Cascade theory of turbulence in a stratified medium[1]

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ABSTRACT

A constant mean thermal gradient is present in a fluid which is assumed incompressible, except for the buoyancy force. The latter couples the momentum equation for the velocity fluctuations to the heat equation for the thermal fluctuations. A single-cascade method, which decomposes a fluctuation into a macroscopic and a random part, resolves in closing the correlation hierarchy, by degenerating a quadruple correlation into a product of two double correlations of different ranks of randomness. The method introduces a preferential pair-coupling between the modes of the quadruple correlation. One of the double correlation appears in the form of an eddy viscosity. The determination of its spectral structure, or equivalently, the problem of the approach to equilibrium, involves a memory-chain of many relaxations. This requires the use of a repeated-cascade. The memory-chain is cutoff on the basis of an isomeric distribution of spectrum in all links of the chain. The solutions of the equations of balance, between the kinetic energy and the potential energy of the thermal field, yield spectral laws k^{-3}, k^{-1} and $k^{-5/3}$ in diverse subranges of the two energy spectra, and the $k^{-7/3}$, k^{-5} laws for the co-spectrum of heat flux.

I. Introduction

The turbulent motions in oceans and in atmosphere have often to deal with the effects of buoyancy. As a dynamical model, we consider a momentum equation which, through the buoyancy force, is coupled to an equation of thermal fluctuations. Except for this buoyancy force, the fluid is assumed incompressible, and presents a constant mean thermal gradient in the vertical direction. The sign of the thermal gradient characterizes a stable or an instable stratification.

It is well verified that the Kolmogoroff law (Kolmogoroff, 1941) of turbulent spectrum is valid in a neutral medium. The cascade transfer across the spectrum is governed by the rate of energy dissipation, and with the use of that single parameter, the similarity method has successfully derived the $k^{-5/3}$ law.

In a stratified medium, the spectral laws are more complex. Experiments in atmosphere and oceans have come up with diverse spectral laws of powers -3, -1 and $-5/3$. It has generally been believed that the -3 power law can be attributed to a buoyancy subrange in a stable stratification (Myrup, 1969; Benton & Kahn, 1958; Pinus et al., 1967; Shur, 1962; Pinus, 1963). Since the Brunt-Väisäla frequency N is the parameter governing the buoyancy force, a dimensional reasoning (Lumley, 1964, 1965; Phillips, 1966) suggests a spectrum N^2k^{-3}.

The dimensional method does not analyse the detailed dynamical processes of turbulence, and, therefore, its results are ambiguous as to the types of fluctuations (velocity or temperature), and as to the types of stratification (stable or unstable). In order to gain a more rational insight into the dynamical processes, we shall develop an analytic theory of turbulence in a stratified medium. The difficulty of analytical theories resides in the closure of correlation hierarchy. The method of cascade decomposition, developed by Tchen, has been able to break the homogeneity of the quadruple correlation in injecting a preferential pair-coupling between the four modes, and hence to degenerate into a product of two double correlations of different ranks. This method has been successfully applied to hydrodynamic turbulence (Tchen, 1973*a*) in a neutral fluid and to plasma turbulence (Tchen, 1973*b*).

In the following, we propose to first apply the method of single cascade to turbulence in a stratified medium (Secs. II–IV). It will close the hierarchy at the quadruple correlations.

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One degenerated double correlation will be in the form of an eddy viscosity. The study of its structure, and, more specifically, the problem of the approach to equilibrium, involve a memory-chain of many relaxation frequencies. Its description will require the use of a repeated-cascade, which decomposes the random motions contributing to the eddy viscosity into many sub-ranks. A cutoff of the memory-chain will be made on the basis of an isomeric distribution of spectrum in all links of the chain (Sec. V). A system of equations, describing the balance between the kinetic energy and the potential energy of the thermal field, will ensue, and will be solved for the diverse subranges in a stably and unstably stratified fluid (Secs. VI–IX).

II. Dynamical equations of a stratified fluid

The fluctuations of velocity $\mathbf{u}$, temperature θ, and pressure p, in a stratified medium, are governed by the following equations of motion and heat conduction:

$$\varrho\left(\frac{\partial}{\partial t}+u_j\frac{\partial}{\partial x_j}\right)u_i=-\frac{\partial p}{\partial x_i}+\varrho\nu\nabla^2 u_i-\varrho g\lambda_i \tag{1a}$$

$$\partial u_i/\partial x_i=0 \tag{1b}$$

$$\left(\frac{\partial}{\partial t}+u_j\frac{\partial}{\partial x_j}\right)\theta=\beta u_j\lambda_j+\varkappa\nabla^2\theta \tag{1c}$$

where ν and $\varkappa$ are the kinematic viscosity and heat conductivity, respectively. The buoyancy force is in the x_3 direction, as indicated by the unit vector $\lambda_i=(0, 0, 1)$, and is proportional to the acceleration of gravity g. The variations of density ϱ and temperature T from their static values ϱ_0, T_0 satisfy the following equation of state

$$\varrho=\varrho_0-\alpha\varrho_0(T-T_0) \tag{2}$$

where α is the coefficient of thermal expansion. In a medium with an adverse temperature gradient β, we can write

$$T=T_0-\beta\lambda_j x_j+\theta \tag{3}$$

The Boussinesq approximation regards the density as constant in all terms of (1) except in the buoyancy term, simplifying (1a) to

$$\left(\frac{\partial}{\partial t}+u_j\frac{\partial}{\partial x_j}\right)u_i=-\frac{\partial q}{\partial x_i}+\nu\nabla^2 u_i+\alpha g\lambda_i\theta \tag{4}$$

where

$$q=\frac{p}{\varrho_0}+g(1+\tfrac{1}{2}\alpha\beta\lambda_s x_s)\lambda_j x_j \tag{5}$$

The system of equations (1b), (1c) and (4) governs the three variables u_i, q and θ, and forms the fundamental framework describing the turbulent motions in the stratified medium. For the purpose of simplification of writing, we introduce the Brunt-Väisäla frequency N, as defined by

$$N^2\equiv-\alpha\beta g \tag{6}$$

where N is real in a stable stratification, i.e. $\beta<0$, but becomes imaginary in an unstable stratification, i.e. $\beta>0$. In the latter case, we shall replace (6) by a new notation

$$M^2=\alpha\beta g\equiv-N^2 \tag{7}$$

giving a real frequency M.

In order to balance the kinetic energy $\frac{1}{2}u_i^2$ with the potential energy $\frac{1}{2}v^2$, which arises from a thermal drift v by gravity, we introduce the notations

$$v=\alpha g\theta/N \tag{8}$$

and

$$v=\alpha g\theta/M \tag{9}$$

for a stable stratification and an unstable stratification, respectively, according to the definition (7). Then we can rewrite (1c) and (4) in the form

$$\frac{du_i}{dt}\equiv\left(\frac{\partial}{\partial t}+u_j\frac{\partial}{\partial x_j}\right)u_i=-\frac{\partial q}{\partial x_i}+Nv\lambda_i+\nu\nabla^2 u_i \tag{10a}$$

$$\frac{dv}{dt}\equiv\left(\frac{\partial}{\partial t}+u_j\frac{\partial}{\partial x_j}\right)v=-\gamma Nu_j\lambda_j+\varkappa\nabla^2 v,\quad \gamma=1 \tag{10b}$$

for a stable stratification, or

$$\frac{du_i}{dt}\equiv\left(\frac{\partial}{\partial t}+u_j\frac{\partial}{\partial x_j}\right)u_i=-\frac{\partial q}{\partial x_i}+Mv\lambda_i+\nu\nabla^2 u_i \tag{11a}$$

$$\frac{dv}{dt} \equiv \left(\frac{\partial}{\partial t} + u_j \frac{\partial}{\partial x_j}\right) v = -\gamma M u_j \lambda_j + \varkappa \nabla^2 v, \quad \gamma = -1 \tag{11 b}$$

for an unstable stratification.

The systems (10) and (11) differ by a change of signs in the terms connected with γ. In view of the formal similarity of the two systems of equations (10) and (11), it suffices to analyse one of them, say the system (10) for a stable stratification, and deduce the results for an unstable stratification by simply changing $\gamma = 1$ into $\gamma = -1$, and N into M.

III. Closure of the correlation hierarchy by means of cascade decomposition

By using the method of the cascade theory of turbulence, we can decompose

$$\mathbf{u} = \mathbf{u}^{(0)} + \mathbf{u}', \quad v = v^{(0)} + v', \quad q = q^{(0)} + q' \tag{12}$$

Here the superscript $(^0)$ represents a macroscopic part contributing to the portion of the spectrum of wavenumber up to k; a variable with a prime represents a random part contributing to the portion of the spectrum of wavenumbers larger than k. An ensemble average $\langle ... \rangle'$, called "rank average", helps in separating the two parts, e.g.

$$\langle \mathbf{u} \rangle' = \mathbf{u}^{(0)}, \quad \langle \mathbf{u}' \rangle' = 0 \tag{13}$$

Other rank averages

$$\langle ... \rangle^{\alpha}, \quad \text{with } \alpha = 0, 1, ... \tag{14}$$

will be introduced later.

By applying the average (13) to the system (10), we obtain a new system governing the macroscopic motions,

$$\left(\frac{\partial}{\partial t} + \mathbf{u}^{(0)} \nabla\right) u_i^{(0)} = -\frac{\partial q^{(0)}}{\partial x_i} + N v^{(0)} \lambda_i + \nu \nabla^2 u_i^{(0)} - \frac{\partial}{\partial x_j} \langle u_i' u_j' \rangle' \tag{15 a}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u}^{(0)} \nabla\right) v^{(0)} = -\gamma N \lambda_j u_j^{(0)} + \varkappa \nabla^2 v^{(0)} - \frac{\partial}{\partial x_j} \langle u_j' v' \rangle' \tag{15 b}$$

satisfying the condition of continuity

$$\nabla \cdot \mathbf{u}^{(0)} = 0 \tag{16}$$

The correlations $\langle u_i' u_j' \rangle'$ and $\langle u_j' v' \rangle'$ represent the statistical effects of the random fluctuations upon their macroscopic correspondents $\mathbf{u}^{(0)}$ and $v^{(0)}$, respectively. They are determined by the rank equations for $\mathbf{u}'$ and v', which can be formally obtained by taking the difference between (10) and (15) to be

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right) u_i' = -(\mathbf{u}' \cdot \nabla) u_i^{(0)} + N v' \lambda_i - \frac{\partial q'}{\partial x_i} + \nu \nabla^2 u_i' \tag{16 a}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla\right) v' = -(\mathbf{u}' \cdot \nabla) v^{(0)} - \gamma N \lambda_j u_j' + \varkappa \nabla^2 v' \tag{16 b}$$

By neglecting the self-damping terms $-\nabla q'$, $\nu \nabla^2 \mathbf{u}'$, $\varkappa \nabla^2 v'$, and only keeping those terms responsible for the exchange between $\mathbf{u}'$, v' and $\mathbf{u}^0$, we approximate (16) to the Lagrangian form

$$\frac{du_i'}{dt} = -(\mathbf{u}' \cdot \nabla) u_i^{(0)} + N v' \lambda_i \tag{17 a}$$

$$\frac{dv'}{dt} = -(\mathbf{u}' \cdot \nabla) v^{(0)} - \gamma N \lambda_j u_j' \tag{17 b}$$

recognized as a generalization of the Langevin equation, useful for the calculation of transport properties from random fluctuations. Upon integrating (17), we find

$$u_i'(t) = -\frac{\partial u_i^{(0)}}{\partial x_s} \int_0^t dt' u_s'(t') + N \lambda_i \int_0^t dt' v(t') \tag{18 a}$$

$$v'(t) = -\frac{\partial v^{(0)}}{\partial x_s} \int_0^t dt' u_s'(t') - \gamma N \lambda_j \int_0^t dt' u_j'(t') \tag{18 b}$$

We have omitted the initial values in (18), which do not contribute to the fluxes in question:

$$\langle u_i' u_j' \rangle' = -\eta_{sj}' \frac{\partial u_i^{(0)}}{\partial x_s} + N \lambda_i \int_0^{t^{(0)}} d\tau' \langle v'(0) u_j'(\tau') \rangle' \tag{19 a}$$

$$\langle v'u_j'\rangle' = -\eta_{sj}'\left(\frac{\partial v^{(0)}}{\partial x_s} + \gamma N\lambda_s\right) \tag{19b}$$

where

$$\eta_{sj}' = \int_0^{t^{(0)}} d\tau' \langle u_s'(0)\, u_j'(\tau')\rangle' K(\tau') \tag{20}$$

is an eddy viscosity, or an eddy conductivity, dependent on the damping factor $K(\tau')$ originated from (16a) or (16b) involving ν or $\varkappa$, respectively. But in the present study, we ignore this distinction by simply taking $K = 1$. Thus (20) reduces to

$$\eta_{sj}' = \int_0^{t^{(0)}} d\tau' \langle u_s'(0)\, u_j'(\tau')\rangle \tag{21}$$

The turbulent motion was considered quasi-stationary and homogeneous, so that

$$\langle u_s'(t_1)\, u_j'(t_1+\tau)\rangle' = \langle u_s'(t_2)\, u_j'(t_2+\tau)\rangle' \tag{22}$$

The upper limit $t^{(0)}$ was set equal to the time scale of the macroscopic motion $\mathbf{u}^{(0)}$.

We can remark that the fluxes (19) consist of a gradient part and a non-gradient part, playing the role of a force, similar to that in the Onsager relation of molecular transport.

Upon differentiating the fluxes (19), we obtain the expressions

$$-\frac{\partial}{\partial x_j}\langle u_i' u_j'\rangle' = \eta_{sj}' \frac{\partial^2 u_i^{(0)}}{\partial x_s \partial x_j} \tag{23a}$$

$$-\frac{\partial}{\partial x_j}\langle v' u_j'\rangle' = \eta_{sj}' \frac{\partial^2 v^{(0)}}{\partial x_s \partial x_j} \tag{23b}$$

which then transform (15) into

$$\left(\frac{\partial}{\partial t} + \mathbf{u}^{(0)}\cdot\nabla\right) u_i^{(0)} = -\frac{\partial q^{(0)}}{\partial x_i} + N\lambda_i v^{(0)} + (\nu\delta_{sj} + \eta_{sj}')\frac{\partial^2 u_i^{(0)}}{\partial x_s \partial x_j} \tag{24a}$$

$$\left(\frac{\partial}{\partial t} + \mathbf{u}^{(0)}\cdot\nabla\right) v^{(0)} = -\gamma N\lambda_j u_j^{(0)} + (\varkappa\delta_{sj} + \eta_{sj}')\frac{\partial^2 v^{(0)}}{\partial x_s \partial x_j} \tag{24b}$$

where the statistical effects of the random fluctuations are shown to shape the eddy transport property η_{sj}'.

Upon multiplying (24a) by u_i^0, and taking the rank average $\langle\ldots\rangle^0$, we find the equation of kinetic energy

$$\frac{1}{2}\frac{\partial}{\partial t}\langle \mathbf{u}^{(0)2}\rangle^\circ = N\lambda_j\langle v^{(0)} u_j^{(0)}\rangle^\circ - (\nu\delta_{sj} + \eta_{sj}')\left\langle \frac{\partial u_r^{(0)}}{\partial x_s}\frac{\partial u_r^{(0)}}{\partial x_j}\right\rangle^\circ \tag{25a}$$

and, in a similar way, the equation of potential energy from (24b)

$$\frac{1}{2}\frac{\partial}{\partial t}\langle v^{(0)2}\rangle^\circ = -\gamma N\lambda_j\langle v^{(0)} u_j^{(0)}\rangle^\circ - (\varkappa\delta_{sj} + \eta_{sj}')\left\langle \frac{\partial v^{(0)}}{\partial x_s}\frac{\partial v^{(0)}}{\partial x_j}\right\rangle^\circ \tag{25b}$$

Here the condition of homogeneity of turbulence permits to negect the terms

$$\frac{\partial}{\partial x_j}\langle u_i^{(0)} u_i^{(0)} u_j^{(0)}\rangle^\circ,\ \frac{\partial}{\partial x_j}\langle q^{(0)} u_j^{(0)}\rangle^\circ,\ \frac{\partial^2}{\partial x_s \partial x_j}\langle u_i^{(0)2}\rangle^\circ$$

$$\frac{\partial}{\partial x_j}\langle v^{(0)2} u_j^{(0)}\rangle^\circ,\ \frac{\partial^2}{\partial x_s \partial x_j}\langle v^{(0)2}\rangle^\circ.$$

IV. Transport functions and energy balance

A. *Transport functions*

The evolution of the kinetic energy, as governed by (25a), involves the following transport functions:

(*a*) The cascade transport across the spectrum, from small into large wavenumbers, is governed by the transfer function

$$T_u^0 \equiv \eta_{sj}'\left\langle \frac{\partial u_r^{(0)}}{\partial x_s}\frac{\partial u_r^{(0)}}{\partial x_j}\right\rangle^\circ \tag{26}$$

(*b*) The dissipation function

$$\varepsilon_u^0 \equiv \nu\langle(\partial u_i^{(0)}/\partial x_j)^2\rangle^0 \tag{27}$$

governs the molecular dissipation of the kinetic energy, and is most effective at large wavenumbers.

The evolution of the potential energy is governed by their own transfer and dissipation functions:

$$T_v^0 \equiv \eta'_{sj}\left\langle \frac{\partial v^{(0)}}{\partial x_s}\frac{\partial v^{(0)}}{\partial x_j}\right\rangle^{\circ} \tag{28}$$

and

$$\varepsilon_v^0 \equiv \varkappa \langle (\partial v^{(0)}/\partial x_j)^2\rangle^{\circ} \tag{29}$$

The coupling function

$$\Lambda^{\circ} \equiv \Lambda_{jj}^0 = -\gamma N \lambda_j \langle v^{(0)} u_j^{(0)}\rangle^{\circ} \tag{30}$$

governs the coupling between the two spectra, and operates at relatively low wavenumbers. It involves a correlation of zeroth rank $\langle v^{(0)} u_j^{(0)}\rangle^0$. We shall first calculate its first rank $\langle v' u'_j\rangle'$, derived in (19b). Unlike the eddy flux in (23b), the coupling function in a homogeneous turbulence should be calculated from (19b) by retaining the term of like rank and not the term under a gradient $\partial v^{(0)}/\partial x_s$ which would vanish after an average $\langle ... \rangle^0$. Thus we find

$$\gamma N \lambda_j \langle v' u'_j\rangle' = -N^2 \lambda_j \lambda_s \eta'_{sj} \tag{31}$$

and hence

$$\Lambda^0 \equiv N^2 \lambda_j \lambda_s \eta_{sj}^0 = N^2 \eta_{33}^0 \tag{32}$$

It has a corresponding random rank

$$\Lambda' = N^2 \eta'_{33} \equiv \Lambda - \Lambda^0 \tag{33}$$

where Λ without superscript denotes the global average. The dissipation functions (27) and (29) have their global averages too:

$$\varepsilon_u = \nu \langle (\partial u_i/\partial x_j)^2\rangle \equiv \nu R \tag{34a}$$

$$\varepsilon_v = \varkappa \langle (\partial v/\partial x_j)^2\rangle \equiv \varkappa J \tag{34b}$$

However, the transfer functions have vanishing global averages

$$T_u^0 = 0, \quad T_v^0 = 0 \tag{35}$$

B. *Evolution of energies*

In the notations (30), (32) and (34), we shall write the energy equation (25) as follows:

$$\frac{1}{2}\frac{\partial}{\partial t}\langle \mathbf{u}^{(0)2}\rangle^{\circ} = -\gamma\Lambda^{\circ} - T_u^0 - \varepsilon_u^0 \tag{36a}$$

$$\frac{1}{2}\frac{\partial}{\partial t}\langle v^{(0)2}\rangle^{\circ} = \Lambda^{\circ} - T_v^0 - \varepsilon_v^0 \tag{36b}$$

We conclude that in a stable stratification the kinetic energy is stabilized and extracted to increase the potential energy. The spectra dissipate at the rates T_u^0 and T_v^0 by eddy motions, and at the rates ε_u^0 and ε_v^0 by molecular motions.

The evolution of spectral distributions in an unstable stratification can be discussed from the same energy equations (36), provided that we take $\gamma = -1$, and replace N by M in Λ^0, according to (7). Thus we see that the effect of gravity in an unstable stratification is to amplify both the kinetic and potential energies.

C. *Universal range*

Spectral theories of turbulence usually deal with the universal range, which occurs at large wavenumbers, on account of its quasi-equilibrium state. In that state, the slow time variation of the spectrum takes place in the variability of the physical parameters, such as Λ, ε_u, and ε_v. Under such a circumstance, we can assume an equilibrium of macroscopic and global ranks, and reduce the energy equations (36) into the following equilibrium form

$$-\gamma\Lambda^0 - T_u^0 - \varepsilon_u^0 = -\gamma\Lambda - \varepsilon_u \tag{37a}$$

$$\Lambda^0 - T_v^0 - \varepsilon_v^0 = \Lambda - \varepsilon_v \tag{37b}$$

or, identically,

$$-\gamma\Lambda' + T_u^0 + \varepsilon_u^0 = \varepsilon_u \tag{38a}$$

$$\Lambda' + T_v^0 + \varepsilon_v^0 = \varepsilon_v \tag{38b}$$

The system (38) can be written in a more explicit form

$$-\gamma N^2 \eta'_{33} + (\eta'_{11} + \nu) R^{\circ} = \varepsilon_u \tag{39a}$$

$$N^2 \eta'_{33} + (\eta'_{11} + \varkappa) J^{\circ} = \varepsilon_v \tag{39b}$$

where

$$R^{\circ} \equiv \langle (\nabla \mathbf{u}^{(0)2}\rangle^{\circ} = 2\int_0^k dk' k'^2 F(k') \tag{40a}$$

$$J^{\circ} \equiv \langle (\nabla v^{(0)})^2\rangle^{\circ} = 2\int_0^k dk' k'^2 G(k') \tag{40b}$$

are vorticity functions expressed in terms of the spectral distributions F and G of the kinetic and potential energies, respectively.

On the basis of the parameters N, R, ε_u, ε_v, γ and $\varkappa$, we can formulate the following critical wavenumbers

$$\left.\begin{matrix} k_L,\ k_{RN},\ k_{R\varepsilon},\ k_\nu \\ K_L,\ K_{RN},\ K_{R\varepsilon},\ K_{\varepsilon\varkappa},\ K_{\varkappa\nu} \end{matrix}\right\} \tag{41a}$$

separating the various subranges of the stable spectral distributions for the kinetic and potential energies, respectively. Here k_L and K_L are the lower bound of the respectively universal ranges. Corresponding wavenumbers exist in an unstable stratification where N and $R^{1/2}$ are changed into M. Thus we have

$$\left.\begin{matrix} k_L,\ k_M,\ k_{M\varepsilon},\ k_\nu \\ K_L,\ K_M,\ K_{M\varepsilon},\ K_{\varepsilon\varkappa},\ K_{\varkappa\nu} \end{matrix}\right\} \tag{41b}$$

for an unstable stratification. We defer the determination of the critical wavenumbers to Sec. IX.

V. Repeated-Cascade as a memory-chain for the derivation of eddy viscosity

The eddy viscosity is found in (21) to be the time integration of the Lagrangian correlation of velocities. For further investigating its spectral composition, it is necessary to analyse the underlying mechanism of degradation into random fluctuations, which governs the way that the transport property approaches to equilibrium. To this end, we need to decompose $\mathbf{u}'$ into subranks

$$\mathbf{u}' = \mathbf{u}^{(1)} + \mathbf{u}^{(2)} + \dots + \mathbf{u}^{(N)} \tag{42a}$$

as a component part of the complete sequence

$$\mathbf{u} = \mathbf{u}^{(0)} + \mathbf{u}^{(1)} + \dots + \mathbf{u}^{(N)} \tag{42b}$$

The wavenumbers k', k'', ..., which are independent variables, serve to separate the ranks $\mathbf{u}^{(1)}$, $\mathbf{u}^{(2)}$, ... The corresponding rank averages $\langle\dots\rangle'$, $\langle\dots\rangle''$, ... will then screen the ranks from the sequence (42b), and derive from (17a) the dynamical equation for the rank $\mathbf{u}^{(1)}$. The method of derivation, which is similar to that used for the derivation of (17a) and has been elaborated by Tchen (1973a) will not be repeated here. We find

$$\frac{d^{(1)}u_i^{(1)}}{dt} + \frac{\partial}{\partial x_j}\langle u_i^{(2)}u_j^{(2)}\rangle'' = -(\mathbf{u}^{(1)}\cdot\nabla)\,u_i^{(0)} + Nv^{(1)}\lambda_i \tag{43}$$

where

$$\frac{d^{(1)}u_i^{(1)}}{dt} \equiv \left[\frac{\partial}{\partial t} + (u^{(0)} + u^{(1)})\,\nabla\right] u_i^{(1)} \tag{44}$$

is the Lagrangian time derivative of rank variable $u_i^{(1)}$.

Ve may note that the correlations

$$\langle u_i^{(1)}(0)\,u_j^{(1)}(\tau)\rangle' = \langle u_i'(0)\,u_j'(\tau)\rangle'$$

$$\langle u_i^{(2)}(0)\,u_j^{(2)}(\tau)\rangle'' = \langle u_i''(0)\,u_j''(\tau)\rangle'', \dots \tag{45}$$

are related to the eddy viscosities of various ranks

$$\eta_{ij}' = \int_0^{t^{(0)}} d\tau'\,\langle u_i'(0)\,u_j'(\tau')\rangle'$$

$$\eta_{ij}'' = \int_0^{t^{(1)}} d\tau''\,\langle u_i''(0)\,u_j''(\tau'')\rangle'', \dots \tag{46}$$

Similarly, we find

$$\frac{d^{(1)}v^{(1)}}{dt} + \frac{\partial}{\partial x_j}\langle v'' u_j''\rangle'' = -(\mathbf{u}^{(1)}\cdot\nabla)\,v^{(0)} - \gamma N\lambda_j u_j^{(1)} \tag{47}$$

The following simplifications will be made:

(*a*) We note that the correlation $\langle u_i' u_j'\rangle'$ will have a rank value of the zeroth rank, since it represents the statistical effects of the random fluctuation $\mathbf{u}'$ upon $\mathbf{u}^{(0)}$. A subsequent Lagrangian integration of such time correlation $\langle u_i'(0)u_j'(\tau)\rangle'$ will further smooth its rank so as to produce a transport property η_{ij}' of rank -1. For this reason, the composition of η_{ij}' should find no contribution from $\mathbf{u}^{(0)}$ and $v^{(0)}$ of equations (43) and (47).

(*b*) The roles of the random fluctuations $u_i'' u_j''$, $u_j'' v''$ upon $\mathbf{u}^{(1)}$ and $v^{(1)}$ in (43) and (47) are analogous to the roles of the random fluctuations $u_i' u_j'$, $u_j' v'$ upon $\mathbf{u}^{(0)}$ and $v^{(0)}$ in (15). Therefore we can write

$$-\frac{\partial}{\partial x_j}\langle u_i'' u_j''\rangle'' = \eta_{sj}''\frac{\partial^2 u_i^{(1)}}{\partial x_s\,\partial x_j} \tag{48a}$$

and

$$-\frac{\partial}{\partial x_j}\langle v'' u_j''\rangle'' = \eta_{sj}''\frac{\partial^2 v^{(1)}}{\partial x_s\,\partial x_j}, \qquad (48\text{b})$$

in analogy with (23a) and (23b).

(c) The system of equations (43) and (47) can be regarded as a generalization of the Langevin equation, with $d^{(1)}/dt$ as the Lagrangian time derivative, as mentioned by (44). The right hand members of (43) and (47) can be treated as noise functions, which may form correlations on their own, so as to damp $\langle u_3^{(1)}(0)\,u_3^{(1)}(\tau)\rangle'$.

With the use of the above simplifications, we reduce (43) and (47) to the following Fourier form

$$\left(\frac{d^{(1)}}{dt} + \omega_k'\right)u_i^{(1)}(t, \mathbf{k}) = N\lambda_i v^{(1)}(t, \mathbf{k}) \qquad (49\text{a})$$

$$\left(\frac{d^{(1)}}{dt} + \omega_k'\right)v^{(1)}(t, \mathbf{k}) = -\gamma N\lambda_j u_j^{(1)}(t, \mathbf{k}), \quad \gamma = 1 \qquad (49\text{b})$$

for a stable stratification, and

$$\left(\frac{d^{(1)}}{dt} + \omega_k'\right)u_i^{(1)} = M\lambda_i v^{(1)} \qquad (50\text{a})$$

$$\left(\frac{d^{(1)}}{dt} + \omega_k'\right)v^{(1)} = 0 \qquad (50\text{b})$$

for an unstable stratification. Here

$$\omega_k' = k_i k_j \eta_{ij}'' \qquad (51\text{a})$$

or

$$\omega_k'' = k_i k_j \eta_{ij}''', \ldots \qquad (51\text{b})$$

is an eddy relaxation frequency.

It is not difficult to solve for $\mathbf{u}^{(1)}$ by an elimination of $v^{(1)}$ between (49a) and (49b), and hence to calculate the eddy viscosity η_{ij}', as defined by (21) and (45). We shall omit the details of the calculations, and write for the vertical transport

$$\eta_{33}'(k) = \frac{\mu\langle|u_{33}'(\mathbf{k})|^2\rangle'}{\omega_k'} \qquad (52)$$

where

$$\left.\begin{aligned}\mu &= [1 + (N/\omega_k')^2]^{-1}, \text{stable stratification}\\ &= 1, \text{unstable stratification}\end{aligned}\right\} \qquad (53)$$

is a damping factor in the vertical transport, and is absent in the horizontal transport. For the horizontal transport we have

$$\eta_{11}'(k) = \eta_{22}'(k) = \langle u_{11}'^2\rangle'/\omega_k' \qquad (54)$$

When we sum over the wavenumbers, we find

$$\eta_{33}' = 2\int_k^\infty dk'\,\frac{\mu F_{33}(k')}{\omega_k'} \equiv c_3\int_k^\infty dk'\,\frac{\mu F(k')}{\omega_k'} \qquad (55\text{a})$$

$$\eta_{11}' = 2\int_k^\infty dk'\,\frac{F_{11}(k')}{\omega_k'} \equiv c_1\int_k^\infty dk'\,\frac{F(k')}{\omega_k'} \qquad (55\text{b})$$

where F_{11}, F_{22} and F_{33} are the spectrum of the kinetic energy in the three components, and F is their sum. The fractional coefficients c_1 and c_3 are left somewhat undermined, as they will require a more precise model of the anisotropic distribution of the kinetic energy.

The above considerations allow to perpetuate the calculation for η_{11}'', obtaining

$$\eta_{11}'' = c_1\int_{k'}^\infty dk''\,\frac{F(k'')}{\omega_k''} \qquad (56)$$

$$\vdots$$

The sequence of equations (51) and (56) describe evidently a chain of memories, with relaxation frequencies ω_k', ω_k'', ... of ever increasing ranks. The simplest closure of the chain at an early stage is to assume an isomeric behavior in all links of the chain; this implies that the spectral distribution, unspecified nevertheless, is assumed to be common to all links. This assumption approximates (51a) to an isotropic relaxation frequency

$$\omega_k' \cong k'^2\eta' \qquad (57)$$

and hence reduces (55b) to the eddy viscosity for the isotropic horizontal transport

$$\eta' \equiv \eta_{11}' = c_1\int_k^\infty dk'\,\frac{F(k')}{k'^2\eta'} \qquad (58)$$

yielding the solution

$$\eta' = \left[2c_1\int_k^\infty dk'k'^{-2}F(k')\right]^{1/2} \qquad (59)$$

It is to be noted that the eddy viscosity η_{33}'

for the vertical transport remains to be given by (55a), but its relaxation frequency ω_k' can now be determined from (57) and (58) on account of its governing large wavenumbers.

VI. Stabilization of the kinetic energy by buoyancy in a stable stratification

A. *Buoyancy subrange* ($k_L < k < k_{RN}$ *for* F, $K_L < K < K_{RN}$ *for* G)

Here and in the following, we divide the subranges by the critical wavenumbers (41).

The kinetic energy is stabilized by the buoyancy force, and is extracted to boost the potential energy. This process of energy balance can be described by (39), which is rewritten approximately as

$$-N^2\eta_{33}' + R^0\eta' = \varepsilon_u \tag{60a}$$

$$N^2\eta_{33}' + J^0\eta' = \varepsilon_v \tag{60b}$$

Since the exchange process occurs without dissipations ε_u and ε_v, we shall take a differential form of (60), which is

$$-N^2\dot{\eta}_{33}' + \eta'\dot{R}^0 + \dot{\eta}'R^0 = 0 \tag{61a}$$

$$N^2\dot{\eta}_{33}' + \eta'\dot{J}^0 \cong 0 \tag{61b}$$

The potential energy spectrum, being the receiving spectrum, possesses a subrange of production. In such an early stage of development in the k space, its vorticity is not fully developed, and therefore permits to assume $J^0 \cong 0$ in (61b).

For the sake of abbreviation of writing we introduce the notations

$$r^0 = R^0/\mu N^2, \quad G = \alpha F \tag{62}$$

and transform (61) into

$$\begin{aligned} c_3 + \frac{2}{1-\mu} - c_1 r^0 &= 0 \\ -c_3 + \frac{2\alpha}{1-\mu} &= 0 \end{aligned} \tag{63}$$

Here we have made use of definitions (55a) and (55b) for η_{33}' and η'.

Since N is the driving frequency of the system, we may approximate that the relaxation frequency to be in a constant ratio to N. It follows then that μ, α and $r^0 \cong r$ become constant. We have

$$\alpha = [(c_1/c_3)r - 1]^{-1} \cong (c_1 r/c_3)^{-1} \tag{64a}$$

$$\mu = 1 - \frac{2c_3^{-1}}{(c_1/c_3)r - 1} \cong 1 \tag{64b}$$

From the definition (53) and (64b), we find

$$(\omega_k'/N)^2 = \tfrac{1}{2}c_3[(c_1/c_3)r - 1 - 2c_3^{-3}] \cong \tfrac{1}{2}r \tag{65}$$

and from the definitions (57) and (59), we find

$$F = A_b N^2 k^{-3} \tag{66a}$$

$$G = \alpha F \tag{66b}$$

where

$$\left.\begin{aligned} A_b &= (c_3/c_1)[(c_1/c_3)r - 1 - 2c_3^{-1}] \\ &\cong r \end{aligned}\right\} \tag{67}$$

The approximations were made in (64), (65) and (67) on the basis of

$$r \gg 1 \tag{68}$$

Under this condition, we can write the solutions (66) as follows:

$$F \cong Rk^{-3} \tag{69a}$$

$$G \cong (c_3/c_1)N^2k^{-3} \tag{69b}$$

The co-spectrum F_{vu}, defined as

$$-\langle v^{(0)}u_3^{(0)}\rangle^\circ = \int_0^k dk' F_{vu}(k') \tag{70}$$

can be obtained from the relations (30) and (32), rewritten as

$$-\gamma\langle v^{(0)}u_3^{(0)}\rangle^0 \equiv N^{-1}\Lambda^0 = N\eta_{33}^0 \tag{71}$$

where Λ^0 is defined by (33). Upon differentiation, we find

$$\left.\begin{aligned} F_{vu} &= (c_3\mu N/\omega_k')F \\ &\cong \sqrt{2}c_3 R^{1/2} N k^{-3} \end{aligned}\right\} \tag{72}$$

We conclude the following:

(*a*) The potential energy spectrum has a production subrange, dependent on the driving frequency N, as found in (69b), while the kinetic energy spectrum, which is extracted by the buoyancy force, is cascading down the spectrum at a constant vorticity R, so that the buoyancy subrange depends on R, as found in (69a).

(*b*) The co-spectrum, which plays the role of a coupling between the two energy spectra, depends on both R and N, as in (72).

(*c*) In order to maintain the flow of energy from the kinetic energy spectrum toward the potential energy spectrum, it is necessary to have a difference of energy levels between the two spectra, requiring $R > N^2$, as consistent to the underlying assumption (68).

B. *Inertia transfer of the potential energy under a buoyant eddy mixing* ($k_L < k < k_{RN}$ *for* F, $K_{RN} < k < K_{R\varepsilon}$ *for* G)

In view of the large conductivity as compared to the kinematic viscosity, the inertia subrange of the potential energy spectrum may have started when the kinetic energy spectrum still remains in the buoyancy subrange.

With an eddy mixing in the buoyancy subrange (69a), the potential energy spectrum in the inertia subrange is governed by

$$\eta' J^0 = \varepsilon_v \tag{73}$$

With the substitution of (69a) into (59), we find the solution

$$G = v_N^2 k^{-1} \tag{74}$$

with

$$v_N^2 = \left(\frac{2}{c_1}\right)^{1/2} \frac{\varepsilon_v}{R^{1/2}} \tag{75}$$

C. *Intermediate subranges governed by the buoyancy and inertia forces* ($k_{RN} < k < k_R$ *for* F, $K_{RN} < k < K_R$ *for* G)

The inertia subranges of the two spectra are governed by the transfer functions, reducing (60) to

$$\eta' R^0 = \varepsilon_u \tag{76a}$$

$$\eta' J^0 = \varepsilon_v \tag{76b}$$

The transfer function consists of an eddy viscosity η' and a vorticity function R^0 or J^0. According to their definitions (59) and (40), η'^2 can be regarded as a k^{-2} moment of the F spectrum, and R^0 and J^0 as the k^2 moment of the F and G spectra, respectively. In the intermediate subrange, we can assume that the low moment originates from the stabilization subrange of a spectral law (69a), and that the high moments lie in the inertia subranges.

With the aid of the spectral law (69a), we have

$$\eta' = (\tfrac{1}{2}c_1)^{1/2} R^{1/2} k^{-2} \tag{77}$$

and transform (76) to

$$\left.\begin{aligned} R^0 &= u_N^2 k^2 \\ J^0 &= v_N^2 k^2 \end{aligned}\right\} \tag{78}$$

Upon differentiation, we obtain

$$F = u_N^2 k^{-1} \tag{79a}$$

$$G = v_N^2 k^{-1} \tag{79b}$$

with

$$u_N^2 = (2/c_1)^{1/2} \varepsilon_u / R^{1/2} \tag{80}$$

and v_N defined by (75).

VII. Production of energy in an unstable stratification

A. *Production subrange* ($k_L < k < k_M$ *for* F, $K_L < k < K_M$ *for* G)

The energy balance in an unstable stratification is described by a system of equations similar to (39), with $\gamma = -1$ and with a change of N into M, according to (7). We have

$$\Lambda' + (\eta' + \nu) R^0 = \varepsilon_u \tag{81a}$$

$$\Lambda' + (\eta' + \varkappa) J^0 = \varepsilon_v \tag{81b}$$

with

$$\Lambda' = M \langle v' u_3' \rangle' = M^2 \eta_{33}' \tag{82a}$$

or

$$\langle v^{(0)} u_3^{(0)} \rangle^0 = M \eta_{33}^0 \tag{82b}$$

where R^0 and J^0 are defined by (40a) and (40b), and η_{33}' and η' are defined by (55a), and (59), respectively.

In the production subranges, the system (81) reduces to

$$M^2\eta'_{33} + \eta' R^0 = \varepsilon_u \tag{83a}$$

$$M^2\eta'_{33} + \eta' J^0 = \varepsilon_v \tag{83b}$$

The flow of energy across each spectrum can be most conveniently demonstrated by a differential form of (83), which is

$$M^2\dot{\eta}'_{33} + \eta' R^0 = 0 \tag{84a}$$

$$M^2\dot{\eta}'_{33} + \eta' \dot{J}^0 = 0 \tag{84b}$$

where we have neglected $\dot{\eta}' R^0$ and $\dot{\eta}' J^\circ$ in view of the small values of vorticities in the production subranges.

The system (84) yields the solution

$$F = (c_3/c_1) M^2 k^{-3} \tag{85a}$$

$$G = (c_3/c_1) M^2 k^{-3} \tag{85b}$$

The co-spectrum F_{vu} can be found by using the definitions (70) and (71), and the solution (85a). We obtain

$$F_{vu} = -C_p M^2 k^{-3}, \quad C_p = (2c_3^3/c_1^2)^{1/2} \tag{86}$$

B. *Inertia transfer of the potential energy under a mixing by eddy movements in the production subrange*
($k_L < k < k_M$ *for* F, $K_M < k < K_{M\varepsilon}$ *for* G)

In view of the large conductivity $\varkappa$ as compared to the kinematic viscosity ν, the potential energy may drop into its inertia subrange, while the kinetic energy still remains in the buoyancy subrange. The inertia subrange of the potential energy has the following equation of spectral balance:

$$\eta' J^0 = \varepsilon_v \tag{87}$$

where η' is calculated from (59), using the spectrum of kinetic energy (85a) in the production subrange, to be

$$\eta' = \left(2c_1 \int_k^\infty dk' k'^{-2} F\right)^{1/2}$$

$$= (\tfrac{1}{2} c_3)^{1/2} M k^{-2} \tag{88}$$

Upon differentiating J° from (87) and substituting (88) we find

$$G = v_M^2 k^{-1}$$

where

$$v_M^2 = \left(\frac{2}{c_3}\right)^{1/2} \frac{\varepsilon_v}{M} \tag{90}$$

C. *Intermediate subrange between the production and inertia forces*
($k_M < k < k_{M\varepsilon}$ *for* F, $K_M < k < K_{M\varepsilon}$ *for* G)

The inertia subrange of the two spectra are governed by their transfer functions in a system of equations identical to (76). We treat the eddy viscosity η' as formed from the spectrum in the production subrange to be (88), and the vorticity functions, being of higher moments, as formed from the inertia subranges of the spectra. In a calculation similar to Subsec. VI C, we find

$$F = u_M^2 k^{-1} \tag{91a}$$

$$G = v_M^2 k^{-1} \tag{91b}$$

where

$$u_M^2 = (\tfrac{1}{2} c_3)^{1/2} \frac{\varepsilon_u}{M} \tag{92}$$

and v_M is defined by (90).

VIII. Inertia transfer and dissipations in stable or unstable stratification

A. *Inertia subrange* ($k_R < k < k_\nu$ *for* F, $K_{R\varepsilon} < k < K_{\varepsilon\varkappa}$ *for* G)

The inertia subranges of the two spectra are governed by the equations of energy balance (76). We find the solutions

$$F = A_i \varepsilon_u^{2/3} k^{-5/3}, \quad A_i = (16/27c_1)^{1/3} \tag{93a}$$

$$G = B_i \varepsilon_v \varepsilon_u^{-1/3} k^{-5/3}, \quad B_i = (16/27c_1 A_i)^{1/2} \tag{93b}$$

The co-spectrum can be obtained from (71), and we find

$$F_{vu} = \gamma C_i \varepsilon_u^{1/3} N k^{-7/3}, \quad C_i = (4c_1 A_i/3)^{1/2} \tag{94}$$

with $\gamma = 1$, for a stable stratification. The result (94) is also valid for an unstable stratification, provided that N is changed into M and $\gamma = -1$, so that the co-spectrum becomes negative.

B. *Dissipation subrange* ($k > k_\nu$ *for* F, $k > K_{\varkappa\nu}$ *for* G)

The dissipation subranges of the two spectra are governed by the molecular dissipation of the cascade transfer processes, as described by the following balance equations, obtained by neglecting the coupling function in (39):

$$(\eta' + \nu) R^0 = \varepsilon_u \tag{95a}$$

$$(\eta' + \varkappa) J^0 = \varepsilon_v \tag{95b}$$

Since a flow of energy from the inertia into the dissipation describes the approach to the dissipation subrange, we rewrite (95) in the differential form

$$\dot{\eta}' R^0 + (\eta' + \nu) \dot{R}^0 = 0 \tag{96a}$$

$$\dot{\eta}' J^0 + (\eta' + \varkappa) \dot{J}^0 = 0 \tag{96b}$$

approximated to

$$\dot{\eta}' R + \nu \dot{R}^0 \cong 0 \tag{97a}$$

$$\dot{\eta}' J + \varkappa \dot{J}^0 \cong 0 \tag{97b}$$

The approximate form (97) is obtained by writing

$$\left.\begin{aligned} &\eta' + \nu \cong \nu, \quad \eta' + \varkappa \cong \varkappa \\ &R^0 \cong R, \quad J^0 \cong J \end{aligned}\right\} \tag{98}$$

since large wavenumbers are involved.

The equations (97) yield the following solutions

$$F = A_d (R/\nu)^2 k^{-7}, \quad A_d = 2^{-8} c_1 \tag{99a}$$

$$G = B_d (RJ/\nu\varkappa) k^{-7}, \quad B_d = 2^{-6} c_1^{3/2} \tag{99b}$$

On account of (71) and (99a), we find the cospectrum

$$\left.\begin{aligned} F_{vu} &= -N\dot{\eta}'_{33} \\ &= \gamma C_d N (R/\nu) k^{-5}, \quad C_d = 2^{-7} c_1 \end{aligned}\right\} \tag{100}$$

With $\gamma = 1$, the result (100) is valid in a stable stratification. The same result is valid for an unstable stratification too, provided that N is changed into M, and $\gamma = -1$, so that the cospectrum becomes negative.

C. *Dissipation of the potential energy with an inertial mixing* ($k_{R\varepsilon} < k < k_\nu$, *for* F, $K_{\varepsilon\varkappa} < k < K_{\varkappa\nu}$ *for* G)

In view of the large conductivity $\varkappa$ as compared to the kinematic viscosity ν, the spectrum of potential energy may start its dissipation subrange earlier (i.e. at smaller k) than the kinetic energy. Thus, with an inertial value of η' given by

$$\begin{aligned} \eta' &= \left(2c_1 \int_k^\infty dk' k'^{-2} F\right)^{1/2} \\ &= (3c_1 A_i/4)^{1/2} \varepsilon_u^{1/3} k^{-4/3} \end{aligned} \tag{101}$$

according to (93a), we can rewrite (95b) as

$$J^\circ = \frac{\varepsilon_v}{\eta' + \varkappa} \tag{102}$$

and differentiate, to obtain

$$2k^2 G = -\frac{\varepsilon_v \dot{\eta}'}{(\eta' + \varkappa)^2} \cong -(\varepsilon_v/\varkappa^2)\dot{\eta}' \tag{103}$$

and the solution

$$G = C_{id} (\varepsilon_u^{1/3} \varepsilon_v/\varkappa^2) k^{-13/3}, \quad C_{id} = 2^{2/3} 3^{-1} c_1^{1/3} \tag{104}$$

IX. Conclusions

A. *Discussions of results*

We summarize the results of the spectral distributions in a stable stratification in Table 1 for a stable stratification. The subranges of the kinetic energy spectrum are: buoyancy, inertia with buoyant mixing, inertia and dissipation, separated by the critical wavenumbers

$$\left.\begin{aligned} k_{RN} &= (R^{1/2} N^2/\varepsilon_u)^{1/2} \\ k_{R\varepsilon} &= (R^{3/2}/\varepsilon_u)^{1/2} \\ k_\nu &= (\varepsilon_u/\nu^3)^{1/4} \end{aligned}\right\} \tag{105a}$$

respectively. The subranges of the potential energy spectrum are: buoyancy, inertia with buoyant mixing, inertia, dissipation with inertia mixing and dissipation, separated by the critical wavenumbers

$$\left.\begin{aligned} K_{RN} &= (R^{1/2} N^2/\varepsilon_v)^{1/2} \\ K_{R\varepsilon} &= (R^{3/2}/\varepsilon_u)^{1/2} \\ K_{\varepsilon\varkappa} &= (\varepsilon_u/\varkappa^3)^{1/4} \\ K_{\varkappa\nu} &= [\varepsilon_u/(\nu\varkappa)^{3/2}]^{1/4} \end{aligned}\right\} \tag{105b}$$

Table 1. *Spectral distributions of kinetic and potential energies and co-spectrum of heat flux (stable stratification)*

Subranges of (F, G)		Kinetic energy spectrum $F(k)$: Buoyancy	Inertia w. buoyancy mixing	Inertia	Dissipation
		k_{RN}	$k_{R\varepsilon}$	k_ν	
Potential energy spectrum G: Buoyancy	K_{RN}	$F = Rk^{-3}$ $G = N^2 k^{-3}$ $F_{vn} = R^{1/2} N k^{-3}$			
Inertia w. buoyancy mixing	$K_{R\varepsilon}$		$F = u_N^2 k^{-1}$ $G = v_N^2 k^{-1}$	$F = \varepsilon_u^{2/3} k^{-5/3}$ $G = v_N^2 k^{-1}$	
Inertia	$K_{\varepsilon\varkappa}$	$F = Rk^{-3}$ $G = v_N^2 k^{-1}$		$F = \varepsilon_u^{2/3} k^{-5/3}$ $G = \varepsilon_v \varepsilon_u^{-1/3} k^{-5/3}$ $F_{vu} = N \varepsilon_u^{1/3} k^{-7/3}$	
Dissipation w. inertia mixing	$K_{\varkappa\nu}$			$F = \varepsilon_u^{2/3} k^{-5/3}$ $G = \varepsilon_u^{1/3} \varepsilon_v \varkappa^{-2} k^{-13/3}$	
Dissipation					$F = \left(\frac{R}{\nu}\right)^2 k^{-7}$ $G = \frac{RJ}{\nu\varkappa} k^{-7}$ $F_{vu} = \frac{NR}{\nu} k^{-5}$

The corresponding critical wavenumbers for an unstable stratification are

$$\left.\begin{aligned} k_M &= (M^3/\varepsilon_v)^{1/2}, & K_M &= (M^3/\varepsilon_v)^{1/2} \\ k_{M\varepsilon} &= (M^3/\varepsilon_u)^{1/2}, & K_{M\varepsilon} &= (M^3/\varepsilon_u)^{1/2} \\ k_\nu &= (\varepsilon_u/\nu^3)^{1/4}, & K_{\varkappa\nu} &= [\varepsilon_u/(\nu\varkappa)^{3/2}]^{1/4} \end{aligned}\right\} \quad (105\text{c})$$

The buoyancy, inertia with buoyant mixing, inertia and dissipation subranges follow the spectral laws of powers -3, -1, $-5/3$ and -7, respectively, for both spectra. The spectral laws from mixed subranges, e.g. a buoyancy kinetic energy with an inertial potential energy, and an inertial kinetic energy with a dissipative potential energy, are also listed. In addition, the co-spectrum F_{vu} in the buoyancy inertia and dissipation subranges predicts the power laws -3, $-7/3$ and -5, respectively.

The spectral laws, as listed in Table 1 for a stable stratification, are also valid for an unstable stratification, after a modification of parameters.

The buoyancy subranges which are characteristic of the mixing by vertical motions assume a -3 power law for stable and unstable stratifications. However, in view of (68), the stable F spectrum must be of a higher magnitude than the unstable F spectrum, in order to perform a flow of energy from the F spectrum into the G spectrum in a stable stratification.

The subranges of inertia with a buoyant eddy mixing predict a spectral law of -1 power for both spectra. However, in a strongly stratified medium which damps out severely the vertical motions, the kinetic energy spectrum may be mainly composed of horizontal components which are not buoyant, and therefore will assume an inertia law of $-5/3$. On the other hand, the potential energy may have a dominant transfer function

$$\eta'_{33}\left\langle\left(\frac{\partial v^{(0)}}{\partial x_3}\right)^2\right\rangle^{\circ} \tag{106}$$

with an eddy viscosity η'_{33} originated from the vertical eddy motions which are subject to the buoyancy force. Under such a circumstance, we may expect the solutions

$$F \cong A_i \varepsilon_u^{2/3} k^{-5/3} \tag{107a}$$

and

$$G = v_N^2 k^{-1} \tag{107b}$$

B. *Comparison with dimensional theories*

The spectral law $k^{-5/3}$ of the kinetic energy, as derived by the present cascade method in (93), is in agreement with the similarity theory of Kolmogoroff (1941).

In a stable stratification, the buoyancy law

$$F = N^2 k^{-3} \tag{108}$$

has been proposed on the basis of a dimensional analysis (Lumley, 1964; 1965; Phillips, 1966), with N as a parameter. Since the dimensional analysis does not introduce a detailed transport process, it cannot make a distinction between a stable and an unstable stratification. Based upon the balance between the kinetic and potential energies, the present cascade theory derives the power law k^{-3} in (69) but finds an internal parameter R.

The k^{-1} law, as found in (79) and (91), cannot rely upon simple dimensional considerations.

C. *Comparison with experiments*

A review of the experimental observations has been given by Pinus et al. (1967). The kinetic energy spectrum has been measured to follow the k^{-3} and $k^{-5/3}$ laws in a stable medium and neutral medium, respectively (Myrup, 1969). The vertical motions have been measured (Benton & Kahn, 1958). They start with a spectral law k^{-3} in the buoyancy subrange, to be followed by an inertia law $k^{-5/3}$. On the other hand, the horizontal motions seem to keep a $k^{-5/3}$ law. Kukharets and Tsvang (1969) have measured a kinetic energy spectrum k^{-1}, together with a co-spectrum $k^{-7/3}$. Measurements have shown that the co-spectrum changes its sign by going from a stable stratification to an unstable stratification (Zubkovskiy & Koprov, 1969). The temperature fluctuations have been measured in oceans, and the spectral laws k^{-3} and $k^{-5/3}$ have been found in the buoyancy (Gastev & Shvachko, 1969) and inertia subranges (Galkin et al. 1971), respectively. Those experimental results conform to some extent to the predictions of the present theory.

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КАСКАДНАЯ ТЕОРИЯ ТУРБУЛЕНТНОСТИ В СТРАТИФИЦИРОВАННОЙ СРЕДЕ

Постоянный средний термический градиент присутствует в жидкости, предпологаемой несжимаемой, за исключением для силы плавучести. Последняя связывает моментное уравнение для флуктуаций скорости с уравнением тепла для температурных флуктуаций. Однокаскадный метод, разделяющий флуктуации на макроскопическую и случайную части, позволяет замкнуть иерархию корреляций путем расщепления корреляции четвергото порядка на произведение двух двойных корреляций различного ранга случайности. Этот метод вводит преимущественные парные корреляции между модами корреляций четвертого порядка. Одна из двойных корреляций появляется в виде вихревйо *вязкости*. Определение ее спектральной структуры или, эквивалентно, проблема приближения к равновесию включает цепочку уравнений для многих релаксаций. Это требует использования повторного каскада. Релаксационная цепочка обрывается на основе *предположения* об изомерном распределении спектра по всем звеньям цепи. Решения уравнения баланса между кинетической энергией и потенциальной энергией термического поля дает спектральные законы k^{-3}, k^{-1} и $k^{-5/3}$ в различных подынтервалах обоих спектров энергии и законы $k^{-7/3}$ и k^{-5} для коспектра потока тепла.