On forced oscillations in a rotating stratified fluid

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ABSTRACT

An attempt is made to study the axisymmetric forced oscillations of an inviscid rotating stratified fluid confined in a finite circular cylinder. An unsteady flow is generated in the fluid by harmonic oscillations of the plane ends of the cylinder. The principal features of the flow phenomena are determined by the relative magnitudes of the angular velocity Ω of rotation, the forcing frequency ω of the imposed oscillations and the Brunt Vaisala frequency N. It is shown that the solution for the disturbance pressure (and hence the velocity field) consists of two components—the periodic solution oscillating harmonically with frequency ω and the normal inertial modes each of which is spatially periodic in the direction of the axis of rotation. The latter contains a doubly infinite set of modes whose frequencies form a discrete spectrum dense in [min $(2\Omega, N)$, max $(2\Omega, N)$]. Both of these solutions are independently modified by the density stratification. The solution is found to have a resonant behaviour when the forcing frequency ω is equal to one of the resonant frequencies ω_{mn}. In contrast to the singular behaviour of the solution at $\omega = 2\Omega$ which usually occurs in the forced motion of a rotating liquid, the present solutions for this critical case $\omega = 2\Omega$ and the case $\omega = N$ represent the inertial modes only. A qualitative assessment of the possible effects of viscosity on the flow has been made. Several interesting features of the rotating stratified flow phenomena have been analyzed. The initial value problem is solved by using the joint Laplace and the Hankel transforms of the second kind.

1. Introduction

One of the most significant features of forced oscillations in an inviscid incompressible rotating fluid is the difference between the motions observed when the forcing frequency ω of the superimposed oscillations is greater or less than twice the angular velocity Ω of rotation of the fluid. More precisely, under the assumptions of steady motion, the governing flow equation for the disturbance pressure field is of elliptic or of hyperbolic type according as $\omega >$ or $< 2\Omega$. In the former case, the equation, and the characteristics of the flows are more or less very analogous to those of the non-rotating liquids. On the other hand, in the latter case, the disturbance caused by a body oscillating with the subcritical frequency propagate as wave motions throughout the flow field. Further, a transition from the elliptic to hyperbolic regime occurs at the critical frequency $\omega = 2\Omega$, the so-called resonant frequency. Numerous interesting properties of the fluid motions related to cases $\omega > = < 2\Omega$ are theoretically and experimentally well understood (Greenspan, 1968).

In the case of stratified fluids, the governing equation for the amplitude function is also of elliptic or of hyperbolic type according as the forcing frequency $\omega >$ or $< N(z)$ where

$$N(z) = \left\{-\frac{g}{\varrho}\frac{d\varrho}{dz}\right\}^{1/2}$$

represents the Brunt-Vaisala frequency and $\varrho(z)$ is the density function of the vertical coordinate z. In this case too, there exists a critical frequency $\omega = N(z)$. Many intersting features of the stratified flows including the internal wave phenomena have also extensively been investigated in various configurations. Based on the Boussinesq approximation, problems concerning internal wave motions have received attention by many authors (Long, 1853; Trustrum, 1964; Yih, 1965; Keller & Mow, 1967).

Recently, Bretherton (1967) has investigated the development of Taylor columns in a homogeneous rotating fluid at a small Rossby number and also the blocking by a two-dimensional obstacle in a stably stratified fluid at a low internal Froude number. Hendershott (1969) has extended the work of Stewartson on oscillations in a rotating stratified fluid of infinite extent due to a spherical source pulsating with a fixed frequency. Both of these authors predicted that the ultimate pattern of the flow may be understood in terms of the inertial-internal waves which carry energy in the far field.

In a recent paper, Baines (1967) has made an initial value investigation of forced oscillations of a uniformly rotating fluid enclosed in a finite circular cylinder whose plane ends are impulsively set into harmonic oscillations with a given frequency ω in the axial direction. The exact solution of the linearized problem is obtained for all values of ω and Ω including the resonant frequency, $\omega = 2\Omega$, for which no physically sensible solution is found by the steady state assumption. He explored the principal characteristics of the flow and included a short discussion on the non-linear and viscous effects. Finally, he reported that many of his theoretical predictions have been confirmed experimentally by A. D. McEwan at Aeronautical Research Laboratories, Fisherman's Bend, Australia.

The present analysis is intended to make some investigation of the rotating stratified flows generated by an impulsive commencement of motion. An unsteady flow is set up in an inviscid, rotating stratified fluid contained in a finite cylinder by the harmonic oscillations of the plane ends of the cylinder. With the aid of the Laplace transform treatment, the principal features of the flow are determined by the relative magnitudes of the various frequencies involved in the problem. An exact solution for the disturbance pressure and the velocity field have been obtained explicitly and their physical implications have been explored. A critical analysis is made of the steady state solution and the normal inertial modes including the effects of density stratification. Attention is also given to the nature of the solution when the forcing frequency ω is equal to one of the resonant frequencies, ω_{mn}, and when $\omega = 2\Omega$ or N. Several interesting features of the flow phenomena including the possible effects of viscosity have been explored.

2. Mathematical formulation

We consider the forced motion of an inviscid, incompressible rotating stratified fluid enclosed in a finite circular cylinder of length $2l$ and radius a. The fluid is in a state of rigid body rotation with constant angular velocity Ω about the axis of symmetry $r = 0$ and satisfies the Boussinesq approximation. The density field of the undisturbed fluid is supposed to be of the form $\varrho_0 = \varrho_{00} \exp(-\beta z)$, $\beta > 0$ so that the Brunt-Vaisala frequency

$$N = \left\{ -\frac{g}{\varrho_0} \frac{d\varrho_0}{dz} \right\}^{1/2}$$

remains constant throughout the flow field. The flow is generated in the stably stratified rotating system by identical harmonic oscillations of the plane ends of the cylinder in the axial direction.

We take the cylindrical polar coordinates (r, θ, z) with the z-axis as the axis of the cylinder and the ends of the cylinder coincide with the planes $z = \pm l$ so that after they set impulsively into motion, the boundaries are at

$$z = \pm l + \alpha H(t) f(r) \sin \omega t \tag{2.1}$$

where $\alpha \ll l$, $H(l)$ is the Heaviside unit function, $f(r)$ is an arbitrary differentiable function of r, and ω is the frequency of oscillations. The assumption of sufficiently small α is a necessity to justify the linearization of the equations of motion and the boundary conditions.

The unsteady motion of the fluid is described by the linearized Euler equation relative to the rotating axes and the equation of continuity, which are in tho usual notations

$$\frac{\partial u}{\partial t} - 2\Omega v = -\frac{1}{\varrho_0} \frac{\partial \chi_1}{\partial r} \tag{2.2}$$

$$\frac{\partial v}{\partial t} + 2\Omega u = 0 \tag{2.3}$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\varrho_0} \frac{\partial \chi_1}{\partial z} - \frac{g\varrho}{\varrho_0} \tag{2.4}$$

$$\frac{\partial \varrho}{\partial t} + w \frac{\partial \varrho_0}{\partial z} = 0 \tag{2.5}$$

and

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0 \tag{2.6}$$

where $\chi_1 \equiv \chi_1(r, z; t)$ is the disturbance pressure including the centrifugal term, ϱ_0 is the undisturbed density and ϱ is the density perturbation.

The linearized boundary conditions for $t \geqslant 0$ are

$$u = 0 \text{ on } r = a \tag{2.7}$$

$$w = \alpha\omega f(r) \cos \omega t \text{ on } z = \pm l \tag{2.8}$$

Taking into account the initial effects of rotation as described by Morgan (1953), the initial motion for $t = 0$ is given by

$$\hat{u} = -\frac{1}{\varrho_0}\frac{\partial \hat{\chi}_1}{\partial r},\ \hat{v} = 0,\ \hat{w} = -\frac{1}{\varrho_0}\frac{\partial \hat{\chi}_1}{\partial z},\ \hat{\varrho} = 0 \tag{2.9}$$

and

$$\frac{\partial \hat{u}}{\partial r} + \frac{\hat{u}}{r} + \frac{\partial \hat{w}}{\partial z} = 0 \tag{2.10}$$

where the hat is used to indicate the value of a quantity at $t = 0$.

3. The solution of the initial value problem

The problem posed above can readily be solved by using the Laplace transform $\bar{\chi}(r, z; s)$ defined by

$$\bar{\chi}(r, z; s) = \int_0^{\infty} e^{-st}\chi(r, z; t)\, dt \tag{3.1}$$

Introducing $\varrho_0 \bar{\chi} = \bar{\chi}_1 + \hat{\chi}_1$, and invoking the Boussinesq approximation and the Laplace transformation, equations (2.2)–(2.6) reduce to the form

$$s\bar{u} - 2\Omega\bar{v} = -\frac{\partial \bar{\chi}}{\partial r} \tag{3.2}$$

$$s\bar{v} + 2\Omega\bar{u} = 0 \tag{3.3}$$

$$s\bar{w} = -\left(\frac{\partial \bar{\chi}}{\partial z} + \frac{g\bar{\varrho}}{\varrho_0}\right) \tag{3.4}$$

$$s\bar{\varrho} + \bar{w}\frac{\partial \varrho_0}{\partial z} = 0 \tag{3.5}$$

$$\frac{\partial \bar{u}}{\partial r} + \frac{\bar{u}}{r} + \frac{\partial \bar{w}}{\partial z} = 0 \tag{3.6}$$

The transformed velocity components in terms of $\bar{\chi}$ are obtained from (3.2)–(3.5) as

$$\bar{u} = -\frac{s}{(s^2 + 4\Omega^2)}\frac{\partial \bar{\chi}}{\partial r}, \tag{3.7 a}$$

$$\bar{v} = \frac{2\Omega}{(s^2 + 4\Omega^2)}\frac{\partial \bar{\chi}}{\partial r}, \tag{3.7 b}$$

$$\bar{w} = -\frac{s}{(s^2 + N^2)}\frac{\partial \bar{\chi}}{\partial z} \tag{3.7 c}$$

Elimination of $\bar{u}$, $\bar{v}$, $\bar{w}$ from (3.6) and (3.7a, b, c) gives a single equation for $\bar{\chi}(r, z; s)$ in the form

$$\frac{\partial^2 \bar{\chi}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{\chi}}{\partial r} + \lambda^2 \frac{\partial^2 \bar{\chi}}{\partial z^2} = 0 \tag{3.8}$$

where

$$\lambda^2 = \left(\frac{s^2 + 4\Omega^2}{s^2 + N^2}\right) \tag{3.9}$$

Equation (3.8) has to be solved subject to the transformed boundary data

$$\frac{\partial \bar{\chi}}{\partial r} = 0 \quad \text{on} \quad r = a \tag{3.10}$$

$$\frac{\partial \bar{\chi}}{\partial z} = -\alpha\omega\left(\frac{s^2 + N^2}{s^2 + \omega^2}\right) f(r) \quad \text{at} \quad z = \pm l \tag{3.11}$$

In order to solve (3.8) with (3.10)–(3.11), we apply the finite Hankel transform of the second kind defined by the integral (Sneddon, 1972)

$$\tilde{\bar{\chi}}(k_m, z; s) = \int_0^a rJ_0(rk_m)\,\bar{\chi}(r, z; s)\, dr \tag{3.12}$$

where, by (3.10), k_m are the roots of the equation $J_1(ak_m) = 0$.

Thus the function $\tilde{\bar{\chi}}$ satisfies the differential system

$$\frac{d^2 \tilde{\bar{\chi}}}{dz^2} - \frac{k_m^2}{\lambda^2}\tilde{\bar{\chi}} = 0 \tag{3.13}$$

$$\frac{d\tilde{\bar{\chi}}}{dz} = -\alpha\omega\left(\frac{s^2 + N^2}{s^2 + \omega^2}\right)\tilde{f}(k_m), \quad \text{at} \quad z = \pm l \tag{3.14}$$

The solution for $\tilde{\bar{\chi}}$ is then given by

$$\tilde{\bar{\chi}} = -\frac{\alpha\omega\tilde{f}(k_m)\{(s^2+N^2)(s^2+4\Omega^2)\}^{1/2}}{k_m(s^2+\omega^2)} \frac{\sinh\left(\dfrac{zk_m}{\lambda}\right)}{\cosh\left(\dfrac{lk_m}{\lambda}\right)} \tag{3.15}$$

The inverse Laplace and Hankel transformations yield

$$\chi(r, z; t) = \frac{2}{a^2}\sum_{m=1}^{\infty}\frac{J_0(rk_m)}{J_0^2(ak_m)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \times e^{st}\tilde{\bar{\chi}}(k_m, z, s)\,ds,\ c>0 \tag{3.16}$$

$$= -\frac{2\alpha\omega}{a^2}\sum_{m=1}^{\infty}\frac{\tilde{f}(k_m)J_0(rk_m)}{k_m J_0^2(ak_m)}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \times\{(s^2+N^2)(s^2+4\Omega^2)\}^{1/2} \times\frac{\sinh\left(\dfrac{zk_m}{\lambda}\right)}{\cosh\left(\dfrac{lk_m}{\lambda}\right)}\frac{e^{st}ds}{(s^2+\omega^2)},\quad c>0 \tag{3.17}$$

where $\tilde{f}(k_m)$ is given by

$$\tilde{f}(k_m) = \int_0^a rJ_0(rk_m)f(r)\,dr \tag{3.18}$$

Making reference to Baines (1967), the asymptotic evaluation of integral (3.18) for large k_m with $f(a)=0$ gives

$$\tilde{f}(k_m) \sim \frac{af'(a)}{k_m^2}J_0(ak_m) + O(k_m^{-7/2}) \sim \left(\frac{2a}{\pi km^5}\right)^{1/2} f'(a)\cos\left(ak_m - \frac{\pi}{4}\right) + O(k_m^{-7/2}) \tag{3.19}$$

provided $f'(a) \neq 0$.

Thus result (3.17) represents the integral solution of the problem for an arbitrary function $f(r)$. However, it may be of special interest to take some particular form of $f(r)$ which has experimental relevance.

4. Solutions at various frequencies

The integral representation of the velocity fields are

$$(u, v) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{(-s, 2\Omega)}{(s^2+4\Omega^2)}\frac{\partial\bar{\chi}}{\partial r}e^{st}ds \tag{4.1 a, b}$$

$$w(r, z; t) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{s\,e^{st}}{(s^2+N^2)}\frac{\partial\bar{\chi}}{\partial z}ds, \tag{4.2}$$

where $\bar{\chi}$ is found from (3.15) by the inverse Hankel transform.

It follows that the structure of the flow pattern is determined by the nature and location of the singularities of the inversion integrals as well as the relative magnitude of the various frequencies associated with the problem. We next turn our attention to the evaluation of the inversion integral (3.17) by the theory of residues to obtain a closed form solution for $\chi(r, z; t)$. To facilitate the subsequent mathematical analysis without loss of generality, we assume that both ω and Ω are non-negative quantities.

The s-integrand of (3.17) has simple poles at $s = \pm i\omega$ and also at the sequence of points $s = \pm i\omega_{mn}$, for all integers n and all positive integers m, with the limit points at $s = \pm 2i\Omega$ for each m; where ω_{mn} are given by

$$\omega_{mn} = \left[\frac{4(2n+1)^2\pi^2\Omega^2 + 4N^2l^2k_m^2}{(2n+1)^2\pi^2 + 4l^2k_m^2}\right]^{1/2} \tag{4.3}$$

The points $\pm 2i\Omega$ must be essential singularities of the s-integrand of (3.17) with the property $|\omega_{mn}| <=> 2\Omega$ for all m and n according as $N <=> 2\Omega$. Thus ω_{mn} are all simple poles unless $\omega = \omega_{mn}$.

We shall now determine the explicit representation of the solution for $\chi(r, z; t)$ related to the following cases of interest:

Case I. $0 < \omega < \delta = \min(2\Omega, N)$

The evaluation of residues of (3.17) gives the solution for $\chi(r, z; t)$ in the form

$$\chi(r, z; t) = \sum_{m=1}^{\infty} B_m J_0(rk_m)\left[\{(N^2-\omega^2)(4\Omega^2-\omega^2)\}^{1/2} \times\frac{\sinh\delta_m z}{\cosh l\delta_m}\frac{\sin\omega t}{\omega} + 2\sum_{n=0}^{\infty} C_{mn}\sin\left\{(2n+1)\frac{\pi z}{2l}\right\}\sin\omega_{mn}t\right] \tag{4.4}$$

where

$$B_m = -\frac{2\omega\alpha\bar{f}(k_m)}{k_m a^2 J_0^2(ak_m)} \tag{4.5}$$

$$C_{mn} = \frac{(-1)^n (N^2-\omega_{mn}^2)(4\Omega^2-\omega_{mn}^2)^2}{l\omega_{mn} k_m(\omega^2-\omega_{mn}^2)(4\Omega^2-N^2)} \tag{4.6}$$

and

$$\delta_m = k_m \left(\frac{N^2-\omega^2}{4\Omega^2-\omega^2}\right)^{1/2} \tag{4.7}$$

The above solution renders valid regardless of $2\Omega < |\omega_{mn}| < N$ or $N < |\omega_{mn}| < 2\Omega$.

It is notable to mention that the first series of (4.4) corresponds to the steady-state solution (or the periodic solution oscillating with the forcing frequency ω). The second term of (4.4) containing the double series represents the normal inertial modes. We return to explain some more implications of these results in a subsequent section.

Case II. $\delta^* = \max(2\Omega, N) < \omega < \infty$

In this case, the forcing frequency of oscillations is greater than the larger of 2Ω or N and the corresponding solution for $\chi(r, z; t)$ is given by the same result (4.4).

Case III. $2\Omega < \omega < N$ *or* $N < \omega < 2\Omega$ *with* $\omega \neq \omega_{mn}$

In contrast to cases I and II, the solution for $\chi(r, z; t)$ has entirely different character and is given by

$$\chi(r,z;t) = \pm \sum_{m=1}^{\infty} B_m J_0(rk_m) \times \left[\{(N^2-\omega^2)(\omega^2-4\Omega^2)\}^{1/2} \frac{\sin\zeta_m z}{\sin\zeta_m l}\left(\frac{\sin\omega t}{\omega}\right) + 2\sum_{n=0}^{\infty} C_{mn} \sin\left\{(2n+1)\frac{\pi z}{2l}\right\}\sin\omega_{mn} t\right] \tag{4.8}$$

where ζ_m is given by

$$\zeta_m = k_m \left(\frac{N^2-\omega^2}{\omega^2-4\Omega^2}\right)^{1/2} \tag{4.9 a}$$

or

$$k_m \left(\frac{\omega^2-N^2}{4\Omega^2-\omega^2}\right)^{1/2} \tag{4.9 b}$$

according as $2\Omega < \omega < N$ or $N < \omega < 2\Omega$.

The above solution consists of the steady state and the normal inertial modes solutions. A discussion with physical implications of the solution will be included in a forthcoming section.

Case IV. $\omega = 2\Omega$ *or* N

In this case, the polar singularities at $s = \pm i\omega$ coalesce into the essential singularities at $\pm 2i\Omega$. Consequently, the solution can directly be evaluated by the theory of residues and has the representation

$$\chi(r,z;t) = 2\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}(-1)^n B_m C_{mn} J_0(rk_m) \times \sin\left\{(2n+1)\frac{\pi z}{2l}\right\}\sin\omega_{mn} t \tag{4.10}$$

It is somewhat surprising that there is no steady-state component in the solution. The double series constitutes the entire solution and represents a series of persistent inertial oscillations of frequency 2Ω or N. This persistent motion is likely to occur in the inviscid fluid unless some dissipation mechanism is included in the analysis.

Case V. $\omega = \omega_{mn}$

In the present case, the s-integrand of (3.17) has simple poles at $s = \pm i\omega_{mn}$, $n \neq N$ and double poles at $s = \pm i\omega$. The evaluation of the residues at the poles gives the solution for $\chi(r, z; t)$ as

$$\chi(r,z;t) = \sum_{m=1}^{\infty} B_m J_0(rk_m) \times \left[c_1 t\cos\omega t \sin(2N'+1)\frac{\pi z}{2l} + \left\{c_2 \frac{z}{l}\cos(2N'+1)\frac{\pi z}{2l} + c_3 \sin(2N'+1)\frac{\pi z}{2l}\right\}\sin\omega t + 2\sum_{\substack{n=0\\ n\neq N'}}^{\infty} C_{mn}\sin(2n+1)\frac{\pi z}{2l}\sin\omega_{mn} t\right] \tag{4.11}$$

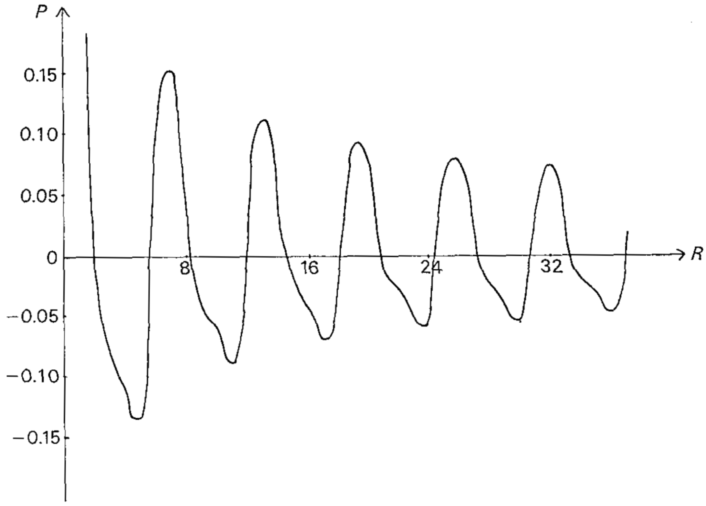

Fig. 1. $T = \pi^3$.

where c_1, c_2 and c_3 are known numerical constant found from the residue calculations.

This solution consists of three terms—the first one involving an infinite series has an amplitude $0(t)$. The second term containing a series represents the period solution oscillating with frequency ω. The third term consists of a double series and describes the persistent inertial oscillations.

5. Structure of the flow field and conclusions

With the aid of the convolution theorem for the Laplace transform, the velocity distributions for all cases except case IV are obtained in the form

$$u = \int_0^t [-\cos 2\Omega(t-\tau)] \left(\frac{\partial \chi}{\partial r}\right) d\tau, \qquad (5.1\,\mathrm{a})$$

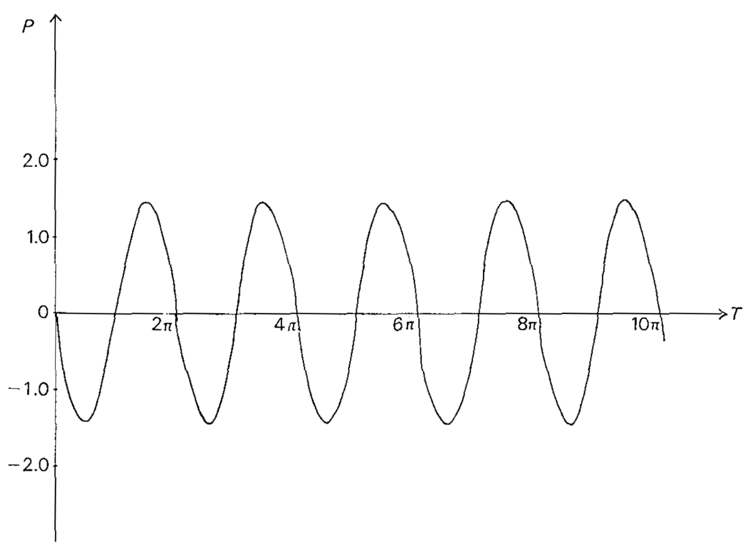

Fig. 2. $R = \pi/10$.

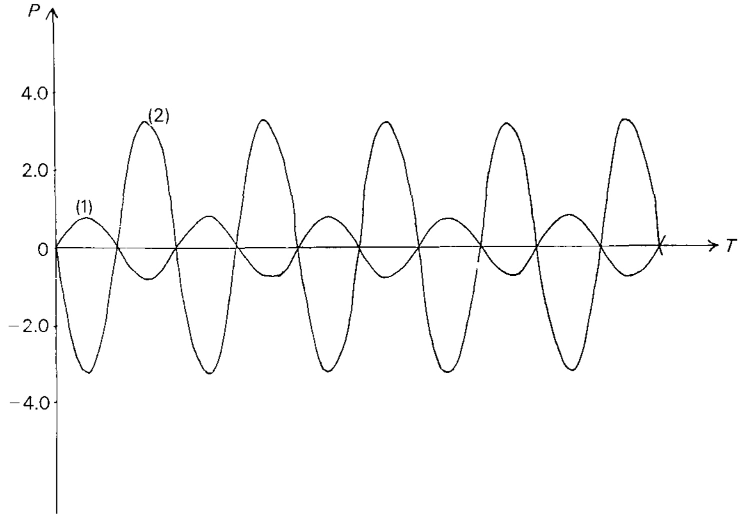

Fig. 3. (1) $\theta = \pi^2$, $Z/c = \pi^2$; (2) $\theta = \pi^2$, $Z/c = \pi/5$.

$$v = \int_0^t [\sin 2\Omega(t-\tau)] \left(\frac{\partial \chi}{\partial r}\right) d\tau, \tag{5.1 b}$$

$$w = -\int_0^t \cos(t-\tau) N \left(\frac{\partial \chi}{\partial z}\right) d\tau \tag{5.2}$$

The solution for the disturbance pressure $\chi(r, z; t)$ (and so the velocity field) consists of the steady-state component and a series of normal inertial modes. The former represents the periodic solution oscillating with the forcing frequency ω. Each component of the normal inertial modes is spatially periodic in z and the frequencies of these modes modified to include the effects of the density stratification and rotation form a discrete spectrum dense in [min $(2\Omega, N)$, max $(2\Omega, N)$]. The inertial modes may be regarded as a manifestation of the prescribed initial conditions. These modes are usually found to be absent in the solution of the steady-state problem. It is also of interest to note that the effects of the density stratification are reflected on the solution.

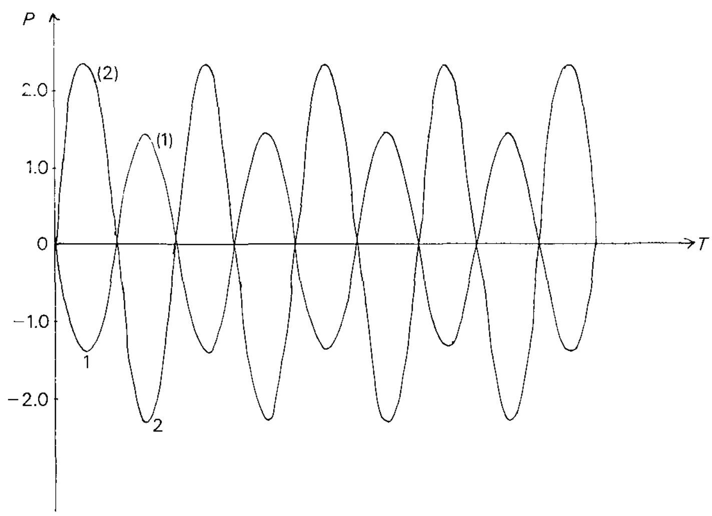

Fig. 4. (1) $\theta = 1$, $Z/c = \pi^2$; (2) $\theta = 1$, $Z/c = \pi/5$.

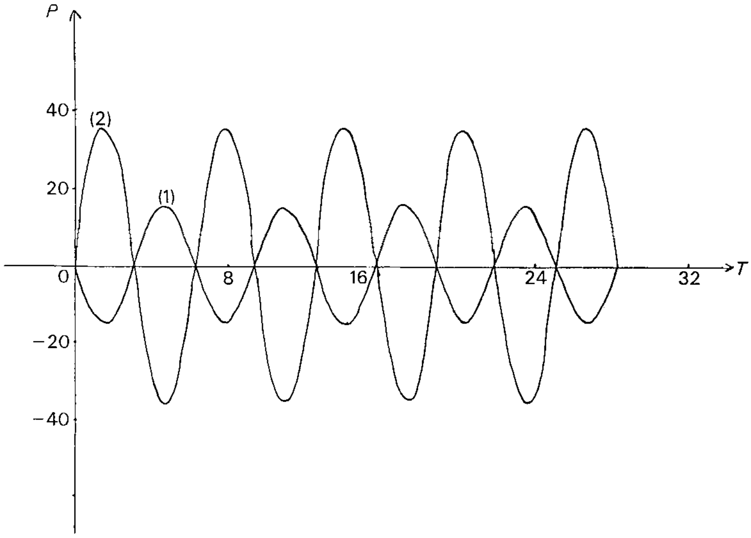

Fig. 5. (1) $\theta = \pi/6$, $Z/c = \pi^2$; (2) $\theta = \pi/6$, $Z/c = \pi/5$.

Apart from the inertial oscillations, the solution for $\chi(r, z; t)$ as well as the flow field related to Case I and II are somewhat similar to that of the irrotational surface waves.

The solution for case III is of special interest and has singificantly different properties. By virtue of (3.19), the steady-state solution for this case has the form

$$\chi_{st} \sim \frac{A \sin \omega t}{\sqrt{r}} \sum_{m=1}^{\infty} \times \frac{(-1)^m \cos\left(rk_m - \frac{\pi}{4}\right) \sin\left(\frac{\theta a z k_m}{l}\right)}{m^3 \cos(a\theta k_m)} \quad (5.3)$$

where A is a constant and θ is equal to

$$\theta = \frac{l}{a}\left(\frac{N^2 - \omega^2}{\omega^2 - 4\Omega^2}\right)^{1/2} \quad (5.4\,a)$$

or

$$\frac{l}{a}\left(\frac{\omega^2 - N^2}{4\Omega^2 - \omega^2}\right)^{1/2} \quad (5.4\,b)$$

according as $2\Omega < \omega < N$ or $N < \omega < 2\Omega$.

According to Wood (1966) and Baines' (1967) analyses, discontinuities may occur in the first and the seond derivatives of the steady-state solution χ_{st}, and these discontinuities are associated with the rational values of θ. Both of these authors discussed the discontinuities of the pressure and the velocity field in relation to the nature of θ. Reference may be made to Wood and Baines' works for the details.

When the forcing frequency of oscillations is equal to the Brunt-Vaisala frequency or twice the angular velocity of rotation, the pressure field (4.10) does not contain any steady-state term and is made up of the inertial modes only. Although this conclusion is quite different from that usually made in all situations concerning the forced oscillations in a rotating non-stratified fluid in which case the solution is found to be singular at the resonant frequency, but the flow field has a steady-state component oscillating with frequency 2Ω in addition to the persistent inertial oscillations.

In case V where the forcing frequency is equal to one of the resonant frequencies, one term of the corresponding solution (4.11) has an amplitude $0(t)$. Hence it oscillates with an unbounded amplitude as $t \to \infty$. Ultimately, the flow appears to be unstable unless viscous or non-linear processes are included.

Although the present analysis has been entirely inviscid, yet some saliant features of the rotating stratified flow have been ascertained. The inclusion of viscous effects would cer-

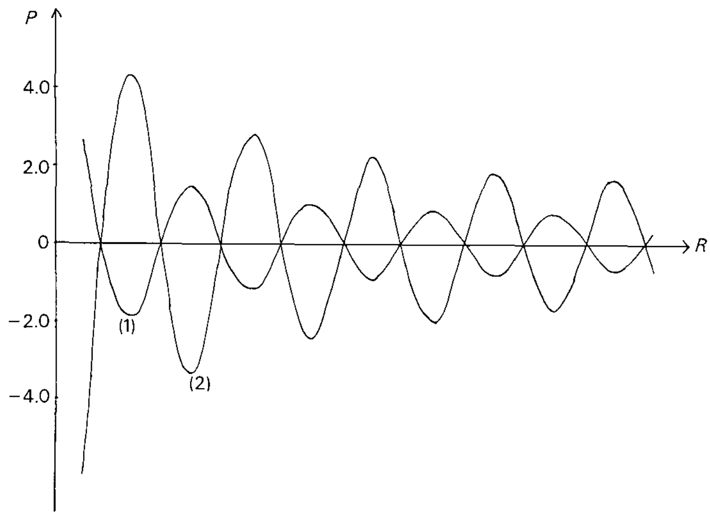

Fig. 6. (1) $\theta = \pi/6$, $Z/c = \pi^2$; (2) $\theta = \pi/6$, $Z/c = \pi/5$.

tainly make the study physically more realistic and would apparently cause the persistent inertial oscillations to decay. Furthermore, the growth of the large amplitude encountered in case V might also be controlled by viscosity. In this connection, it may be relevant to mention an excellent work of Greenspan (1964, 1965) on the time dependent problems in a viscous rotating fluid enclosed in an arbitrary axisymmetric container. His analysis suggests that the inertial modes decay exponentially in the uniform time scale $t = O(E\Omega^2)^{\frac{1}{2}}$ where $E = \nu/\Omega l^2$ is the Ekman number. Wood (1966) has also investigated certain aspects of the effects of viscosity on the forced motion and found the existence of shear layers of thickness $O(E^{1/3})$ which originated due to viscosity and replace the surfaces of discontinuity in the velocity gradients.

Without entering into the analytical details, some qualitative assessment of the possible effects of viscosity on the present linearized motion can similarly be made. It is very likely that the steady-state as well as the inertial

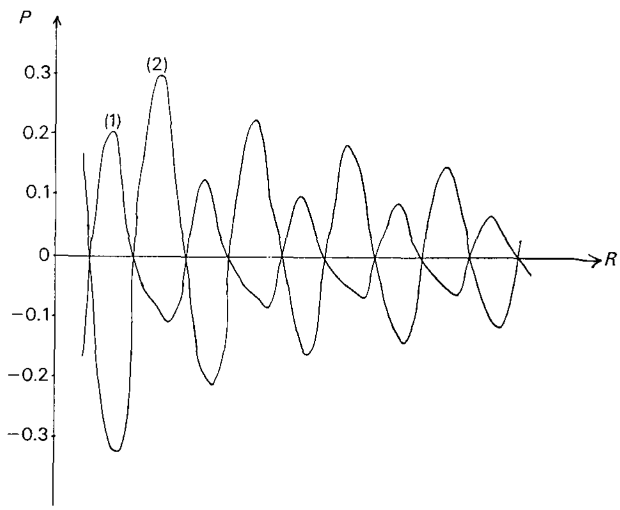

Fig. 7. (1) $\theta = \pi^2$, $Z/c = \pi^2$; (2) $\theta = \pi^2$, $Z/c = \pi/5$.

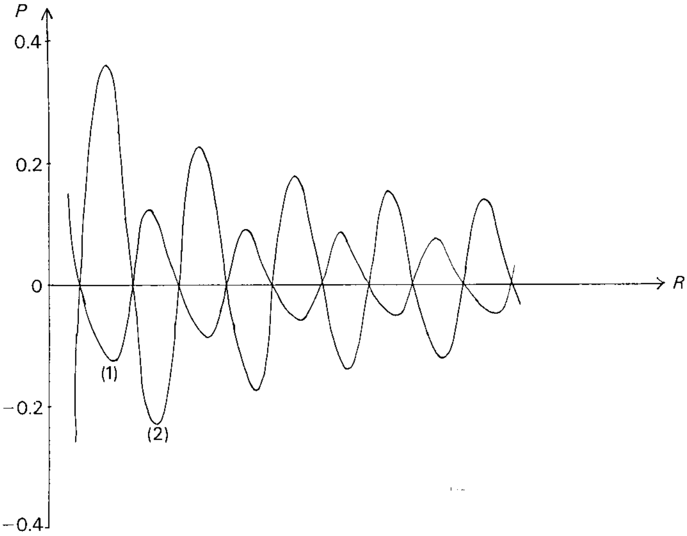

Fig. 8. (1) $\theta = 1$, $Z/c = \pi^2$; (2) $\theta = 1$, $Z/c = \pi/5$.

modes solution will independently be affected by viscosity. It may be presumed that the inertial modes will ultimately die out due to the action of viscosity.

Finally, in the absence of the density stratification, Baines' analysis emerges as a limiting case of our analysis.

6. Numerical results

To illustrate the solution by diagrams, we introduce the non-dimensional flow variables R, Z, T and P by $(R, Z) = 1/a(r, z)$, $T = \omega t$ and $P = (a/A)\chi_{st}$. In terms of these dimensionless quantities, solution (5.3) can be written as

$$\chi_{st} = \frac{A \sin T}{\sqrt{aR}} \left[\frac{\cos\left(aRk_2 - \frac{\pi}{4}\right) \sin(ak_2 bZ)}{8 \cos(ak_2)} - \frac{\cos\left(ak_1 R - \frac{\pi}{4}\right) \sin(ak_1 bZ)}{\cos(ak_1)} + \dots \right] \tag{6.1}$$

where

$$\frac{a\theta}{l} = \left(\frac{b^2 - 1}{1 - c^2}\right)^{1/2} \sim b$$

with

$$b^2 = \frac{N^2}{\omega^2} \gg 1$$

and

$$c^2 = \frac{4\Omega^2}{\omega^2} \ll 1.$$

When

$$\theta = \frac{l}{c}\left(\frac{1 - b^2}{c^2 - 1}\right)^{1/2} \sim \frac{l}{ac}$$

with $b^2 \ll 1$ and $c^2 \gg 1$, result (5.3) takes the form

$$\chi_{st} = \frac{\sqrt{\pi} \sin T}{\sqrt{\pi R}} \left[\frac{\cos\left(aRk_2 - \frac{\pi}{4}\right) \sin\left(ak_2 \frac{Z}{c}\right)}{8 \cos(ak_2 \theta)} - \frac{\cos\left(ak_1 R - \frac{\pi}{4}\right) \sin\left(\frac{ak_1 Z}{c}\right)}{\cos(ak_1 \theta)} \dots \right], \tag{6.2}$$

Setting $ak_1 \sim \pi$, $ak_2 \sim 2\pi$ and $bZ = \pi^2$, the function P from (6.1) is plotted against R or T and exhibited in Figs. 1–2.

With the same value for ak_1 and ak_2, the result P obtained from (6.2) is drawn against T in Figs. 3–5 for a fixed $R = \pi/10$ and different value of θ and Z/c.

The function $P = P(R)$ from (6.2) is shown in Figs. 6–8 for fixed $T = \pi^3$, $ka_1 \sim \pi$, $ka_2 \sim 2\pi$ and various values of θ and Z/c.

7. Concluding remarks

The primary purpose of this study has been to develop a theory of a special flow configuration which might be modeled in the laboratory. An experimental verification of the theory would be far more valuable, and of more interest. Such an experimental study will be made and published in a subsequent paper.

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О ВЫНУЖДЕННЫХ КОЛЕБАНИЯХ ВО ВРАЩАЮЩЕЙСЯ СТРАТИФИЦИРОВАННОЙ ЖИДКОСТИ

Сделана попытка изучить вынужденные осесимметричные колебания невязкой вращающейся стратифицированной жидкости, ограниченной конечным круговым цилиндром. Нестационарное течение генерируется в жидкости гармоническими колебаниями плоских торцов цилиндра. Основные черты течения определяются относительными величинами угловой скорости вращения Ω, вынуждающей частотой ω наложенных колебаний и частотой Брента–Вяйсяля N. Показано, что решение для возмущений давления (и, следовательно, поля скорости) состоит из двух компонент — периодического решения, гармонически осциллирующего с частотой ω и нормальных инерционных мод, каждая из которых пространственно периодична в направлении оси вращения. Последние состоят из двойного бесконечного ряда мод с частотами, образующими дискретный спектр в интервале [min $(2\Omega, N)$, max $(2\Omega, N)$]. Оба эти решения независимо модифицируются стратификацией плотности. Найдено, что решение имеет резонансное поведение, когда вынуждающая частота равна одной из резонансных частот ω_{mn}. В противоположность сингулярному поведению решения при $\omega = 2\Omega$, которое обычно имеет место при вынужденном движении вращающейся жидкости, настоящие решения для этого критического случая $\omega = 2\Omega$ и случая $\omega = \Omega$ представляют только инерционные моды. Сделана качественная оценка возможных эффектов вязкости на течение. Проанализирован ряд интересных особенностей течений вращающейся стратифицированной жидкости. Задача с начальными данными решена с использованием преобразований Лапласа и Ханкеля второго рода.