have a given covariance function.

Most technical studies of the theory of weakly stationary processes result in non-ergodic applications. This is a highly unsatisfactory situation and we find it natural to look for a solution, where we make our predictions with the use of auxililary "unknown observations" which are determined in such a way that they for a prescribed covariance function give optimal predictions, that are identical with the given observations.

Problem. A set of n discrete observations is given. It is required to find a set of unknown observations which have the given observation as optimal predictions for a prescribed covariance function.

Solution. From the Wiener–Hopf approach we recall the general prediction formula

$$\boldsymbol{f} - \boldsymbol{\varepsilon} = \boldsymbol{A}\boldsymbol{h} \tag{5}$$

with

$\boldsymbol{h}$ = set of observation $E(h) = 0$ $(n \times 1)$
$\boldsymbol{f}$ = unknown prediction (1×1)
$\boldsymbol{\varepsilon}$ = prediction error (1×1)
$\boldsymbol{A}$ = predictor (vector) $(1 \times n)$

Wiener wanted to determine the predictor that gives minimum variance for the selected covariance function. We have

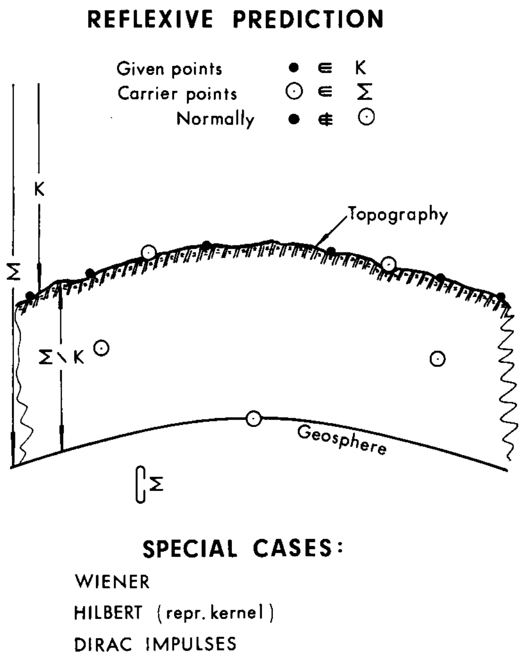

Fig. 3

$$\varepsilon\varepsilon^T = \boldsymbol{A}\boldsymbol{h}\boldsymbol{h}^T\boldsymbol{A}^T - 2\boldsymbol{A}\boldsymbol{h}\boldsymbol{f}^T + \boldsymbol{f}\boldsymbol{f}^T$$

with the expectations

$$\sigma^2 = \boldsymbol{AQA}^T - 2\boldsymbol{AK}^T + E\{\boldsymbol{ff}^T\}$$

where

$$E\{\boldsymbol{hh}^T\} = \boldsymbol{Q};\ E\{\boldsymbol{fh}^T\} = \boldsymbol{K}$$

$2\boldsymbol{AQ} - 2\boldsymbol{K} = 0$ for a minimum

$$\boldsymbol{A} = \boldsymbol{KQ}^{-1}$$

and we obtain the optimal prediction

$$\hat{\boldsymbol{f}} = \boldsymbol{KQ}^{-1}\boldsymbol{h} \quad (\hat{\boldsymbol{f}} = \text{vector } \boldsymbol{K} = \text{matrix}) \tag{6}$$

We now *reverse* this problem and introduce the following modifications:

(1) All observations f are considered "given predictions".

(2) The h-values are considered unknown for selected carrier points.

(3) The prescribed covariance function is valid for the unknown h-values.

We can now solve our problem with the use of a system of linear equations

$$\hat{\boldsymbol{f}} = \boldsymbol{K}\hat{\boldsymbol{x}}$$

with

$$\hat{\boldsymbol{x}} = \boldsymbol{Q}^{-1}\boldsymbol{h}$$

In an application we study the prediction of gravity when using the relation

$$\Delta g_j = \frac{r_j^2 - r_0^2}{4\pi r_j} \iint \frac{\Delta g^*}{r_{ji}^3} dS \tag{7}$$

Δg_j = gravity anomalies at the fixed point P_j
r_j = geocentric distance to the fixed point
r_0 = radius of the internal sphere
r_{ji} = distance between the fixed point P_j and the moving point of integration at the internal sphere
Δg^* = gravity at the internal sphere
dS = surface element at the sphere

We select a solution where the covariance function is given for the gravity points at the sphere. Then we have the following matrix equation

$$\Delta\boldsymbol{g} = \boldsymbol{KQ}^{-1}\Delta\boldsymbol{g}^* \tag{8}$$

$\Delta\boldsymbol{g}^*$ = unknown reduced gravity anomalies for the surface of the internal sphere $(m \times 1)$
$\boldsymbol{Q}$ = auto-covariance matrix of the unknown Δg^*-values $(m \times m)$
$\boldsymbol{Q}$ = $E\{\Delta\boldsymbol{g}^*(\Delta\boldsymbol{g}^*)^T\}$
$\boldsymbol{K}$ = cross-covariance matrix
$\boldsymbol{K}$ = $E\{\Delta\boldsymbol{g}(\Delta\boldsymbol{g}^*)^T\}$ $(n \times m)$
$\Delta\boldsymbol{g}$ = given gravity anomalies $(n \times 1)$

It is anticipated that $E\{\Delta g^*\} = 0$

We can identify the following types of solutions:

Case 1. Prediction without filtering

$n = m$ and full rank of $\boldsymbol{K}$

$$\Delta \boldsymbol{g} = \boldsymbol{K}\boldsymbol{x} \quad \text{where } \boldsymbol{x} = \boldsymbol{Q}^{-1}\Delta \boldsymbol{g} \tag{9}$$

$$\hat{\boldsymbol{x}} = \boldsymbol{K}^{-1}\Delta \boldsymbol{g}$$

Case 2. Prediction and filtering:
$n > m$ and full rank of $\boldsymbol{K}$
Prediction

$$\hat{\boldsymbol{x}} = (\boldsymbol{K}^T\boldsymbol{P}\boldsymbol{K})^{-1}\boldsymbol{K}^T\boldsymbol{P}\Delta \boldsymbol{g} \tag{10}$$

with

$$(\Delta \boldsymbol{g} - \boldsymbol{K}\hat{\boldsymbol{x}})^T\boldsymbol{P}(\Delta \boldsymbol{g} - \boldsymbol{K}\hat{\boldsymbol{x}}) = \min$$

$\boldsymbol{P}^{-1}$ = covariance matrix from the noise of $\Delta \boldsymbol{g}$

Case 3. Pure prediction with redundant carrier points:
$n < m$ and $\boldsymbol{K}$ full rank

$$\hat{\boldsymbol{x}} = \boldsymbol{P}^{-1}\boldsymbol{K}^T(\boldsymbol{K}\boldsymbol{P}^{-1}\boldsymbol{K}^T)^{-1}\Delta \boldsymbol{g} \tag{11}$$

with

$$\boldsymbol{x}^T\boldsymbol{P}\hat{\boldsymbol{x}} = \min$$

Case 4. $\boldsymbol{K}$ of any rank and $\boldsymbol{R}$ = covariance matrix:

$$\hat{\boldsymbol{x}} = \boldsymbol{R}\boldsymbol{K}^T(\boldsymbol{K}\boldsymbol{R}\boldsymbol{K}^T)^{-1}\boldsymbol{K}(\boldsymbol{K}^T\boldsymbol{P}\boldsymbol{K})^{-1}\boldsymbol{K}^T\boldsymbol{P}\Delta \boldsymbol{g} \tag{12}$$

minimizing (using generalized inverses)

$$(\boldsymbol{K}\hat{\boldsymbol{x}} - \Delta \boldsymbol{g})^T\boldsymbol{P}(\boldsymbol{K}\hat{\boldsymbol{x}} - \Delta \boldsymbol{g})$$

and

$$\hat{\boldsymbol{x}}^T\boldsymbol{R}^{-1}\hat{\boldsymbol{x}}$$

In case 1 we choose the covariance function

$$K_{ji} = \sum_{n=0}^{\infty} \sigma_n^2(2n+1)\left(\frac{r_0^2}{r_j^2}\right)^{(n+2)} P_n(\cos \omega_{ji}) \tag{13}$$

The classical formula for the disturbance potential T is in the infinitesimal approach

$$T_j = \frac{r_0^2}{4\pi r_j}\iint \Delta g^* \sum_{n=2}^{\infty}\left(\frac{r_0}{r_j}\right)^n\left(\frac{2n+1}{n-1}\right)P_n(\cos \omega_{ji})\, d\sigma \tag{14}$$

$d\sigma$ = surface element on the unit sphere

or

$$T_j = \frac{r_0}{4\pi}\iint \Delta g^* S(\omega_{ji})\, d\sigma \tag{15}$$

where

$$S(\omega_{ji}) = \sum_{n=2}^{\infty}\left(\frac{r_0}{r_j}\right)^{n+1}\left(\frac{2n+1}{n-1}\right)P_n(\cos \omega_{ji}) \tag{16}$$

For the "reflexive solution" we obtain the predictions

$$\Delta\hat{\boldsymbol{g}}_j = E\{\Delta \boldsymbol{g}_j \boldsymbol{x}^T\}\,\hat{\boldsymbol{x}}$$

$$\hat{T}_j = E\{T_j \boldsymbol{x}^T\}\,\hat{\boldsymbol{x}}$$

The vertical deflection is obtained directly after differentation of $\hat{T}$

We easily identify the following special cases:

(1) Carrier points in the given points. Classical solution according Wiener–Hopf.
(2) Carrier points in the given points with $\sigma_n^2 = 1$. Classical Hilbert space solution with reproducing kernel.
(3) Carrier points on the reference sphere. Dirac approach.
(4) Carrier points on the surface of the earth outside the given points.
(5) Carrier points between the surface of the earth and the reference sphere.
(6) Cases 1–5 with redundant carrier points and minimization of $x^T x$.
(7) Cases 1–5 with least squares filtering.

With any of these choices of carrier points we can combine the four different cases of solution.

Filtering. We can still use the classical filtering according to Wiener. However, a considerable reduction of the computational work is obtained when using least squares filtering according to case 2. If the number of carrier points is reduced to 50% of the total number of observations, then the computational work is reduced by approximately 80%.

Non-ergodicity. When our weakly stationary stochastic process is not ergodic, then the classical solution according to Wiener is not relevant. If we instead, for a given covariance function, choose a solution where the carrier points do not coincide with the given points, then we obtain such values for the carrier points, that an optimal prediction from the carrier points gives exactly the given points (eventually with filtering). *Thus we let our carrier points adapt themselves to the covariance function, instead of adapting the covariance function to the given points.* One can expect that this method is most powerful if we select the carrier points not too close to the given points. When the data for the carrier points have been computed, then the wanted predictions are computed with the use of the "cross-variance" between the carrier points and the unknowns.

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ДИСКРЕТНЫЕ РЕШЕНИЯ ГРАНИЧНОЙ ЗАДАЧИ В ФИЗИЧЕСКОЙ ГЕОДЕЗИИ

Обсуждаются решения дискретной граничной задачи физической геодезии. Сравниваются предсказания, полученные с помощью импульсной функции Дирака, метода Винера–Хопфа и «рефлексивные предсказания».