An alternative formulation of the problem of density jump

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ABSTRACT

An alternative formulation of the problem of density jump, first considered by Wilkinson & Wood (1971), is presented. A first order ordinary differential equation results for the crest height of a controlling weir as a function of the depth of the flowing layer downsteam of the transition, with a boundary condition corresponding to the immiscible jump. An outline of the solution is presented.

Introduction

Examples of density jumps are found in many geophysical phenomena such as Föhn and Katabatic winds, and have been frequently cited in the existing literature (see, for example, Ball, 1957; Wilkinson & Wood, 1971. The latter will hereafter be referred to as WW.) It was WW who first investigated the role of a downstream control in the phenomenon of density jump. This note presents an alternative formulation of the problem considered by them and should best be read in conjunction with their paper. The present formulation is somewhat more general, since it does not invoke the Boussinesq approximation. A first order ordinary differential equation for the crest height of a controlling weir as a function of the depth of the flowing layer downstream of the transition results, with a boundary condition corresponding to the immiscible jump. A closed form solution of the differential equation is possible. An explicit expression of the solution is, however, not presented here.

Formulation

We consider a layer of upstream density ϱ_1 depth h_1 and discharge q_1 flowing under a stationary[1] ambient fluid of density $\varrho_0 (\varrho_0 < \varrho_1)$, and going through an abrupt thickening at a transition.

A schematic of the density jump is shown in Fig. 1. The transition comprises an entrainment zone in the upstream and a roller region in the downstream portion of it, the relative extent of each depending on the crest height of a controlling weir. WW have shown that, for a given upstream flow, there is a maximum height of the crest corresponding to which the transition comprises entirely of the roller region. Under this condition the density jump reduces to the classical immiscible internal jump. Above this crest height the upstream flow is flooded. When the crest height is lowered from its maximum value, the entrainment zone increases at the expense of the roller region, until, when the crest height is reduced to zero, the transition is composed entirely of the entrainment zone.

The formulation is based on the conservation of flow force and mass deficit across the transition and the conditions that the flow downstream of the transition is energy preserving and that it is critical at the crest. The above considerations are sufficient to yield a first order ordinary differential equation for the height of the crest as a function of the depth of the flowing layer downstream of the transition. The relationship between the maximum height of the crest and the parameters of the upstream flow is known from classical internal jump theory. This relationship serves as the boundary condition fo the differential equation.

[1] The induced motion due to entrainment of the ambient fluid is neglected.

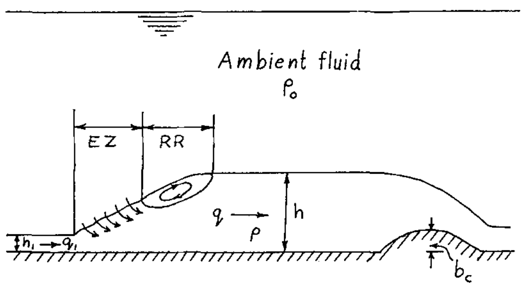

Fig. 1. Schematic of the density jump. *EZ*, entrainment zone; *RR*, roller region.

The density, depth, and discharge downstream of the transition are, respectively, ϱ, h and q, and b_c is the height of the crest. For simplicity we shall, from the very first, assume uniform velocity and density distributions across the flowing layer both upstream and downstream of the transition. Non-uniformities in these can be accounted for in the manner of WW.

With a suitable datum pressure the flow force $\mathcal{S}$, of the flowing layer can be written as

$$\mathcal{S} = \varrho s g h^2\left(\frac{q^2}{sgh^3} + \frac{1}{2}\right) = \mathcal{S}_1 \tag{1}$$

where $s = (\varrho - \varrho_0)/\varrho$, and g is the acceleration due to gravity. (Suffix 1 denotes upstream values.)

The requirements that the Bernoulli head behind the transition is constant and that the flow is critical at the crest can be summed up in the following equation

$$\frac{b_c}{h} = \frac{q^2}{2sgh^3} + 1 - \frac{3}{2}\left(\frac{q^2}{sgh^3}\right)^{1/3} \tag{2}$$

The change in the density of the flowing layer downstream of the transition as a result of entrainment of the ambient fluid is governed by

$$\frac{d\varrho}{\varrho_0 - \varrho} = \frac{dq}{q} \tag{3}$$

whence the equation of the conservation of mass deficit is obtained:

$$q(\varrho - \varrho_0) = q_1(\varrho_1 - \varrho_0) = \mathcal{M}_1, \quad \text{say} \tag{4}$$

Eqs. (1), (2) and (4) fully define solutions to the problem under consideration.

From (1) and (4), eliminations of ϱ yields the following cubic equation for q:

$$\varrho_0 q^3 + \mathcal{M}_1 q^2 - \mathcal{S}_1 h q + \tfrac{1}{2}\mathcal{M}_1 g h^3 = 0 \tag{5}$$

It can be shown that (5) has two positive real roots (the third one is then a real negative root), one of which is characterized by $q < q_1$, for $h < h'$, where h' is the immiscible jump height. Eq. (5) yields a functional relationship between q and h.

From (4)

$$\varrho = \frac{\mathcal{M}_1 + \varrho_0 q}{q} \tag{6}$$

and we also have a functional relationship between ϱ and h.

Now taking total differentials of (1) and (2) with

$$d\mathcal{S} = 0$$

$$d\varrho = \frac{\varrho^2}{\varrho_0} ds$$

$$dq = -\frac{q}{s}\frac{\varrho}{\varrho_0} ds,$$

and eliminating ds, we get upon simplification

$$\frac{db_c}{dh} - \frac{b_c}{h} = \left[\mathcal{F}^2(1 - \mathcal{F}^2)\left\{\frac{q_0 + 2\varrho}{\varrho - 2\mathcal{F}^2(\varrho_0 + \varrho)}\right\} - \frac{3}{2}\mathcal{F}^2\right] \tag{7}$$

where $\mathcal{F}^2 = q^2/sgh^3$ is the square of the modified Froude number of the flowing layer. Since both q and ϱ (and hence s) are known functions of h, we can write (7) as

$$\frac{db_c}{dh} - \frac{b_c}{h} = \mathcal{G}(h; q_1, \varrho_1, h_1, \varrho_0) \tag{8}$$

where $\mathcal{G}$ is a known function of h.

Boundary condition

When the roller region occupies the entire transition the density jump reduces to an immiscible jump. Then

$$\frac{b_c}{h'} = \frac{1}{2}\frac{q_1^2}{s_1 g h'^3} + 1 - \frac{3}{2}\left(\frac{q_1^2}{s_1 g h'^3}\right)^{1/3} \tag{9}$$

where

$$h' = \frac{h_1}{2}\{-1+\sqrt{1+8\mathcal{F}_1^2}\};\ \mathcal{F}_1 \geq 1 \tag{10}$$

We write

$$b_c|_{h=h'} = \mathcal{L}_1(q_1, \varrho_1, h_1, \varrho_0),\ \text{say} \tag{11}$$

Eq. (9) is same as Eq. (2), but now the depth of the flowing layer downstrean of the transition is related to the upstream depth by the classical relationship (10). The quantity $\mathcal{L}_1$, is, of course, the maximum height of the crest, above which the upstream flow is flooded.

Eq. (8), a first order ordinary differential equation for b_c as a function of h, is to be solved subject to the boundary condition (11). A solution to the homogeneous problem corresponding to (8) is, by inspection

$$b_c = h$$

Using the method of variation of parameter, let $b_c = h\mathcal{N}(h)$ be the solution to (8) satisfying (9). Then $\mathcal{N}$ satisfies

$$\frac{d\mathcal{N}}{dh} = \frac{\mathcal{G}(h)}{h} \tag{12}$$

Integration of (12) at once yields

$$\mathcal{N} = \int^h \frac{\mathcal{G}(h)}{h}\, dh + \text{Constant} \tag{13}$$

The solution to (8) satisfying (11) is easily found to be

$$b_c = h\int^h \frac{\mathcal{G}(h)}{h}\, dh + \frac{h}{h'}\left(\mathcal{L}_1 - h'\int^{h'} \frac{\mathcal{G}(h')}{h'}\, dh'\right);\quad 0 \leq b_c \leq \mathcal{L}_1 \tag{14}$$

Eq. (14) gives the relationship between the height of the crest of a controlling weir and the depth of the flowing layer downstream of the transition of a density jump. Corresponding relationships between the crest height and other flow variables can be similarly obtained.

Conclusions and discussions

An alternative to WW's formulation of the density jump has been presented. However, the analysis is not restricted to Boussinesq fluids. The relation between the crest height of a controlling weir and the depth of the flowing layer downstream of the transition of a density jump is governed by a first order ordinary differential equation, with a boundary condition corresponding to the immiscible jump. Although the problem is amenable to a closed form solution, an explicit expression of the solution has not been presented.

Consistent with WW's findings, it is to be expected that the solution to the problem formulated here will not be unique when the upstream Froude number exceeds a certain value. WW have shown that two different flow states downstream of the transition are possible if the upstream Froude number exceeds 13.2 for Boussinesq fluids. However, only one of these states is stable.

For a lighter layer flowing over a heavier ambient fluid, a similar formulation can be used.

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ДРУГАЯ ФОРМУЛИРОВКА ПРОБЛЕМЫ СКАЧКА ПЛОТНОСТИ

Предлагается другая формулировка проблемы скачка плотности, впервые рассмотренной Уилкинсоном и Вудом (1971). Выводится обыкновенное дифференциальное уравнение первого порядка для высоты гребня водосливной плотины как функции глубины текущего потока вниз по течению с граничным условием, соответствующим несмешивающемуся скачку. Дается краткое описание решение.