Channel flow driven by a stationary thermal source

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ABSTRACT

It is shown by analytic means that it is possible to produce mean horizontal flows in a channel from a periodic distribution of stationary heat sources which are placed upon a varying topographic structure. Past analyses and experiments of channel flows are reviewed. The conditions necessary for production of mean horizontal flows are derived using a number of boundary conditions. The heat balance and momentum balance of this fluid system are discussed.

Introduction

The idea that a heat source is capable of generating mean horizontal flow in a region of fluid when applied to a boundary of a given shape comes from the realization that Reynolds stresses can supply energy and momentum from spatially fluctuating flows to the mean flow. Such a transport of energy from spatially fluctuating to mean flows has been observed in the general circulation pattern of the atmosphere, as well as in other naturally occurring systems.

Until recently, all the examples of transfer of energy from fluctuating to mean flows which have been discussed in the literature have involved some element of motion within the boundary conditions. These boundary conditions have provided the region of fluid with a certain orientation or specified direction of motion through such agents as rotation of the fluid boundaries or the movement of the heat source. Nevertheless, there is no *a priori* reason why a system cannot maintain a net horizontal flow, even if it does not have a traveling thermal source and its boundaries are stationary. Such a flow could be maintained by appropriate placement of stationary heat sources. It would be necessary, however, to place some orientation upon the system as to a preferred direction, and this orientation could be provided by the introduction of suitably shaped topographic features as boundaries, upon which would be placed stationary sources of constant temperature. It would be desirable for these features to be able to maintain a streamline pattern with a net zonal component. This pattern could be maintained only by the presence of tilted troughs, which would transfer momentum from one horizontal level to another.

In order to understand better the mean flow processes, one would like to analyze a flow in two dimensions only. Therefore, the fluid considered will be confined to a channel which is independent of the lateral y-direction. The flow will be dependent only upon the length down the channel (x-direction) and the channel height (z-direction). Furthermore, since both the bottom topography and the bottom temperature boundary condition are periodic in the x-direction with wavelength R, it will be assumed that the resulting flow has the same periodic structure.

A steady-state zonal flow can exist in a fluid system such as this if the temperature field, together with appropriate boundary conditions and structures provide the correct type of flow patterns so as to maintain the zonal momentum against dissipation. It is therefore the purpose of this work to obtain conditions necessary for such a flow, in terms of correct interaction between the thermal boundary conditions and the topographic structure.

A method will be used which turns out to be a weak non-linear one. The aspect ratio, which is the ratio of the height of the fluid to the wavelength, will be considered a small number. As a consequence of this assumption, the non-linear terms, although present, are taken as smaller in order of magnitude than the linear terms. Some of these non-linear terms have a non-zero zonal average and act as driving terms of the next order flow. Their presence therefore provides an orientation to the fluid and can provide for non-zero zonally averaged flows.

Mean channel flows can be maintained when momentum transport is balanced between the spatially fluctuating and mean flows. Both the Reynolds stress and the stress due to the vertical gradient of horizontally averaged velocity must be in balance in order to maintain a steady state.

The presence of a shear of horizontally averaged velocity $\partial \bar{u}/\partial z$ transports momentum down the gradient at the rate $-\varrho \nu(\partial \bar{u}/\partial z)$ while the Reynolds stress transports momentum by a correlation in horizontal and vertical flows at a rate $\varrho \overline{uw}$. This correlation means that tilted streamline patterns occur. If the flow is closed then tilted cells will exist, but if the flow is not closed there will be tilted troughs.

The situation of alternating cold and heat sources on the side of topographic "hills" will be demonstrated to produce a mean zonal flow. When the streamline pattern is such that the fluid has a u–w correlation, as will be seen in Fig. 8, then this Reynolds stress necessarily transports momentum upward, this being due to the tilted streamlines. In the steady state, this then must be balanced by a vertical gradient in the zonally averaged horizontal flow, and in this manner mean flows can occur.

Past experiments

In recent years, much work has been done to investigate fluid motions due to a traveling source of heat. The rationale behind most of these investigations was the development of crude, but at least qualitatively correct, models of the motions of certain planetary atmospheres. The generation of mean horizontal fluid motions has been studied experimentally using laboratory models. In addition, there have been attempts at mathematical analyses and numerical models of such systems.

Fultz et al. (1959) describe an experiment in which they rotated a flame underneath the bottom rim of a cylindrical container which was filled with water. The basic result was that, with the fluid starting from rest, a net angular momentum was obtained in the opposite direction as the motion of the flame. Later Stern (1959) performed a similar experiment, this time using a cylindrical annulus, with a small ratio of width to height of water. By doing this, he tried to eliminate radial currents which were considerable in the experiments of Fultz. A fluid motion opposite to that of the flame was observed by Stern by the use of appropriate markers in the fluid.

Stern's analysis was considered in a rigid-boundary annulus and a moving temperature distribution in the fluid. Davey (1967) conducted more computations on a different set-up including the effects of a free upper surface as a boundary.

The next experiments came when Schubert & Whitehead (1969) tried to revolve a heat source under the rim of an annulus which this time contained mercury instead of water. The results in this study were much magnified over those of previous studies. Mean flow motions as much as four times as fast as the flame speed arose, while in the previous experiments using water, a counterflow of only as much as 1 % of the flame speed was observed. Using mercury much enhanced the thermal effect because of its high conductivity and relatively low viscosity.

Douglas et al. (1972) performed experiments in an annulus using water in which was dissolved an electrolyte. This solution was heated by passing an electric current across the fluid. The two electrodes were rotated around the annulus by means of a slip ring, thus heating the fluid internally. Again there was a mean flow counter to the flame direction. This method was significantly more effective in producing the mean current than was the moving heater because of the larger amounts of internal heat which could be absorbed by the water. However, when they stopped the motion of the flame, they invariably found that mean motion would always disappear, regardless of the initial flows.

In all these experiments, the zonal motion of the fluid was discovered to be in the opposite direction as that of the flame. An order of magnitude estimate of the results of the mercury experiment is as follows. For imposed flame speeds of near 10 mm/sec, and for a temperature difference ranging from 8.5°C to 62°C, steady state zonal flow was found to be between 0.01 and 4.5 times the flame speed and was achieved after about five minutes.

Past mathematical analyses

Many of the small-scale models have been used to simulate general features of planetary flow regimes. While the earlier annulus experiments attempted to illustrate many terrestrial flow phenomena, some of the more recent experimental and mathematical models have been used to simulate some features of the circulation of the planet Venus. Venus is known to have a circulation pattern which is different in many respects from those of other planets.

Schubert & Whitehead were first to discuss the moving flame experiment as a model for the zonal motions on Venus. This planet rotates rather slowly with a period of 243 days. The sun's being above a slowly rotating planet has been viewed as a periodic thermal source overhead, which, although small, is still not stationary. On Venus, the upper atmosphere has been observed to move around the planet in four days and to have speeds of 300 km/hour relative to the planet's surface. The atmosphere of Venus has a gas (CO_2) which absorbs radiation in the near infrared, and so the heat is absorbed throughout the atmosphere. One reason why the mercury experiment was compared to the Venus zonal circulation was primarily that it was the first experiment which, from a moving thermal source, could produce horizontal motions which were faster than the traveling source.

Both Malkus (1970) and Thompson (1970) developed models to provide explanations for the zonal flows of systems similar to the Venus atmosphere based upon slowly traveling and quasi-stationary heat sources.

Malkus tried to extend the linear solution for a moving flame problem to the non-linear regime. After discussing the linear problem with a boundary condition of a moving heat source S (as opposed to a moving temperature source), he obtained flows which were dependent upon the speed of the heat source. Furthermore, Malkus analyzed his mean flow results in the limiting cases of both high and low speeds.

For the present purposes, the low speed limit which Malkus obtained is of primary interest. Malkus got non-vanishing values for the mean flow at the top of the model (a rigid slippery top surface)

$$\bar{u} = \frac{4}{6!}\left(\frac{11\,\mathrm{Pr}+91}{175}\right)\left(\frac{g\bar{\alpha}S}{k}\right)^2\frac{(kh)^8}{(\nu k)^4}\,\frac{1}{U}\times[1+o(U^2)+\ldots] \tag{1}$$

where Pr is the Prandtl number, h is the height, $\bar{\alpha}$ is the coefficient of thermal expansion, k is the wavelength of the disturbance, S is the amplitude of the heat source and $U=\omega/k$ is its speed. However, this low speed limit result is only valid for conditions where the speed is still large enough so that the wavelength is much larger than the boundary layer.

However, in the more extreme case where the boundary layer is of greater magnitude than the wavelength, i.e., the whole flow is within a boundary layer, Malkus obtained a vanishing value for $\bar{u}_{\mathrm{top}}$. This is understandable because without applied thermal motion or topography there can be no orientation to the flow. Malkus' results can then be interpreted to mean that a thermal source on a planet will generate a mean horizontal flow if the heat source and the lower rigid boundary have a relative velocity in excess of the viscous diffusion velocity. Malkus further suggests that a mean horizontal velocity will occur even if no such differential motion exists as a finite-amplitude instability.

Malkus used a non-linear approximation known as the "Oseen approximation" to obtain a further solution to the mean field equations. This method is actually the first iteration of a non-linear approach, which consists of modifying the linear problem by the addition of a non-linear advection velocity at the top surface of the fluid. As a solution, upon consideration of $\bar{u}_{\mathrm{top}}$ to be much larger than the imposed source speed, U, the solution for this top zonal flow becomes

$$\bar{u} = \left(\frac{g\bar{\alpha}S}{k}\right)^{1/3}\left(\frac{1}{\mathrm{Pr}}\right)^{1/6}\left[\frac{1}{4}+\frac{\mathrm{Pr}}{2(\mathrm{Pr}^{1/2}+1)^2(\mathrm{Pr}+1)}\right]^{1/6} \tag{2}$$

which is just a function of the heat source strength, the wavelength, and of the Prandtl number, and is independent of the source speed. In his estimation of the zonal wind speed of Venus, Malkus obtained from various parameters $\bar{u}=4.2\times10^3\pm100\%$ cm sec^{-1} which overlaps reported results of the movement of clouds.

Thompson took a different approach in the analysis of the zonal flow problem. He used a numerical model of flow based upon the heating throughout the fluid with a solid boundary and observed that convective flows would commence. At a certain point in time, the fluid was subjected to a perturbation of small but finite amplitude zonal flow. The perturbation contained a zonal velocity which tilted the convection cells and thus caused the transference of momentum via a velocity correlation $\overline{u'w'}$ into the mean flow itself. By this method, the kinetic energy of the mean flow began to increase. It seems that there was some kind of natural, almost steady state solution obtained of small oscillations about the mean. At a certain value of shear, however, an instability set in, which tilted the cells in the other direction. At this value, apparently the Reynolds number was too high to maintain a stable flow.

In the analysis of his moving flame in mercury experiment, Whitehead (1972) used a non-linear method in order to determine what types of flows would occur as direct results of both buoyancy and surface tension differences. He came to the conclusion that mean driven flows were not to be found until order a^4, where a is the aspect ratio. Furthermore, he obtained the fact that the surface tension driven flows were more effective in the generation of mean flows from the moving heater than are flows generated by buoyancy.

Young et al. (1972) analyzed by numerical techniques the moving flame experiments of fluid regions having flat surfaces and they were quite adequately able to simulate the tilted cells which maintain the mean flow. They experimented with several parameters, including the ratio of buoyancy to inertial forces, the ratio of viscous diffusion time across the fluid layer to the period of the wave, and the Prandtl number. They were able to develop steady state motions, and among them the expected counter-flame mean flows. In addition, they developed by numerical methods mean flows by non-symmetric placement of stationary heat sources. However, their equations were scaled with the values for a moving flame speed. It is not obvious that this method truly corresponds to the situation for a stationary heat source. Malkus did make the distinction between the "low-speed" limit of the moving source, and the solution for a truly stationary source.

Young et al. conclude that it must be the non-linear terms which are causing the mean flow in this stationary state. In the case of the moving thermal source, it is the thermal diffusion which creates tilted convection cells because it takes time for the information that the thermal source has moved to travel up the fluid. Such tilted cells are then responsible for the mean fluid motions. Nevertheless, neither Whitehead nor Young et al. considered the effects of bottom topography in their studies. A method similar to Whitehead's will be used in this study in order to determine the flows due to the effects of spatially varying temperature structures and topographic features.

Mathematical scheme

In order to formulate the problem of the fluid subjected to stationary heat sources over a topographic perturbation structure, the Boussinesq approximations to the conservation of mass, momentum and energy will be used:

$$\nabla \cdot \mathbf{u} = 0 \tag{3}$$

$$\partial \mathbf{u}/\partial t + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\varrho} \nabla p + \nu \nabla^2 \mathbf{u} + g\bar{\alpha} T \hat{\mathbf{k}} \tag{4}$$

$$\partial T/\partial t + \mathbf{u} \cdot \nabla T = \varkappa \nabla^2 T \tag{5}$$

where $\mathbf{u}$ is the two-dimensional velocity (in the x- and z-directions), p is the deviation from hydrostatic pressure, T is the temperature, and ν, $\varkappa$, g, $\bar{\alpha}$ are kinematic viscosity, thermal diffusivity, force of gravity, and the volumetric coefficient of expansion, respectively.

The fluid is confined to a region whose top surface is flat and whose bottom surface has an arbitrary periodic structure. The equation for the top surface of the fluid will be $z = 0$, and that of the bottom surface is $z = z_B \equiv [-1 + \alpha\gamma(x)]L$. Here α is a small constant parameter and is a function of x of order one. In this way, the height of the topographic ridges will be small compared with the height of the tank itself.

It will be assumed that a certain temperature will be imposed at the bottom boundary z_B, although using a source of heat is also a reasonable alternative. The temperature at z_B is $T(z_B) = \theta(x) \cdot \Delta T$ where ΔT is a characteristic temperature difference and $\theta(x)$ is a function of order one.

Following Whitehead, the scheme for non-dimensionalization of the above equations is the use of the transforms:

$$u_d = \frac{\nu}{R}\frac{\partial \psi_n}{\partial z_n} \qquad w_d = -\frac{\nu L}{R^2}\frac{\partial \psi_n}{\partial x_n}$$

$$t_d = \frac{R^2}{\nu} t_n \qquad T_d = \Delta T \cdot T_n$$

$$x_d = R x_n \qquad z_d = L z_n$$

where L and R are the height of the tank (not including the topographic perturbation), and the wavelength of the perturbation, respectively, d and n stand for dimensional and non-dimensional values, and u, w, and t are the horizontal and vertical velocities and the time. When these transforms are applied to eqs. (4)–(5) and the symbol n is deleted, the following non-dimensional equations are obtained:

$$\left[a^2 \frac{\partial}{\partial t} - \nabla_1^2\right] \nabla_1^2 \psi + G\frac{\partial T}{\partial x} = a^2 J(\psi, \nabla_1^2 \psi) \tag{6}$$

$$a^2 \Pr \left[\frac{\partial T}{\partial t} - J(\psi, T)\right] = \nabla_1^2 T \tag{7}$$

where

$$\nabla_1^2 \equiv a^2 \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}, \qquad a = L/R$$

$$G = g\bar{\alpha} L^3 \Delta T/\nu^2, \qquad \Pr = \nu/\varkappa$$

The bulk of analysis of this study will be in the determination of steady-state solutions to the set of eqs. (6)–(7). Therefore, from the results, it will be seen what kinds of non-varying oriented flows can result.

The procedure used to solve the above equations subject to boundary conditions which soon will be examined, is to expand the non-dimensional stream-function ψ and temperature T in parameters a^2, the square of the aspect ratio, and in α, the topographic ratio, as follows:

$$\psi = (\psi_0^{(0)} + \alpha\psi_0^{(1)} + \ldots) + a^2(\psi_1^{(0)} + \alpha\psi_1^{(1)} + \ldots) + \ldots \tag{8}$$

$$T = (T_0^{(0)} + \alpha T_0^{(1)} + \ldots) + a^2(T_1^{(0)} + \alpha T_1^{(1)} + \ldots) + \ldots \tag{9}$$

Also, the following notation will be used to represent the terms in parentheses of the above expansions:

$$\psi_0 = \psi_0^{(0)} + \alpha\psi_0^{(1)} + \ldots \qquad \psi_1 = \psi_1^{(0)} + \alpha\psi_1^{(1)} + \ldots \tag{10}$$

$$T_0 = T_0^{(0)} + \alpha T_0^{(1)} + \ldots \qquad T_1 = T_1^{(0)} + \alpha T_1^{(1)} + \ldots \tag{11}$$

If ψ is considered to be $O(1)$, then the smallness of the aspect ratio leads to the subordination of the non-linear terms in magnitude to the linear terms of the eqs. (6) and (7), provided that the Prandtl number is not too large.

In this study three separate cases will be considered, each with its own set of boundary conditions. These cases are: (I) fluid with a closed top boundary and hence a no-slip condition at the top and bottom; (II) fluid with an upper free surface which, however, can conduct heat to the air above it; and (III) fluid with an upper free surface which conducts no heat to the air.

The following notation will be used in order to average a quantity A over the wavelength R:

$$\bar{A}_d = \frac{1}{R}\int_0^R A\,dx_d \qquad A_n = \int_0^{2\pi} A\,dx_n \tag{12}$$

The periodicity enables us to define a mean or zonal stream function and hence a velocity as follows:

$$\bar{\psi} = \int_0^{2\pi} \psi\,dx \qquad \bar{u} = \int_0^{2\pi} u\,dx \tag{13}$$

Because the imposed temperature $\theta(x)$ and the imposed topography $\gamma(x)$ are given as analytic functions, the following analysis will also yield the stream-function and temperature fields as analytic functions. The above method of zonal averaging can thus be applied at all horizontal levels. Near the bottom boundary, the averaged quantities $\bar{\psi}$ and $\bar{u}$ are defined for the analytic functions of ψ and u so obtained from the mathematical solution, even if some part of the horizontal level is not enclosed within the fluid volume. This method can be used for the reason that the non-dimensional parameter α is a small quantity. Therefore the average integrals are within the fluid volume for almost all of the height levels (for the fraction $1-\alpha$ of the height levels), and thus, at the last step, the error will be small.

Case I

For the case of the rigid surface boundaries, we adopt the no-slip condition at $z=0$ and $z=z_B$. At z_B, the no-slip condition says that both the normal and tangential components of $\nabla\psi$ are zero. Since the x- and z-components of $\nabla\psi$ are just linear combinations of the normal and tangential components, they can be set to zero:

$$\frac{\partial\psi}{\partial x}=\frac{\partial\psi}{\partial z}=0 \quad \text{at } z=0 \text{ and } z_B. \tag{14}$$

The imposed zero order non-dimensional temperature field is $T_0=\theta(x)$ at $z=z_B$ and $T_0=0$ at $z=0$. The first and higher order temperature fields will have zero boundary conditions:

$$T_1=T_2=\ldots=0 \text{ at both } z=0 \text{ and } z_B. \tag{15}$$

The lowest order equations in the zeroth power of a^2 are as follows:

$$\frac{\partial^2 T_0}{\partial z^2}=0 \tag{16}$$

$$-\frac{\partial^4\psi_0}{\partial z^4}+G\frac{\partial T_0}{\partial x}=0 \tag{17}$$

Using the temperature boundary conditions (14) and (15), eq. (16) yields a solution for the temperature:

$$T_0=\frac{-\theta(x)\cdot z}{1-\alpha\gamma(x)}=-\theta(x)z[1+\alpha\gamma+\ldots]$$
$$\approx-\theta(x)z[1+\alpha\gamma] \tag{18}$$

So we come to the conclusion that $T_0^{(0)}=-\theta(x)z$ and $T_0^{(1)}=-\theta\gamma z$.

Setting (18) into eq. (17), we obtain

$$\frac{\partial^4\psi_0}{\partial z^4}=G\frac{\partial T_0}{\partial x}=z\{-G[\theta'(1+\alpha\gamma)+\alpha\theta\gamma']\} \tag{19}$$

Here a prime denotes a derivative with respect to x, e.g. $\theta'\equiv\partial\theta/\partial x$, etc. Solving eq. (19) we obtain

$$\psi_0=(z^5+Az^3+Bz^2+Cz+D)$$
$$\times\left\{-\frac{G}{5!}[\theta'(1+\alpha\gamma)+\alpha\theta\gamma']\right\} \tag{20}$$

where A, B, C, and D are constants yet to be determined by the boundary conditions. The conditions at $z=0$ imply $C=D=0$, and those at z_B yield the following two relations:

$$-(1-\alpha\gamma)^5-A(1-\alpha\gamma)^3+B(1-\alpha\gamma)^2=0 \tag{21}$$

$$5(1-\alpha\gamma)^4+3A(1-\alpha\gamma)^2-2B(1-\alpha\gamma)=0 \tag{22}$$

Expanding the coefficients in the parameter α, we obtain values of

$$A=-3+6\alpha\gamma \tag{23}$$

$$B=-2+6\alpha\gamma \tag{24}$$

and therefore, expanding only up to the first power of α:

$$\psi_0=[z^5+(-3+6\alpha\gamma)z^3+(-2+6\alpha\gamma)z^2]$$
$$\times\left\{-\frac{G}{5!}[\theta'(1+\alpha\gamma)+\alpha\theta\gamma']\right\} \tag{25}$$

The terms in the zero power of a^2 are thus divided in the following manner:

$$\psi_0^{(0)}=(z^5-3z^3-2z^2)\left\{-\frac{G}{5!}\theta'\right\} \tag{26}$$

$$\psi_0^{(1)}=[(z^5-3z^3-2z^2)\theta\gamma'+(z^5+3z^3+4z^2)\theta'\gamma]$$
$$\times\left\{-\frac{G}{5!}\theta'\right\} \tag{27}$$

Both terms of the stream-function in the zero power of a^2 evaluated at 0 and -1 have the value zero. Therefore, there can be no net flow across any vertical $y-z$ plane, and thus a net mean flow to order a^0 cannot exist.

With the first order equations in a^2, the first non-linear terms appear. The vorticity and heat equations in a^2 are

$$\frac{\partial}{\partial t}\frac{\partial^2\psi_0}{\partial z^2}-\frac{\partial^4\psi_1}{\partial z^4}-2\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial z^2}\psi_0+G\frac{\partial T_1}{\partial x}=J\left(\psi_0,\frac{\partial^2\psi_0}{\partial z^2}\right) \tag{28}$$

$$\mathrm{Pr}\left[\frac{\partial}{\partial t}T_0-J(\psi_0,T_0)\right]=\frac{\partial^2}{\partial x^2}T_0+\frac{\partial^2}{\partial z^2}T_1 \tag{29}$$

In order to investigate the steady state mean flow in one wavelength of x, the vorticity equation (28) will be integrated over the wavelength

$$\frac{\partial^4 \bar{\psi}_1}{\partial z^4} = -2\overline{\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial z^2}\psi_0} + G\overline{\frac{\partial T_1}{\partial x}} - \overline{J\left(\psi_0, \frac{\partial^2 \psi_0}{\partial z^2}\right)} \qquad (30)$$

Now the first two quantities on the right hand side are zero since the quantities to be integrated are derivatives of functions which are periodic in the wavelength. Breaking up the vorticity equation into powers of α, the following two non-linear relations are obtained:

$$\frac{\partial^4 \overline{\psi_1^{(0)}}}{\partial z^4} = -\overline{J\left(\psi_0^{(0)}, \frac{\partial^2 \psi_0^{(0)}}{\partial z^2}\right)} \qquad (31)$$

$$\frac{\partial^4 \overline{\psi_1^{(1)}}}{\partial z^4} = -\overline{J\left(\psi_0^{(0)}, \frac{\partial^2 \psi_0^{(1)}}{\partial z^2}\right)} - \overline{J\left(\psi_0^{(1)}, \frac{\partial^2 \psi_0^{(0)}}{\partial z^2}\right)} \qquad (32)$$

It is the presence of these non-linear terms which leads to the first non-zero values of mean motion with a net mass transport in the horizontal.

The boundary conditions for the mean flow first must reflect the fact that there can be no net external force on the fluid of the tank. Therefore, the average tangential stress in the x-direction on the boundaries must be zero.

$$\frac{\partial \bar{u}}{\partial z_{\text{top}}} - \frac{\partial \bar{u}}{\partial z_{\text{bottom}}} = 0$$

or

$$\frac{\partial^2 \bar{\psi}}{\partial z^2_{\text{top}}} - \frac{\partial^2 \bar{\psi}}{\partial z^2_{\text{bottom}}} = 0 \qquad (33)$$

Secondly, the no-slip condition yields the two requirements

$$\frac{\partial \bar{\psi}}{\partial z_{\text{top}}} = \frac{\partial \bar{\psi}}{\partial z_{\text{bottom}}} = 0 \qquad (34)$$

Next, it is easy to demonstrate that

$$\overline{J\left(\psi_0^{(0)}, \frac{\partial^2 \psi_0^{(0)}}{\partial z^2}\right)} = 0.$$

Since

$$\psi_0^{(0)} = P(z)\,\theta' = (z^5 - 3z^3 - 2z^2)\left(-\frac{G}{5!}\right)\theta$$

and

$$\frac{\partial^2 \psi_0^{(0)}}{\partial z^2} = P''(z)\,\theta',$$

therefore, we find that

$$\overline{J\left(\psi_0^{(0)}, \frac{\partial^2 \psi_0^{(0)}}{\partial z^2}\right)} = \overline{[P(z)\,P'''(z) - P'(z)\,P''(z)]\,\theta'\theta''}$$

$$= Q(z)\,\overline{\theta'\theta''} = Q(z)\frac{d}{dx}\frac{\overline{\theta'^2}}{2} = 0.$$

Thus $\partial^4 \bar{\psi}_0/\partial z^4 = 0$.

With the quantities $\psi_0^{(0)}$ and $\psi_0^{(1)}$ known, and also using the facts that $\overline{\theta'\theta\gamma''} + \overline{\theta''\theta\gamma'} = -\overline{\theta'^2\gamma'} = \overline{2\theta'\theta''\gamma}$, the value for $\partial^4 \overline{\psi_1^{(1)}}/\partial z^4$ is therefore:

$$\frac{\partial^4 \overline{\psi_1^{(1)}}}{\partial z^4} = (504z^5 + 540z^4 - 72z^2)\,\overline{\theta'\theta''\gamma}\left(\frac{G}{5!}\right)^2 \qquad (35)$$

In order to determine the exact values for $\overline{\psi_1^{(0)}}$ and $\overline{\psi_1^{(1)}}$, we must evaluate appropriate boundary conditions at $z = 0$ and at $-1 + \alpha\gamma \approx -1$, respectively. After determining the appropriate constants, the mean value of $\psi_1^{(0)}$ is seen to be just an arbitrary constant, which can be taken to be zero:

$$\overline{\psi_1^{(0)}} = 0 \quad \overline{u_1^{(0)}} = \frac{\partial \overline{\psi_1^{(0)}}}{\partial z} = 0 \qquad (36)$$

The form of the mean value of $\psi_1^{(1)}$ is that of a polynomial in z multiplied by a number which is the average of functions of x evaluated over one wavelength:

$$\overline{\psi_1^{(1)}} = (70z^9 + 135z^8 - 84z^6 + 27z^2 + c_1) \times \frac{1}{420}\,\overline{\theta(x)\,\theta''(x)\,\gamma(x)} \qquad (37)$$

where c_1 is an arbitrary constant. c_1 will be taken so that $\overline{\psi_1^{(1)}} = 0$ at $z = -1$:

$$\overline{\psi_1^{(1)}} = (70z^9 + 135z^8 - 84z^6 + 27z^2 - 8) \times \frac{1}{420}\left(\frac{G}{5!}\right)^2 \overline{\theta'\theta''\gamma} \qquad (38)$$

It can therefore be seen that there exists mean values for $\psi_1^{(1)}$ and hence for $u_1^{(1)}$, which are proportional to $a^2\alpha$, if the functions θ and γ are so chosen to have a correlation $\int_0^1 \theta'\theta''\gamma dx \neq 0$.

Various combinations of functions may be used, but it can be shown that θ must have more than one Fourier component. If two components are present in the Fourier expansion of θ, then their sum and difference will be components of the expansion $\theta'\theta''$. Therefore, if γ has the sum or difference component in its representation, then the combination $\theta'\theta''\gamma$ will have a non-zero mean.

If, for example, θ is chosen as $\theta = \cos x + \sin x$, then $\theta'\theta'' = (-\sin x + \cos x)(-\cos x - \sin x) = -(\cos^2 x - \sin^2 x) = -\cos 2x$. If $\gamma = \cos 2x$, the mean value of $\theta'\theta''\gamma = -1/2$. Physically, this situation is sinusoidal "bumps" with alternating hot and cold regions (see Fig. 1).

Case II

In this case, the upper boundary is considered a free surface which can conduct heat to the air above it. The top surface is assumed to be at the temperature of the air, which is $T=0$. The temperature boundary conditions are thus identical to those of Case I.

However, the dynamic boundary conditions are different because of the free surface. For the laterally varying flow, the boundary conditions are

$$\frac{\partial \psi}{\partial x} = \frac{\partial^2 \psi}{\partial z^2} = 0 \quad \text{at } z=0 \quad \text{(free surface)} \tag{39}$$

and

$$\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial z} = 0 \quad \text{at } z=z_B \quad \text{(solid surface)} \tag{40}$$

For the mean flow, neither the bottom surface nor the free surface can transfer mean momentum to the fluid. This consideration leads to the condition $\partial^2\bar{\psi}/\partial z^2 = 0$ at both $z=0$ and z_B. In addition, the requirement that there be no average flow at the bottom boundary yields the expression $\partial\bar{\psi}/\partial z = 0$. However, this is no longer true at the top free boundary.

A similar analysis to that of Case I leads to the same zero order temperature fields:

$$T_0^{(0)} = -\theta(x)z \quad T_0^{(1)} = -\theta\gamma z \tag{41}$$

and yields the following zero order stream function:

$$\psi_0^{(0)} = (z^5 - 2z^3 + z)\left(-\frac{G}{5!}\theta'\right) \tag{42}$$

$$\psi_0^{(1)} = [(z^5 - 2z^3 + z)\theta\gamma' + (z^5 + 2z^3 - 3z)\theta'\gamma]\left(-\frac{G}{5!}\right) \tag{43}$$

Again these stream functions have values of zero at $z=0$ and -1, so that there is no net horizontal transport of fluid from order zero in the aspect ratio.

The next order flow leads to the presence of the non-linear terms

$$\frac{\partial^4 \overline{\psi_1^{(1)}}}{\partial z^4} = -\overline{J\left(\psi_0^{(0)}, \frac{\partial^2 \psi_0^{(1)}}{\partial z^2}\right)} - \overline{J\left(\psi_0^{(1)}, \frac{\partial^2 \psi_0^{(0)}}{\partial z^2}\right)}$$

$$= (336z^5 - 320z^3 + 48z)\overline{\theta'\theta''\gamma} \tag{44}$$

which, with the application of the above mentioned boundary conditions and the selection of the arbitrary constant to make $\psi = 0$ at -1, leads to

$$\overline{\psi_1^{(1)}} = (35z^9 - 120z^7 + 126z^5 - 105z - 64)\overline{\theta'\theta''\gamma} \times \left(\frac{G}{5!}\right)^2 \frac{1}{315} \tag{45}$$

The same correlation must exist as in Case I, i.e. that $\overline{\theta'\theta''\gamma} \neq 0$ for mean flow to occur, and again, the values are multiplied by $a^2\alpha$.

Case III

The assumption of a free surface which cannot conduct heat across it leads to some different considerations. To first order, then, the temperature at the free surface will be identical to the temperature of the bottom boundary directly below it. Thus there is no net transport of heat from the bottom of the fluid to the top, but the heat flow is strictly from the hot regions of the bottom boundary to the cold region.

In addition, the maintenance of a thermal gradient along a surface leads to the maintenance of a stress due to a variation of surface tension. If δ is the surface tension coefficient (see Landau & Lifschitz, p. 234) the force equation along a surface is

$$\left[p_1 - p_2 - \delta\left(\frac{1}{R_1} + \frac{1}{R_3}\right)\right] n_i = (\sigma'_{1ik} - \sigma'_{2ik}) n_k + \frac{\partial \delta}{\partial x_i} \tag{46}$$

where the subscripts 1 and 2 on p (pressure) and σ'_{ij} (stress) represent medium 1 (the fluid) and medium 2 (the air above), R_1 and R_2 are the principal radii of curvature, and $\mathbf{n}$ is the unit normal vector directed into medium 1. There is no surface curvature assumed in this system and no jump in pressure across the boundary. Furthermore, medium 2 is assumed to be negligible in the assumption of the force.

The upper boundary condition then becomes modified in that there is no longer zero stress at the top surface, but that the stress at $z=0$ is that due to surface tension variations:

$$\varrho \nu \frac{\partial u}{\partial z} = \frac{\partial \delta}{\partial x} \tag{47}$$

The surface tension coefficient decreases with temperature, and it can be represented by $\delta = S_0(1-\beta T)$. The non-dimensional form for the upper boundary condition becomes

$$\frac{\partial^2 \psi}{\partial z^2} = -S\frac{\partial T}{\partial x} \tag{48}$$

for laterally varying flows, where S is defined as $S = S_0 \beta \Delta T L/\varrho \nu^2$. The remaining dynamical boundary conditions are identical to those in Case II. The thermal boundary conditions are as follows:

$$\frac{\partial T_0}{\partial z} = \frac{\partial T_1}{\partial z} = 0 \quad \text{at } z=0 \tag{49}$$

$$T_0 = \theta(x) \qquad \text{at } z = z_B \tag{50}$$

$$T_1 = 0 \qquad \text{at } z = z_B \tag{51}$$

Solving for the zero order temperature field, it is found that $T_0^{(0)} = \theta(x)$ and $T_0^{(1)} = 0$. Using the notation $S^* = 4!\,S/2G$, the zero order streamfunctions can be written, after solution, as

$$\psi_0^{(0)} = \left[z^4 + \left(\frac{3-S^*}{2}\right)z^3 - S^* z^2 + \left(\frac{-1-S^*}{2}\right)z\right] \times \theta'\left(\frac{G}{4!}\right) \tag{52}$$

$$\psi_0^{(1)} = \left[\left(\frac{-3-S^*}{2}\right)z^3 + \left(\frac{3+S^*}{2}\right)z\right]\gamma\theta'\left(\frac{G}{4!}\right) \tag{53}$$

The following expression is then used for the determination of the mean flow:

$$\frac{\partial^4 \overline{\psi_1^{(1)}}}{\partial z^4} = [(-45z^4 + 54z^2 + 18z) + (-15z^4 + 9z^2 - 3)S^* + (-6z^2 - 6z - 1)S^{*2}]\left(\frac{G}{4!}\right)^2 \overline{\theta'\theta''\gamma} \tag{54}$$

After integration and final application of the boundary conditions, the following result is obtained:

$$\overline{\psi_1^{(1)}} = \left[(-15z^8 + 84z^6 + 84z^5 - 36z - 21)\frac{G^2}{4!4!}\frac{1}{560} + (-5z^8 + 14z^6 - 56z^5 - 70z^4 + 70z^3 - 166z - 91) \times \frac{GS}{4!}\cdot\frac{1}{1120} + (-2z^8 - 6z^5 - 5z^4 - 2z - 1)S^2\frac{1}{480}\right]\overline{\theta\theta''\gamma} \tag{55}$$

Discussion of heat transport

It is possible to compare the convective and diffusive heat transports from the heat conduction equation

$$\frac{\partial T}{\partial t} + \nabla\cdot\mathbf{u}T = \varkappa\nabla^2 T \tag{56}$$

and using the results which have been obtained from the power series expansions obtained in the previous section for the three cases. The quantity $\overline{wT} \approx -\overline{(\partial\psi_0/\partial x)\,T}$ is proportional to the vertical convective heat transport. It is responsible for the transport of heat from high temperature boundaries to lower temperature boundaries. The convective heat flux in dimensional terms is as follows:

$$\overline{F}_{\text{conv}} = \varrho c_p \overline{w_d T_d} = -\frac{\nu L \Delta T \varrho c_p}{R^2}\overline{\frac{\partial \psi_{n0}}{\partial x_n} T_n} \tag{57}$$

An order of magnitude estimate can be obtained for the convective heat flux by determining its value for the lowest order solution at the level $z = -1/2$ for Case I: If

$$\psi_0^{(0)} = (z^5 - 3z^3 - 2z^2)\left(-\frac{G}{5!}\right)\theta'$$

and $T_0^{(0)} = -\theta z$ then

$$\overline{\frac{\partial \psi_0^{(0)}}{\partial x} T_0^{(0)}} = (z^6 - 3z^4 - 2z^3)\frac{G}{5!}\overline{\theta''\theta} \tag{58}$$

For $|\overline{\theta\theta''}| = 1/2$, expression (58) yields, when evaluated at $z = -1/2$,

$$|\overline{F}_{\text{conv}}| = \frac{\nu L \Delta T \varrho c_p}{R^2}\frac{5}{128}\left(\frac{G}{5!}\right) = \frac{\varkappa L \Delta T \varrho c_p}{R^2}\frac{5}{128}\frac{\text{Ra}}{5!} \tag{59}$$

where the Rayleigh number Ra is equivalent to the Grashof number G multiplied by the Prandtl number Pr.

The diffusive heat transport is proportional to the quantity $-\overline{\varkappa \nabla T}$ and in the vertical direction the diffusive heat flux in dimensional terms is as follows:

$$|\overline{F}_{\text{diff}}| = \left|\varrho c_p \varkappa \frac{\partial T_d}{\partial z_d}\right| = \frac{\varrho c_p \varkappa \Delta T}{L}\left|\frac{\partial T_n}{\partial z_n}\right| \tag{60}$$

Evaluating quantity (60) for Case I at level $z = -1/2$, we obtain

$$|\overline{F}_{\text{diff}}| = \frac{\varrho c_p \varkappa \Delta T}{L} \tag{61}$$

Furthermore, a ratio can be obtained between the two types of heat transport:

$$F_{\text{conv}}/F_{\text{diff}} = \frac{5}{128}\frac{\text{Ra}}{5!}a^2 \tag{62}$$

In order for the convective transport to remain smaller than the diffusive transport, the following order of magnitude must exist:

$$\frac{5}{128}\frac{\text{Ra}}{5!}a^2 < o(1) \tag{63}$$

Hence, for $F_{\text{conv}}/F_{\text{diff}} = o(a)$, we conclude that $\text{Ra} \lessapprox 3\,000/a$. For an aspect ratio of near 0.3, we reach the conclusion that for $F_{\text{conv}}/F_{\text{diff}} \lessapprox o(a)$, the Rayleigh number must be smaller than 10 000. The approach used in this study is thus contingent upon the Rayleigh number's being 10 000 at most, since the mathematical analysis used depends upon the non-linear terms being of smaller magnitude than the linear ones.

The net transport mean flows have been shown to be proportional to the square of the Grashof number ($\sim G^2$). Since $G = \text{Ra}/\text{Pr}$, if the fluid has the maximum possible Rayleigh number of 10 000, then the maximum possible value for G^2 is dependent upon the Prandtl number. The Prandtl number of mercury is 0.027, while that of water is near 7. Therefore, for equivalent Rayleigh numbers, the non-dimensional velocity will be 7×10^4 times as great for mercury as for water. However, since the dimensional velocity is the non-dimensional velocity u multiplied by ν, we can see that the velocity of mercury will be 7×10^3 times that of water for the same Rayleigh number. It is certainly clear that mercury is the better fluid for testing this flow phenomenon.

Discussion of momentum balance

The understanding of the momentum balance of a fluid system such as the one in this study is instrumental to the realization that mean channel flows are capable of being maintained. Starting with the x-component of the Navier-Stokes equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\varrho}\frac{\partial p}{\partial x} + \nu\nabla^2 u \tag{64}$$

and the continuity equation $\nabla \cdot \mathbf{u} = 0$ and taking zonal averages, we see that

$$\frac{\partial \bar{u}}{\partial t} + \frac{\partial}{\partial z}\overline{uw} = \nu\frac{\partial^2 \bar{u}}{\partial z^2} \tag{65}$$

Therefore, a temporal change in the mean flow must be caused by either a Reynolds stress, or by a stress due to the gradient in the vertical direction of the zonally averaged velocity. For steady state to be maintained, the following relationship must hold:

$$\frac{\partial}{\partial z}\overline{uw} - \nu\frac{\partial^2 \bar{u}}{\partial z^2} = 0 \tag{66}$$

In the lowest order flow, there is closed cell motion, as one would expect, with rising motion above the heated regions and sinking motion above the colder regions. In Case I and Case II, the lowest order flow provides the cell pattern with the solution $\psi_0^{(0)} = (z^5 - 3z^3 - 2z^2)(-G/5!)\theta'$ and $(z^5 - 2z^3 + z)(-G/5!)\theta'$, respectively, both of which look like the two-cell pattern above hot and cold regions, as in Fig. 2. Flow of order αa^0, as has been shown, provides no net flow between top and bottom boundaries because the average stream-function is the same at the two boundary levels; however, there is mean flow for each horizontal level.

The stream-function of order αa^0 has the mean value

$$\overline{\psi_0^{(1)}} = [(z^5 - 3z^3 - 2z^2)\overline{\theta\gamma'} + (z^5 + 3z^3 + 4z^2)\overline{\theta'\gamma}]\left(-\frac{G}{5!}\right)$$
$$= (6z^3 + 6z^2)\overline{\theta'\gamma}\left(\frac{G}{5!}\right) \quad (67)$$

When the functions θ and γ have a positive correlation for $\overline{\theta'\gamma}$, then a mean flow of order α results. As an example, if $\theta = \sin x$, $\gamma = \cos x$, and $\overline{\theta'\gamma} = 1/2$, this would induce a flow for which $\bar{u} = \partial\bar{\psi}/\partial z > 0$ in the top of the region and $\bar{u} < 0$ in the bottom of the region, but each transporting the same amounts of fluid. This corresponds to the situation illustrated in Fig. 3.

Since $\partial\bar{u}/\partial z = \partial^2\bar{\psi}/\partial z^2 < 0$ at the top and $\partial\bar{u}/\partial z > 0$ at the bottom and in the middle, we find that the original cells must tilt—have $\overline{uw} > 0$ at the bottom and middle and $\overline{uw} < 0$ near the top. Therefore the cells would tilt somewhat as illustrated in Fig. 4.

It is not until the term in $a^2\alpha$ with the presence of the non-linear terms, that net horizontal mean motion occurs, if $\overline{\theta'\theta''\gamma} \neq 0$. Figs. 5, 6, and 7 correspond to the results of Cases I, II, and III, respectively. In all of these diagrams the net flow is $U > 0$ in the case where $\theta = \sin x + \cos x$, $\gamma = \cos 2x$.

The most efficient of those cases considered is Case III—that of a free non-conducting top surface. This is evident upon a comparison of results in Figs. 5, 6, and 7. In all the cases, where $\partial\bar{u}/\partial z > 0$, this implies that $\overline{uw} > 0$ and so the cells must be tilted in the direction from bottom left to right. Conversely, where $\partial\bar{u}/\partial z < 0$, this implies $\overline{uw} < 0$, and the cells must tilt in the other direction.

On the right side of the ridges are the maxima and minima in temperature. There the streamlines must be nearly vertical above these sources, with the area above the heat source having upward flow, and with the area above the colder source having downward flow. The region which

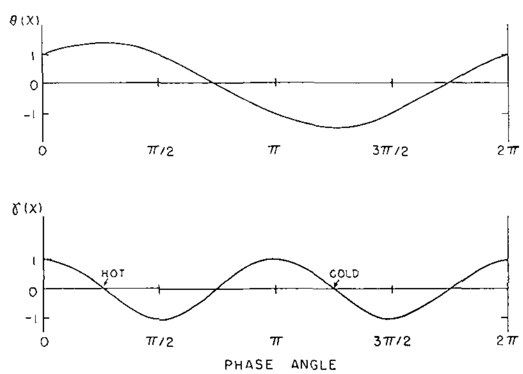

Fig. 1. Non-dimensional heat sources and topography for $\theta = \sin x + \cos x$, $\gamma = \cos 2x$.

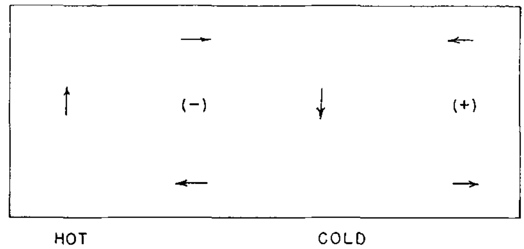

Fig. 2. Two-cell motion of lowest order flow.

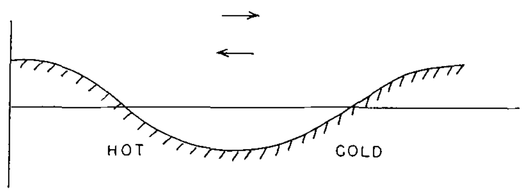

Fig. 3. Mean flow of order α for $\theta = \sin x$, $\gamma = \cos 2x$.

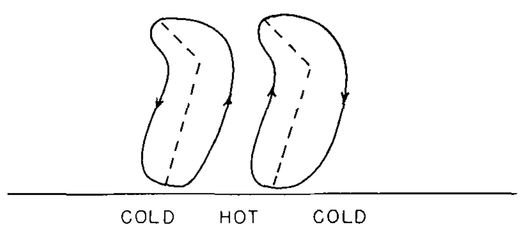

Fig. 4. Resulting tilted cells for $\theta = \sin x$, $\gamma = \cos 2x$.

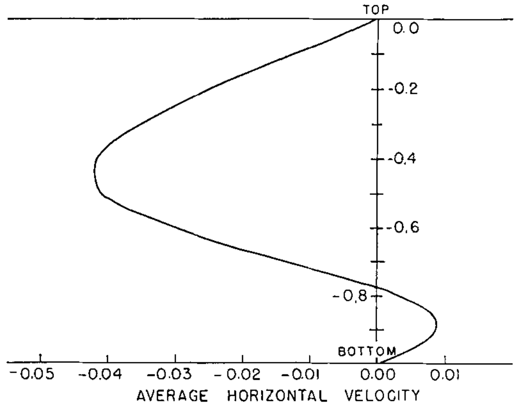

Fig. 5. Case I: Rigid, conducting top—terms leading to mean horizontal flow. Average horizontal velocity $\bar{u} = \partial\bar{\psi}/\partial z$ in non-dimensionalized units multiplied by $(G/5!)^2\, \overline{\theta'\theta''\gamma a^2\alpha}$.

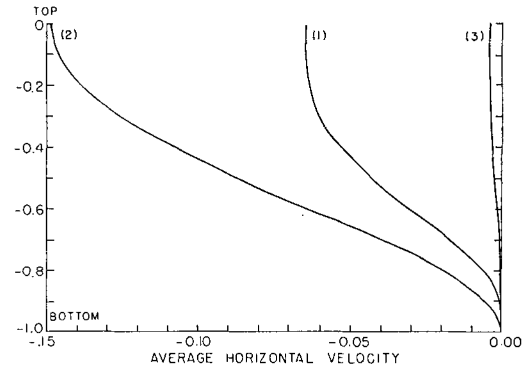

Fig. 7. Case III: Free, non-conducting top—terms leading to mean horizontal flow. Average horizontal velocity $\bar{u} = \partial\bar{\psi}/\partial z$ in non-dimensionalized units multiplied in (1) by $(G/4!)^2 \overline{\theta'\theta''\gamma a^2\alpha}$, in (2) by $(GS/4!)\overline{\theta'\theta''\gamma a^2\alpha}$ and in (3) by $S^2\overline{\theta'\theta''\gamma a^2\alpha}$.

connects these flows is on the left of the ridges, and the streamlines, in connecting from the hot to the cold region, and vice versa, must gently slope up the left side of the ridges. Thus these streamlines are tilted such as to carry positive horizontal momentum from the bottom of the tank to the top of the tank. This type of flow pattern is illustrated in Fig. 8.

Results

Since it has been shown that mercury would be much more suitable in providing net flows than would be water, the results of this study will be analyzed with the mercury as the fluid

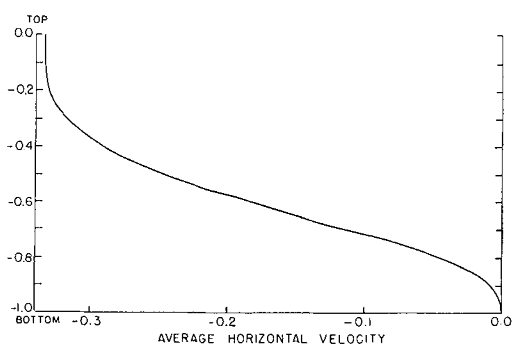

Fig. 6. Case II: Free, conducting top—terms leading to mean horizontal flow. Average horizontal velocity $\bar{u} = \partial\bar{\psi}/\partial z$ in non-dimensionalized units multiplied by $(G/5!)^2\, \overline{\theta'\theta''\gamma a^2\alpha}$.

in the hypothetical apparatus. It has been derived that the Rayleigh number must be less than or equal to order of magnitude 10 000. The appropriate physical constants for mercury in cg units are as follows:

$\bar{\alpha} = 1.8 \times 10^{-4}\ (^\circ\mathrm{C})^{-1}$

$\nu = 1.16 \times 10^{-3}\ \mathrm{cm^2\ sec^{-1}}$

$\varkappa = 4.3 \times 10^{-2}\ \mathrm{cm^2\ sec^{-1}}$

It is then computed that $\mathrm{Ra} = (g\bar{\alpha}/\varkappa\nu)L^3\Delta T = 3\,500\ L^3\Delta T$. Therefore $L^3\Delta T \lessapprox 3$, and so both L and ΔT in a tank experiment are of order one. For $\mathrm{Ra} = 10\,000$, the Grashof number, G, will be $\mathrm{Ra}/\mathrm{Pr} = 10\,000/0.027$, and so $G = 3.7 \times 10^5$.

In order to determine the value of the surface tension parameter S, it is necessary to know the change in the surface tension of a mercury–

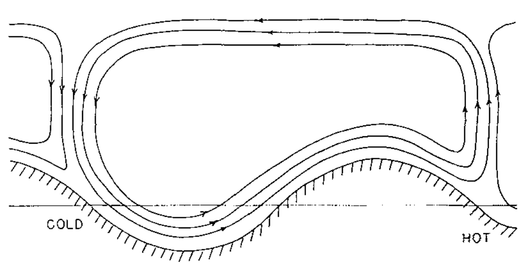

Fig. 8. Schematic streamlines for $\theta = \sin x + \cos x$, $\gamma = \cos 2x$.

Table 1. *Fluid transport of mean channel flows for* $G = 3.7 \times 10^5$, $S = 5.5 \times 10^3$, $\theta = \cos x + \sin x$, $\gamma = \cos 2x$.

Case	Fluid transport	Numerical value
I	$-\frac{8}{420}\left(\frac{G}{5!}\right)^2 \frac{L}{G} \nu a^2 \alpha \overline{\theta'\theta''\gamma}$	$= 0.31$ cm³ sec⁻¹ cm⁻¹
II	$-\frac{64}{315}\left(\frac{G}{5!}\right)^2 \frac{L}{R} \nu a^2 \alpha \overline{\theta'\theta''\gamma}$	$= 3.3$ cm³ sec⁻¹ cm⁻¹
III	$-\frac{21}{560}\left(\frac{G^2}{4!\,4!}\right) \frac{L}{R} \nu a^2 \alpha \overline{\theta'\theta''\gamma}$	$= 15.0$ cm³ sec⁻¹ cm⁻¹
	$-\frac{91}{1120}\left(\frac{GS}{4!}\right) \frac{L}{R} \nu a^2 \alpha \overline{\theta'\theta''\gamma}$	$= 11$ cm³ sec⁻¹ cm⁻¹
	$-\frac{1}{480} S^2 \frac{L}{R} \nu a^2 \alpha \overline{\theta'\theta''\gamma}$	$= 0.11$ cm³ sec⁻¹ cm⁻¹

air interface with respect to temperature. This value, which is denoted by S_0, is approximately 0.1 dyne cm^{-1} $(°C)^{-1}$. With the length and temperature scales both of order 1, and $\varrho = 13.6$ g cm^{-3}, the value for S of a mercury-air interface is as follows:

$$S = \frac{S_0 \beta \Delta T L}{\varrho \nu^2} = 5.5 \times 10^3$$

A total fluid transport U represents the difference between the values of the stream function at the points 0 and $z_B \approx -1$:

$$U = \int \bar{u}_d \, dz = \frac{L}{R} \nu (\bar{\psi}_{\text{top}} - \bar{\psi}_{\text{bottom}}) \tag{68}$$

The results obtained for the term which provides net horizontal fluid transport in Cases I, II, and III are shown in Table 1. The results in this table are taken at $G_{\max}$ which is 3.7×10^5. It is not consistent with the method to take G any larger than this value. Case I, with the rigid top, has a much smaller horizontal mass transport than either case of a free boundary.

It is questionable which case—that whose top surface is at the same temperature as the air (Case II), or that of no heat conduction to the air (Case III)—is valid for a tank. Since mercury is such a good conductor of heat, it is probable that it will be able to pick up heat from below, but at the same time adjust to some extent to the temperature of the air above. Thus, it will be assumed that the free surface flow is some combination of Case II and Case III. In any event, the mass transfer is far greater with a free surface than with a rigid boundary. One would have expected this, since a rigid boundary acts to decrease the flow.

The term in G^2 of Case III is 1.4 times as great as the term in GS, which in turn is 100 times as great as the term in S^2. However, these results are only approximate because of the non-exactness of the surface tension term. In addition, smaller values of G would reduce the differences between the terms.

Whitehead, in his moving heater analysis decided that surface tension is the primary cause for the mean flow. With a stationary heater above topographic ridges, it appears that the buoyancy terms remain more important even for no conduction across the top surface. If heat conduction is permitted, so as to smooth out top temperature patterns, then the surface tension induces even less net transport of fluid relative to buoyancy induced transport.

Conclusions

It can thus be seen that stationary heat sources are capable of producing mean motion in fluids provided that the topography of the boundaries are constructed properly. When the temperature sources are so arranged to have certain mean components in combination with the topography, this produces the mean motions.

Although the analysis here assumed a Cartesian x-y-z space, since one would want a reentrant channel, an annulus could be used provided that its curvature effects are not too great. Further work could be carried out to determine the optimum conditions which would produce a mean channel flow. Instead of a temperature boundary condition, a heat boundary condition can be used.

Unfortunately, it seems unlikely that such a mean fluid flow will be observed in the natural planetary atmospheres, since such regular topography and correlation of heat sources appear improbable. However, the great theoretical interest of the mean fluid motions can lead to useful applications. The fact that horizontal flow can be produced from purely stationary heat sources is a topic in fluid mechanics worth studying.

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ТЕЧЕНИЕ В КАНАЛЕ, ВОЗБУЖДАЕМОЕ СТАЦИОНАРНЫМ ИСТОЧНИКОМ ТЕПЛА

Аналитически показано, что периодическое распределение стационарных источников тепла на изменяющейся топографической структуре дна канала может возбудить в нем средние горизонтальные потоки. Дается обзор теоретических и экспериментальных исследований течений в канале, выполненных в прошлом. Для ряда граничных условий выводятся условия, необходимые для генерации средних горизонтальных потоков. Обсуждаются балансы тепла и количества движения в таких жидких системах.