Interaction of vortices in a fluid on the surface of a rotating sphere

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ABSTRACT

The interaction of 3 point vortices, initially situated on the same circle of latitude on the surface of a rotating sphere, is studied under the assumption that the deviation in latitude is small. A linear stability analysis of the equilibrium motion is performed. The purpose of this work is to simulate and understand the motion of anticyclones and cyclones in the atmosphere: the results may help explain observed phenomena. Under certain boundary conditions a linear analysis shows that a ring of sub-tropical anticyclones is stable, however a ring of cyclones would be unstable. This result is due to the variation of Coriolis force with latitude over the surface of a sphere. The "β-plane" effect induces a rotation of a ring of anticyclones, but opposing the direction of rotation of a ring of cyclones. Unless the strength of the cyclonic vorticity is very strong, this effect causes instability in the system of cyclones. The arguments show that in the atmosphere an anticyclone system can exist at subtropical latitudes, whereas cyclonic vorticity will be confined to the polar regions. These results agree with the observed distribution of the major high and low pressure systems in the earth's atmosphere.

Introduction

In an attempt to simulate the system of semi-permanent sub-tropical high pressure areas Stewart (1943) examined the fluid motions of two-dimensional point vortices on a rotating disc. More recently Morikawa & Swenson (1971) studied the interaction of N rectilinear geostrophic (Bessel) vortices of equal strength γ initially distributed on the perimeter of a circle with a single vortox of strength γ_0 placed initially at the center of the circle. This configuration models the ring of sub-tropical anticyclones and the polar cyclonic vorticity. They performed the linearized stability analysis of the eigenvalue problem and a numerical computation of the non-linear motions in order to interpret the stability results.

The intention of this present investigation is to extend the above work to the case of interacting vortices on a circle of latitude of a rotating sphere. We will not include a polar vortex. In spherical geometry, which is obviously the most relevant to the atmosphere, the Coriolis parameter is not constant but varies with latitude. The effects of a non-constant Coriolis force in a rotating fluid have been widely studied in a number of contexts. It is well known (Greenspan, 1968a) that small deviations from a constant of the Coriolis parameter can produce most marked effects in the fluid motion. For example, the quasi-steady geostrophic mode decomposes into an infinite set of Rossby, or planetary waves. In this present investigation we find that the effect of a variable Coriolis force produces results distinct from those obtained by the previous investigators who considered the problem on a rotating disc. We will use the "β-plane" approximation to model the variation in Coriolis force with latitude: i.e., we approximate $\cos\theta$ by a linear variation in θ. We will also assume that the latitudinal deviation of the vortices from the initial circle of latitude is small. In the appendix we comment on the justification of such an approach.

Fig. 1 indicates that on the spherical earth

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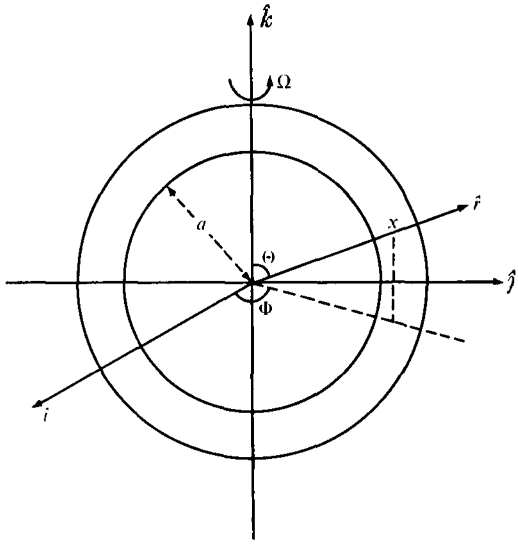

Fig. 1

the two independent variables ϕ and θ are the longitude and co-latitude respectively of the point x, $\hat{k}$ is a unit vector along the axis of rotation, $\Omega\hat{k}$ is the angular velocity vector of the spherical earth, $\hat{r}$ is a unit azimuthal vector at the point x. The orthonormal vectors ($\hat{\imath}$, $\hat{\jmath}$ $\hat{k}$) are affixed to the rotating sphere with $\hat{\imath}$ and $\hat{\jmath}$ in the equatorial plane of the sphere. We will use $\hat{\phi}$ and $\hat{\theta}$ to represent unit tangential vectors in the ϕ and θ directions.

Equations of motion and equilibrium

In this approach, as in the work of Stewart (1943) and Morikawa & Swenson (1971), it is assumed that the atmosphere can be treated as a single layer of fluid of constant density in which the effects of friction are neglected. This layer is assumed to be shallow with respect to both the radius of the earth and the lateral scale of typical planetary motions.

The model equations governing the depth-averaged tangential velocity vector $\mathbf{Q}$ are

$$\frac{\partial H}{\partial t} + \nabla \cdot H\mathbf{Q} = 0 \tag{1}$$

$$\frac{\partial \mathbf{Q}}{\partial t} + \mathbf{Q} \cdot \nabla \mathbf{Q} + 2\Omega \cos\theta \hat{r} \times \mathbf{Q} = -\frac{1}{\varrho} \nabla P \tag{2}$$

where $\mathbf{Q} = (1/H)\int_0^H \mathbf{q}_H dr$, and $\mathbf{q}_H$ is the horizontal component of velocity. H is the atmospheric depth, $\Omega \cos\theta\hat{r}$ is the component of the Coriolis vector $\Omega\hat{k}$ in the direction of the spherical normal, and ∇ denotes the surface gradient. A reference to a fairly complete discussion of the derivation of these equations is given in Greenspan (1968*b*). We will also assume that vertical velocities are small so the pressure can be determined from the hydrostatic equation.

Combining $\nabla \times (2)$ with eq. (1) we can replace eq. (1) by the potential vorticity equation,

$$\frac{d}{dt}\left[\frac{f+\xi}{H}\right] = 0 \tag{3}$$

where the Coriolis parameter is given by $f = 2\Omega \cos\theta$, and $\xi = \hat{r} \cdot \nabla \times \mathbf{Q}$.

We will now assume that the solution (U, V, H) can be expressed in a power series expansion in ε perturbed about the atmosphere at rest, where ε is a small parameter that will be the Rossby number of the flow. Hence,

$$(U, V, H) = (O, O, H_0) + \varepsilon(u, v, h) + O(\varepsilon^2)$$

where $\mathbf{Q} \equiv U\hat{\phi} + V\hat{\theta}$. To $O(\varepsilon)$, eq. (2) in component form becomes

$$\frac{\partial v}{\partial t} - uf + \frac{g}{a}\frac{\partial h}{\partial \theta} = 0 \tag{4}$$

$$\frac{\partial u}{\partial t} + vf + \frac{g}{a \sin\theta}\frac{\partial h}{\partial \phi} = 0 \tag{5}$$

and eq. (1) becomes

$$\frac{\partial h}{\partial t} + \frac{H_0}{a \sin\theta}\left(\frac{\partial v}{\partial \theta}\sin\theta + \frac{\partial u}{\partial \phi}\right) = 0 \tag{6}$$

We note that eqs. (4) to (6) are Laplace's tidal equations which have been studied in detail by Longuet-Higgins (1968) and to whose results we will refer later.

In the case of stable motion of the geostrophic vortices on a disc, Morikawa & Swenson (1971) show that the N circle vortices deviate only slightly from the equilibrium positions. In our examination of the spherical problem we will therfore make the following assumption: the vortices initially lie on a circle of co-latitude θ_0 making only small perturbations from this circle.

We wish to consider motion where the "β-

plane" effect is of order one with respect to the velocity field induced by the vortices. We will therfore assume that the latitudinal width of the band in which the vortices move and the wind speed are of the same order of magnitude. The Rossby number ε is the order of the perturbation from the initial state of rest, hence the latitudinal variation of Coriolis force will also be $0(\varepsilon)$.

We will now investigate motion within this band by writing,

$$\theta = \theta_0 + y \quad \text{where} \quad -\varepsilon \leqslant y \leqslant \varepsilon$$

Within this band the Coriolis parameter $f = 2\Omega \cos\theta$ can be approximated by

$$f = f_0 + f_1 y$$

where $f_0 = 2\Omega \cos\theta_0$, $f_1 = -2\Omega \sin\theta_0$.

We have, in effect, made the classical β-plane approximation by allowing the Coriolis parameter to vary with latitude in such a way that the northward derivative is constant.

We are primarily interested in oscillations of very long period, thus we will consider motions on a long-time scale by writing

$$\varepsilon t = \tau$$

Eqs. (4) and (5) then yield the relations

$$u = \frac{g}{af}\frac{\partial h}{\partial \theta}, \quad v = -\frac{g}{a\sin\theta f}\frac{\partial h}{\partial \phi} \tag{7}$$

Thus the vorticity to $0(\varepsilon)$ becomes $\xi = \varepsilon\xi_1$, with

$$\xi_1 = \hat{r}\cdot\nabla\times(u\hat{\phi}, v\hat{\theta})$$

$$= \frac{g}{f}\overline{\nabla}^2 h \tag{8}$$

where

$$\overline{\nabla}^2 = \frac{1}{a^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{a^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{a^2\sin\theta\cos\theta}\frac{\partial}{\partial\theta}$$

Expanding the potential vorticity eq. (3) in powers of ε gives, with $\mathbf{q} = u\hat{\phi} + v\hat{\theta}$,

$$\varepsilon\left(\frac{\partial}{\partial\tau} + \mathbf{q}\cdot\nabla\right)\left[\frac{f_0 + f_1 y + \varepsilon\xi_1 + \dots}{H_0 + \varepsilon h + \dots}\right] = 0$$

Thus

$$\frac{d}{d\tau}\left[-\frac{f_0}{H_0}h\varepsilon + \varepsilon\xi_1 + f_1 y + \dots\right] = 0 \tag{9}$$

We have assumed that $-\varepsilon \leqslant y \leqslant \varepsilon$, thus within the strip the term $f_1 y$ is $0(\varepsilon)$. Hence to the first order eq. (9) becomes

$$\frac{d}{d\tau}\left[-\frac{f_0}{H_0}h + \frac{g}{f_0}\overline{\nabla}^2 h + \beta y\right] = 0 \tag{10}$$

where $\beta = f_1/\varepsilon$ and within the band

$$\overline{\nabla}^2 h \simeq \frac{1}{a^2\sin^2\theta_0}\frac{\partial^2 h}{\partial\phi^2} + \frac{1}{a^2}\frac{\partial^2 h}{\partial y^2} + \frac{1}{a^2\sin\theta_0\cos\theta_0}\frac{\partial h}{\partial y}$$

We will now use eq. (10), which represents to the first order the conservation of potential vorticity, to study the interaction between the vortices in the ring of sub-tropical anticyclones. We will neglect the interaction between the northern and southern hemispheres and for the present we will also neglect the effects of the polar cyclonic vorticity. In the northern hemisphere the Pacific and Atlantic high pressure regions are well defined and there is evidence of a third system over India equidistant from the other two. In the southern hemisphere there are also three such systems. We will therefore consider the interaction of 3 vortices initially on the same circle of co-latitude θ_0.

We will first determine the solution to eq. (10) with a single point vortex at $\phi = \phi_0$, $y = 0$. We consider

$$\frac{1}{a^2\sin^2\theta_0}\frac{\partial^2 h}{\partial\phi^2} + \frac{1}{a^2}\frac{\partial^2 h}{\partial y^2} + \frac{1}{a^2\cos\theta_0\sin\theta_0}\frac{\partial h}{\partial y} - \alpha^2 h$$

$$+ \frac{\beta f_0}{g}y = 0 \tag{11}$$

where

$$\alpha^2 = \frac{f_0^2}{gH_0},$$

together with the condition that there exists a simple point vortex at $\phi = \phi_0$, $y = 0$ [i.e. $\theta = \theta_0$].

In this equation the variables are separated in the following way:

$$h(y, \phi) = A(y, \phi) + B(y)$$

where

$$\frac{1}{a^2 \sin^2\theta_0}\frac{\partial^2 A}{\partial\phi^2} + \frac{1}{a^2}\frac{\partial^2 A}{\partial y^2} + \frac{1}{a^2 \cos\theta_0 \sin\theta_0}\frac{\partial A}{\partial y}$$

$$-\alpha^2 A = 0 \qquad (12)$$

and A has a vortex singularity at $(0, \phi_0)$, and

$$\frac{1}{a^2}\frac{\partial^2 B}{\partial y^2} + \frac{1}{a^2 \cos\theta_0 \sin\theta_0}\frac{\partial B}{\partial y} - \alpha^2 B + \frac{\beta f_0}{g} y = 0 \qquad (13)$$

The solution of (13) is given by

$$B(y) = c_+ e^{(q-p)y} + c_- e^{-(q+p)y}$$

$$+ \frac{\beta f_0}{g\alpha^2}\left[y + \frac{1}{\alpha^2 a^2 \cos\theta_0 \sin\theta_0}\right] \qquad (14)$$

where

$$p = \frac{1}{2\cos\theta_0 \sin\theta_0}, \quad q = (p^2 + \alpha^2 a^2)^{1/2}$$

We now seek the solution to eq. (12) that represents a vortex at $(0, \phi_0)$. We expect this solution to be a function of the distance from the center of the vortex $(0, \phi_0)$. Since we are restricting our consideration to the narrow strip $-\varepsilon \leqslant y \leqslant \varepsilon$, the distance of a point (y, ϕ) from the vortex centered at $(0, \phi_0)$ is approximately given by

$$r = a\,(\sin^2\theta_0(\phi - \phi_0)^2 + y^2)^{1/2}$$

We will therefore rewrite (12) in the form

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial A}{\partial r} + \frac{1}{r^2}\frac{\partial^2 A}{\partial \psi^2} + \frac{1}{a^2 \cos\theta_0 \sin\theta_0}\frac{\partial A}{\partial y} - \alpha^2 A = 0 \qquad (15)$$

where $\sin\psi = ay/r$.

Seeking the vortex solution we consider an expansion for A of the form $A = \Sigma_{n=0}^{\infty} y^n A_n(r)$.

Substituting this representation into eq. (15) and collecting terms in y gives,

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial A_0}{\partial r} - \alpha^2 A_0 + \frac{A_1}{a^2 \cos\theta_0 \sin\theta_0} + \frac{2A_2}{a^2}$$

$$+ \sum_{n=1}^{\infty} y^n \left[\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial A_n}{\partial r} - \alpha^2 A_n + \frac{(n+1)A_{n+1}}{a^2 \cos\theta_0 \sin\theta_0}\right.$$

$$+ \frac{(n+2)(n+1)A_{n+2}}{a^2} + \frac{1}{\cos\theta_0 \sin\theta_0}$$

$$\left.\times \frac{1}{r}\frac{\partial A_{n-1}}{\partial r}\right] = 0 \qquad (16)$$

The following vortex solution representing to $0(\varepsilon)$ flow in circles around the origin and decaying at infinity is derived from,

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial A_0}{\partial r} - \alpha^2 A_0 = 0$$

hence

$$A_0(r) = -\frac{\gamma}{2\pi} K_0(\alpha r)$$

where K_0 is the modified Bessel function of the second kind (Stewart, 1943; Morikawa & Swenson, 1971).

The functions $A_1(r)$ and $A_2(r)$ are required by the boundary conditions to be identically zero, and the functions $A_n(r)$ are determined successively in terms of $A_0(r)$ by setting the bracketed coefficient of each power of y to zero.

Hence

$$A(y, \phi) = -\frac{\gamma}{2\pi}\left[K_0(\alpha r) + \frac{y^3 a^2 \alpha K_1(\alpha r)}{3!\cos\theta_0 \sin\theta_0\, r}\right.$$

$$- \frac{y^4 a^2 \alpha K_1(\alpha r)}{4!(\cos\theta_0 \sin\theta_0)^2 r}$$

$$+ \frac{y^5 a^2 \alpha}{5!(\cos\theta_0 \sin\theta_0)^3}\frac{K_1(\alpha r)}{r}$$

$$\left.\times\left[1 + \frac{a^2}{r^2}(\cos\theta_0 \sin\theta_0)^2\right] + \ldots\right]$$

Here γ represents the strength of the vortex, with γ positive corresponding to a counterclockwise rotating vortex, i.e. a cyclone or low pressure center. In this context γ has the dimensions of length.

It can be shown (Courant Institute report) that this function $A(y, \phi)$ has the correct logarithmic singularity at the origin $r = 0$, and decays away from the origin.

The solution we have obtained for the flow on the sphere generated by a simple vortex at

$(0, \phi_0)$ is only valid within the strip $-\varepsilon \leqslant y \leqslant \varepsilon$. We therefore note that as r becomes large, i.e. $r = O(a)$ the successive terms in the expansion for $A(y, \phi)$ decay as increasing powers of ε. Thus within the narrow band $-\varepsilon \leqslant y \leqslant \varepsilon$ the function $A(y, \phi)$ is given by

$$A(y, \phi) = -\frac{\gamma}{2\pi} K_0(\alpha r) + 0(\varepsilon^3)$$

Hence the perturbation height $h(y, \phi)$ is given by,

$$h(y, \phi) = -\frac{\gamma}{2\pi} K_0(\alpha r) + B(y) + 0(\varepsilon^3)$$

where $B(y)$ is defined by eq. (14). And the velocity components for the motion generated by a vortex centered at $(0, \phi_0)$ are given by the geostrophic relations

$$u = \frac{g}{af_0}\frac{\partial h}{\partial y}, \quad v = -\frac{g}{af_0 \sin\theta_0}\frac{\partial h}{\partial \phi}$$

We can now consider the problem of 3 interacting vortices of equal strength initially on the same subtropical circle of co-latitude θ_0. The velocity components of the center of the ith vortex, considered as a point under the action of the other two vortices, are given by

$$u_i = \frac{g}{af_0}\left[\frac{\gamma}{2\pi}\sum_{\substack{k=1\\k\neq i}}^{3} K_1(\alpha r_{ik}) \cdot \alpha \frac{\partial}{\partial y_i} r_{ik} + \frac{\partial B_i}{\partial y_i}\right] + (\varepsilon^2)$$

$$v_i = -\frac{g}{af_0 \sin\theta_0}\frac{\gamma}{2\pi}\sum_{\substack{k=1\\k\neq i}}^{3} K_1(\alpha r_{ik}) \cdot \alpha \frac{\partial}{\partial \phi_i} r_{ik} + 0(\varepsilon^2)$$

where

$$r_{ik} = a\,(\sin^2\theta_0(\phi_i - \phi_k)^2 + (y_i - y_k)^2)^{1/2}$$

Now within the strip $-\varepsilon \leqslant y \leqslant \varepsilon$ the distance r_{ik} between the ith and kth vortices is approximated to $O(\varepsilon^2)$ by the distance around the circle of latitude, thus

$$r_{ik} \simeq a \sin\theta_0 |\phi_i - \phi_k|$$

We have, of course, assumed in making this approximation that the vortices are so spaced that their distance apart is greater than $O(\varepsilon)$. Thus to our order of accuracy the velocity components of the center of the ith vortex are approximately given by

$$v_i = -\frac{g}{af_0 \sin\theta_0}\frac{\gamma}{2\pi}\sum_{\substack{k=1\\k\neq i}}^{3}$$

$$\times \left[K_1(\alpha a \sin\theta_0 |\phi_i - \phi_k|) \cdot \alpha a \sin\theta_0 \frac{\partial}{\partial \phi_i} |\phi_i - \phi_k|\right] \tag{17}$$

and from eq. (14)

$$u_i = \left[\frac{g}{af_0} A_+ e^{(q-p)i} + A_- e^{-(q+p)y_i} + \frac{\beta f_0}{g\alpha^a}\right]$$

$$+ \frac{g}{af_0}\frac{\gamma}{2\pi}\sum_{\substack{k=1\\k\neq i}}^{3} \alpha K_1(\alpha a|\phi_i - \phi_k|\sin\theta_0)$$

$$\times a|\phi_i - \phi_k|\left[\cos\theta_0 + \frac{y_i - y_k}{|\phi_i - \phi_k|^2 \sin\theta_0}\right] \tag{18}$$

where the constants $A_+ = (q-p)C_+$ and $A_- = -(q+p)C_-$, will be determined by the boundary conditions on the velocity field. We note that the first bracketed expression in (18) represents the contribution to the east–west velocity from the "β-plane" effect. The second bracketed expression represents the contribution from the north–south vortex interaction which will be small unless we approach the pole and the sin θ_0 singularity becomes important.

From eqs. (17) and (18) we see that for three vortices of the same strength γ, equally spaced on a circle of latitude, the vortices move around the circle with uniform angular velocity ω. For this equilibrium configuration, the vortices satisfy the following conditions:

$$y = 0 \qquad v_i = 0$$

$$\phi_i = \omega t + \frac{2\pi i}{3} \qquad u_i = a \sin\theta_0, \text{ constant.}$$

From eq. (18), the velocity

$$u = a \sin\theta_0 = \frac{g}{af_0}\left[A_+ + A_- + \frac{\beta f_0}{g\alpha^2}\right]$$

where the term $(\beta/a\alpha^2) = -(2\Omega/a^2\alpha^2)a \sin\theta_0$ is the contribution due to the Rossby wave phase speed. We note that as expected the "β-plane" effect produces a velocity component towards the west. This opposes the direction of rotation of a ring of cyclones and will cause instability in such a configuration.

Linear stability analysis

We will now make a perturbation expansion of the position vector (y_i, ϕ_i) about the equilibrium solution. The linearized equations of motion are obtained in the usual way considering eqs. (17) and (18) as differential equations for $\dot{y}_i$ and $\dot{\phi}_i$. The motion of the ith vortex in terms of the perturbation parameter δ is described by,

$$y_i = \delta y_i^{(1)} + 0(\delta^2) \qquad \dot{y}_i = \delta \dot{y}_i^{(1)} + 0(\delta^2)$$

$$\phi_i = \omega t + \frac{2\pi i}{3} + \delta\phi_i^{(1)} + 0(\delta^2) \qquad \dot{\phi}_i = \omega + \delta\dot{\phi}_i^{(1)} + 0(\delta^2)$$

Substituting in eqs. (17) and (18) and retaining only first-order terms in δ yields six simultaneous differential equations for the displacements. The resulting linear differential equations for $(y^1_{(i)}, \phi_i^{(1)})$ with constant coefficients are,

$$a \sin\theta_0 \dot{\phi}^{(1)} = \frac{g}{af_0}[(q-p)A_+ - (q+p)A_-]y_i^{(1)} - \frac{g}{af_0}\sum_{\substack{k=1\\k\neq i}}^{3}[y_i^{(1)} - y_k^{(1)}]\frac{\gamma}{2\pi}F_1(\theta_0) - \omega a\cos\theta_0 y_i^{(1)} \qquad (19)$$

$$a\dot{y}_i^{(1)} = \frac{g}{af_0}\frac{\gamma}{2\pi}\sum_{\substack{k=1\\k\neq i}}^{3} F_1(\theta_0)[\phi_i^{(1)} - \phi_k^{(1)}] \qquad (20)$$

where

$$F_1(\theta_0) = \alpha a\frac{2\pi}{3}\left[K_0''\left(\alpha a\frac{2\pi}{3}\sin\theta_0\right)\cdot \alpha a\frac{2\pi}{3}\cos^2\theta_0 + K_1\left(\alpha a\frac{2\pi}{3}\sin\theta_0\right)\left(\sin\theta_0 - \frac{9}{(2\pi)^2\sin\theta_0}\right)\right]$$

and

$$F_2(\theta_0) = \alpha^2 a^2 \sin\theta_0 K_0''\left(\alpha a\frac{2\pi}{3}\sin\theta_0\right)$$

Adding the three equations in each of the sets (19) and (20) gives

$$a\sin\theta_0\sum_{i=1}^{3}\dot{\phi}_i^{(1)} = \left(\frac{g}{af_0}[(q-p)A_+ - (q+p)A_-] - \omega a\cos\theta_0\right)\sum_{i=1}^{3} y_i^{(1)} \qquad (21)$$

and

$$a\sum_{i=1}^{3}\dot{y}_i^{(1)} = 0 \qquad (22)$$

These results illustrate that the impulse of a system having no external forces remains constant. We observe that if the displacements are chosen so that $\Sigma_{i=1}^3 y_i^{(1)} = 0$ initially, then from (21) and (22) both $\Sigma_{i=1}^3 y_i^{(1)}$ and $\Sigma_{i=1}^3 \phi i^{(1)}$ will remain constant. From symmetry conditions, the displacement in each normal mode must be of the form

$$y_n^{(1)} = \exp\left[\frac{2\pi n i}{3}\right]y^{(1)}, \qquad \phi_n^{(1)} = \exp\left[\frac{2\pi n i}{3}\right]\phi^{(1)}$$

where $n = 1, 2, 3$ and in this context $i = \sqrt{-1}$.

Substituting in eqs. (19) and (20) gives two simultaneous equations for the perturbations $y^{(1)}$ and $\phi^{(1)}$,

$$a\sin\theta_0\dot{\phi}^{(0)} = \frac{g}{af_0}[(q-p)A_+ - (q+p)A_-]y^{(1)} - \omega a\cos\theta_0 y^{(1)} - \frac{3g}{af_0}\frac{\gamma}{2\pi}F_1(\theta_0)y^{(1)} \qquad (23)$$

$$a\dot{y}^{(1)} = \frac{3g}{af_0}\frac{\gamma}{2\pi}F_2(\theta_0)\phi^{(1)} \qquad (24)$$

Performing the usual exponential stability analysis, we assume that the amplitudes $\phi^{(1)}$ and $y^{(1)}$ vary with time as $e^{\lambda t}$. From eqs. (23) and (24), we then obtain the expression for λ^2 as

$$\lambda^2 = \frac{3g^2}{a^2 f_0^2}\frac{\gamma}{2\pi}F_2(\theta_0)\left[(q-p)A_+ - (q+p)A_- - \frac{3\gamma}{2\pi}F_1(\theta_0) - \frac{f_0}{g}\omega\cos\theta_0\right] \qquad (25)$$

Thus in the case of 3 vortices, the expression for λ^2 is real and the linear stability criteria depends on the sign of λ^2.

We note that $F_2(\theta_0)$ is positive, thus the sign of

$$\gamma\left[(q-p)A_+ - (q+p)A_- - \frac{3\gamma}{2\pi}F_1(\theta_0) - \frac{f_0\omega}{g}\cos\theta_0\right] \tag{26}$$

determines linear stability. When expression (26) is negative, the eigenvalues are imaginary and the configuration is linearly stable. When (26) is positive, we have exponential growth with time and the configuration is linearly unstable.

We will now evaluate the constants A_+ and A_- in the case of several physically plausible boundary conditions, and thus determine the sign of expression (26).

Condition 1

It is known that the effect of the polar cyclonic vorticity on the sub-tropical ring of anticyclones is to fix the vortices in a stationary position relative to the rotating earth. In order to simulate this effect, we will impose the boundary condition on the east–west component of the velocity field, namely that the angular velocity with which the vortices rotate around the circle of latitude is zero. Hence,

$$\omega a \sin\theta_0 = \frac{g}{af_0}\left[A_+ + A_- + \frac{\beta f_0}{g\alpha^2}\right] = 0 \tag{27}$$

As the further determining boundary condition, we will model the circular structure of the vortices by writing $u(\varepsilon) = -u(-\varepsilon)$. In other words, the speeds are numerically equal but opposite in sign at the edges of the band. From the expression for the velocity at a regular point, this second boundary condition, to first order, becomes

$$A_+ e^{(q-p)\varepsilon} + A_- e^{-(q+p)\varepsilon} + \frac{f_0\beta}{g\alpha^2}$$

$$= -\left[A_+ e^{-(q-p)\varepsilon} + A_- e^{(q+p)\varepsilon} + \frac{f_0\beta}{g\alpha^2}\right] \tag{28}$$

Eqs. (27) and (28) determine the constants A_+ and A_-, and thus expression (26) for the stability criteria can be evaluated. Substituting for p and q from (14), the approximate evaluation of (26) is

$$\gamma\left[\frac{4\Omega^2 a^2 \cos^2\theta_0 \sin^2\theta_0}{g} - \frac{3\gamma}{2\pi}F_1(\theta_0)\right] \tag{29}$$

We note that the first term in expression (28) is the contribution from the "β-plane" effect, and the second term is due to the north–south vortex interaction: Hence; if we neglect the variation is Coriolis force with latitude, the stability criteria becomes

λ^2 proportional to $-\gamma^2$

and we obtain again the results of Stewart (1943) and Morikawa & Swenson (1971) who found that for 3 vortices the system was linearly stable for both cyclones and anticyclones.

However when we include the β-plane effect, expression (28) shows that a system of anticyclones where γ is negative will always be linearly stable; but for weak cyclones where γ is small and positive, the sign of λ^2 will be positive, implying linear instability. A system of cyclones will be unstable when

$$\frac{2\Omega a \cos\theta_0 \sin^3\theta_0 \sqrt{H_0/g}}{z\cos^2\theta_0 K_0(z) + K_1(z)\,(1-(3/2\pi)^2)} > \gamma \tag{30}$$

where

$$z = \frac{2\pi\alpha}{3}a\sin\theta_0$$

and

$$\alpha = 2\Omega\cos\theta_0/\sqrt{gH_0}.$$

Estimating the order of magnitude of these terms at subtropical latitudes where $\theta_0 = 60°$, we find that the left hand side of the inequality (30) is an order of magnitude larger than an estimate for the strength of a Pacific or Azores vortex (Stewart, 1943). Thus we have shown that at subtropical latitudes a system of cyclones would be unstable. This instability is caused by the latitudinal shear in Coriolis force and the resulting interaction of the Rossby wave with the vortex motion.

Further inspection of the inequality (30) shows that as $\theta_0 \to 0$, i.e., for polar latitudes, the left hand side becomes rapidly smaller. Thus at polar latitudes, the cyclone system would be stable even for weak vortices where γ is small. This result is to be expected, since it reflects the fact that at polar latitudes the "β-plane" effect is weak, and the strong north–south vortex interaction contributes to the stability of the cyclone system.

In conclusion, our analysis leads us to expect quasi-steady systems of anticyclones at subtropical latitudes and no equivalent system of cyclones. The major low pressure systems in the atmosphere will be confined to polar latitudes where the gradient of Coriolis force is small. In these regions the vortex interaction will overcome the destabilising "β-plane" effect; and the cyclone configuration need not be destroyed by instability.

Further boundary conditions

We have also determined the sign of the stability criteria given by expression (26) in the case of several other boundary conditions (Courant Institute Report). Allowing the vortices to precess around the circle of latitude with a constant angular velocity ω, once again gives the result that only an anticyclone system is stable at subtropical latitudes. Here ω is positive for cyclones and negative for anticyclones. Similarly, if we model the decaying structure of the vortices by considering the boundary condition where the velocity is set to zero at the edge of the strip, we find that the "β-plane" term causes an instability in a system of cyclones.

Additional vortices

The above arguments are valid in the case of 3 vortices. If we were to consider N vortices centered on the circle of colatitude θ_0, certain terms in the stability criteria could change sign. The equivalent expression for $F_1(\theta_0)$ becomes

$$\frac{2\pi}{N}\frac{\alpha a}{\sin\theta_0}[z\cos^2\theta_0 K_0(z)+K_1(z)(1-(N/2\pi)^2)] \quad (31)$$

where

$$z=\frac{2\pi}{N}\alpha a\sin\theta_0.$$

We observe that when $N \leqslant 6$ the coefficient of $K_1(z)$ is positive and the function $F_1(\theta_0)$ remains positive. Thus the stability arguments are the same for $2 \leqslant N \leqslant 6$. However, when $N=7$, we find that $F_1(\theta_0)$ becomes negative in the limits $\theta_0 \to 0$ and $\theta_0 \to \pi/2$. From the stability criteria (29) it then follows that a system of cyclones would be unstable, and a system of strong anticyclones would also be unstable at these latitudes. In the case of many vortices, the term $-(N/2\pi)^2 K_1(z)$ will dominate the function $F_1(\theta_0)$ for all values of θ_0 and the approximate stability criteria becomes

$$\gamma^2\frac{\alpha a}{\sin\theta_0}(N/2\pi)^2K_1(z).$$

Thus the configuration of many cyclones or many anticyclones will be unstable at all latitudes.

These results conform with the linear stability conclusions of Morikawa & Swenson (1971).

Conclusion

We have investigated the "β-plane" effect in the context of a latitudinal ring of vortices on the surface of a rotating sphere. Under the assumption that the vortices are confined to a strip about the initial circle of latitude, we can obtain explicit expressions for the motion of the vortices. The effect of the variation of Coriolis force with latitude is to cause a precession of the vortices towards the west. At subtropical latitudes, this effect is sufficiently strong to induce an instability in a ring of cyclones, whereas the effect enhances the stability of a ring of anticyclones. Our results show that the quasi-steady ring of high pressure systems in the atmosphere will be stable, and that the major low pressure areas will be found in polar latitudes.

In the appendix, we comment on the plausibility of the "strip" assumption.

Appendix

The assumption that the vortices do not move far from the initial circle of latitude enabled us to obtain simple and explicit solutions to the problem. We now give reasons why this assumption is plausible.

In their work on the planar equivalent to this problem Morikawa & Swenson (1971) found that, even in a non-linear analysis, for a wide range of values of the parameters, the circle vortices deviated only slightly from the initial circle. This leads us to expect a similar result will hold in the spherical geometry.

As we previously mentioned Laplace's tidal equations (eqs. (4) to (6)) have been studied in detail by Longuet-Higgins (1964). In his paper, Planetary waves on a rotating sphere, he considers the problem of Rossby waves on the surface of a rotating sphere when the upper surface is rigid hence the motion is divergence free. He compares the results obtained through a "β-plane" analysis and the solutions to the spherical problem where the curvature is fully taken into account. He finds that in most regions the results are very similar, thus justifying the β-plane approximation.

The fundamental solution for the stream function of the motion on a sphere is given by the spherical harmonic

$$\psi = P_n^s\,(\cos\theta)^{is\phi}$$

By examining the asymptotic form of this spherical harmonic as $n\to\infty$, and for different values of the ratio s/n, Longuet-Higgins shows that for $0 < s/n < 1$ the character of the solution depends upon whether

$$\sin\theta \gtrless \frac{s}{n} = \sin\bar{\theta}$$

When $\sin\theta > s/n$, i.e. $\bar{\theta} < \theta < \pi - \bar{\theta}$, the solution ψ is oscillatory in the θ direction and approximately satisfies the local β-plane equations. However when $\sin\theta < s/n$ it is found that the function is no longer oscillatory. Hence for each value of s/n the Rossby waves are effectively trapped within a latitudinal band.

The equation for the stream function for the motion of fluid on a rotating sphere with a rigid surface is

$$\frac{\partial}{\partial t}\nabla^2\psi + 2\Omega\frac{\partial\psi}{\partial\phi} = 0$$

Considering now the solution of this equation when there exists a point vortex at (θ_0, ϕ_0) we can verify that

$$\psi = Q_n\left[\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos\left(\phi-\phi_0+\frac{2\Omega t}{n(n+1)}\right)\right]$$

Hence ψ can be represented in the following form (see Bateman manuscript, Higher Transcendental Functions, Vol. I).

$$\psi = P_n(\cos\theta_0)\,Q_n(\cos\theta_0) + 2\sum_{\infty}^{m=1}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta_0)\,Q_n^m(\cos\theta) \times\cos m\left(\phi-\phi_0+\frac{2\Omega t}{n(n+1)}\right)$$

Thus the asymptotic form of ψ as $n\to\infty$ is given by

$$\lim_{n\to\infty}\psi \propto \lim_{n\to\infty}P_n^m(\cos\theta_0)\times\lim_{n\to\infty}Q_n^m(\cos\theta)$$

Repeating the analysis of Longuet-Higgins we find that the solution is that of undamped oscillation in the θ direction only when θ and θ_0 both satisfy,

$$\sin\theta_0 > \frac{m}{n} = \sin\bar{\theta},$$

$$\sin\theta > \frac{m}{n} = \sin\bar{\theta}.$$

Thus for values of m such that $\lim\,(m/n) < \sin\theta_0$ we have Rossby waves trapped in a latitudinal band that includes the circle of latitude of the point vortex.

In a further paper, Longuet-Higgins (1968) treats the divergent set of Laplace's tidal equations (see eqs. (4) to (6)) and obtains the eigenfunctions for these equations on a sphere. For certain parameter ranges the solutions again show that the energy is trapped in an equatorial band. The eigenfunctions for this problem are harder to analyze than the Legendre polynomials that describe the stream function for the non-divergent problem. However we would again expect trapping of energy in a latitudinal band, containing the circle of latitude of the vortex, when a vortex is introduced.

We feel that the results of Longuet-Higgins justify the use of the β-plane approximation to account for spherical curvature. We should note however that his results indicate that the approximation may not be valid at the boundaries of the strip. The results also suggest that restricting the motion of the vortices to a latitudinal band is a reasonable assumption.

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ВЗАИМОДЕЙСТВИЕ В ЖИДКОСТИ НА ПОВЕРХНОСТИ ВРАЩАЮЩЕЙСЯ СФЕРЫ

Изучается взаимодействие трех точечных вихрей, вначале расположенных на одном и том же круге широты на поверхности вращающейся сферы в предположении, что отклонения по широте малы. Анализируется линейная устойчивость равновесного движения. Цель настоящей работы — промоделировать и понять движение антициклонов и циклонов в атмосфере: результаты могут помочь объяснению наблюдаемых явлений. Линейный анализ показывает, что при некоторых граничных условиях цепочка субтропических антициклонов устойчива, тогда как кольцо циклонов может быть неустойчиво. Этот результат является следствием изменения силы Кориолиса с широтой на поверхности сферы. Эффект β-плоскости приводит к вращению кольца антициклонов, в направлении, противоположном вращению кольца циклонов. Пока интенсивность циклонических вихрей велика, этот эффект приводит к неустойчивости системы циклонов. Эти соображения показывают, что в атмосфере антициклоническая система может существовать в субтропических широтах, тогда как циклонические вихри ограничены полярными районами. Результаты согласуются с наблюдаемым в земной атмосфере распределением крупных систем высокого и низкого давления.