Some kinematic and dynamic properties of the atmosphere

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(Manuscript received March 9, 1973; revised version February 25, 1974)

ABSTRACT

Given only the most general conditions present in the atmosphere, it is demonstrated by theoretical means that the circulation regime must be of the eddy type rather than being axisymmetric. This is done by showing that the mean meridional circulations cannot transport angular momentum across surfaces of constant mean absolute angular momentum. In the course of the development of the argument a quantity named the rotational Rossby number is defined, this nondimensional parameter being somewhat distinct from the conventional Rossby number. The kinematic part of the discussion is applicable to a variety of other specific quantities in addition to angular momentum. Mention is made of the possible application of the results to systems other than the terrestrial atmosphere.

Introduction

The general circulation of the atmosphere and its internal dynamics have come under closer and closer scrutiny in recent years both from the observational aspects and from rather complete numerical models. It is to be hoped that the main features of it are now rather well in hand as far as descriptions go. Their explanation, however, will no doubt still occupy us for a long period of time. The importance of all this is heightened due to the fact that the atmospheres of other planets and of the sun require much more study—a process aided by comparisons with the one member of this group of celestial bodies for which we have a wealth of data (see Starr, 1973).

For these reasons it appears that what follows may be of some utility. To the writer's knowledge the main points to be made have not been treated in the literature. However, even if they have, more attention should be given to them.

Kinematical analysis

We assume that the surface of the earth is a figure of revolution about the polar axis, and that its meridional cross section is spherical. Some attention to these simplifications will be given later. We assume that at some level H above the surface essentially all the atmospheric mass lies below, as far as the present purposes are concerned. Considering the meridional cross section in Fig. 1, we draw a simple arbitrary geometric curve from A to B. Using a spherical polar system of coordinates R, λ, ϕ where R is radius, λ longitude counted positive eastward in absolute space, and ϕ is latitude, our curve is given by

$$\phi = F(R) \tag{1}$$

where F is arbitrary, ϕ being independent of time.

We now generate a surface of revolution, S, by rotating the curve AB about the polar axis. Since A is at $R=R_0$, the surface of the earth, and B is at $R=R_0+H$, the simple annular surface S separates the atmosphere into two portions. For our purposes we may assume that no appreciable mass flows from one portion to the other across S at time t progresses. That is

$$\int_S \varrho V_n \, dS = 0 \tag{2}$$

where ϱ is air density and V_n is taken as the component of velocity normal to S and positive from the equatorial toward the polar side.

We next consider the flux across the (fixed) surface S of some specific property of the atmosphere having a value L per unit mass. The

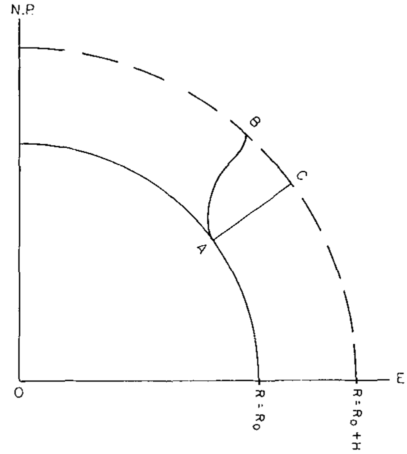

Fig. 1. A meridional cross-section through the atmosphere illustrating in schematic form the problems discussed in the text.

northward flux T of L is then

$$T = \int_S L\varrho V_n \, dS \tag{3}$$

It immediately follows, however, that if we consider the special case where L has a uniform value over the entire surface S, then

$$\int_S L\varrho V_n \, dS = L \int_S \varrho V_n \, dS = 0 \tag{4}$$

because of (2). However, most physical quantities as directly measured in the atmosphere do not have the symmetric distribution required by (4), so that added procedures are necessary.

Let us define the (instantaneous) zonal means of L and ϱV_n at a given latitude and level on S as

$$[L] = \frac{1}{2\pi} \oint L \, d\lambda; \quad [\varrho V_n] = \frac{1}{2\pi} \oint \varrho V_n \, d\lambda \tag{5}$$

We may note that $[L]$, unlike $[\varrho V_n]$, can be defined without regard to any particular surface S. Also we define the departures from these means at individual longitudes by primes so that

$$L = [L] + L'; \quad \varrho V_n = [\varrho V_n] + (\varrho V_n)' \tag{6}$$

We now write, after the manner of Starr & White (1951), that

$$\overline{[L\varrho V_n]} = \overline{[L]}\,\overline{[\varrho V_n]} + \overline{[L'(\varrho V_n)']} \tag{7}$$

where bars have been placed distributively over each term to indicate linear time averages over an interval t_1. The time averaging leads to definitions analogous to (5) and (6). We indicate departures from time averages by superscript asterisks and expand the first term on the right of (7) so that

$$\overline{[L\varrho V_n]} = \overline{[L]}\,\overline{[\varrho V_n]} + \overline{[L]^*[\varrho V_n]^*} + \overline{[L'(\varrho V_n)']} \tag{8}$$

where the first term on the right gives the contribution of the so-called mean meridional circulation.

We may now integrate (8) over the entire fixed annular surface S. It is clear that the left-hand member will give us the time average rate $\bar{T}$ of the flux of L across S poleward, i.e.,

$$\bar{T} = \int_S \overline{[L\varrho V_n]} \, dS = \frac{1}{t_1} \int_0^{t_1} \int_S L\varrho V_n \, dS \, dt \tag{9}$$

since the brackets are redundant in the integration over S.

Next we make the crucial selection of a definition of the surface S. We choose S so as to be a surface of constant $[\bar{L}]$, assuming that $[\bar{L}]$ is so distributed as to make this possible. The contribution to $\bar{T}$ of the first term on the right of (8) now becomes

$$\int_S \overline{[L]}\,\overline{[\varrho V_n]} \, dS = \overline{[L]} \int_S \overline{[\varrho V_n]} \, dS$$
$$= \overline{[L]} \left\{ \frac{1}{t_1} \int_0^{t_1} \int_S \varrho V_n \, dS \, dt \right\} = 0 \tag{10}$$

since the mean mass flux across S is zero for all present considerations as already stated in (2). We have now reached the important kinematical conclusion that the mean meridional circulations (of mass) cannot cause any contribution to the transport of a specific property of the atmosphere across complete mean zonal time average surfaces of that property. Regardless of the physical nature of this property, its transport across such fixed surfaces must be due to the time and space eddy terms in (8).

Were the eddy motions absent as in a steady zonally symmetric flow, $\bar{T}$ would then be zero except for possible molecular effects temporarily not included in (3). On this understanding we write that

$$\bar{T} = \int_S \overline{[L]^*[\varrho V_n]^*}\, dS + \int_S \overline{[L'(\varrho V_n)']}\, dS \qquad (11)$$

We may note that in the discussion by Starr & White (1952) it was shown that the sum of the last two terms in (8) could be expressed also as the sum of the standing and transient eddy effects. In our present scheme we may therefore write alternatively that

$$\bar{T} = \int_S [\overline{L'}\,\overline{(\varrho V_n)'}]\, dS + \int_S [\overline{L^*(\varrho V_n)^*}]\, dS \qquad (12)$$

giving these two effects in the stated order.

Dynamical considerations

Perhaps the most important dynamical result follows from the equations obtained if we let $L = M$, where M is the specific absolute angular momentum of a parcel of air, measured with reference to the polar axis, i.e.,

$$M = r^2\Omega + ru \qquad (13)$$

in which Ω is the earth's angular velocity, u is the eastward relative velocity of a particle and r is its normal distance to the central axis.

As indicated by (13) if $u \equiv 0$, then the surfaces of constant M are coaxial cylinders. Since the angular velocity u/r of the relative motions in the atmosphere is small compared with Ω, the surfaces of constant M will not depart too much from the basic distribution corresponding to Ω alone (see Starr, 1948). This is difficult to show in a diagram of measured values of $[M]$ or $[\bar{M}]$ because inevitably we must exaggerate the vertical scale of the atmosphere as compared to the size of the earth. With this in mind we may consider the long term average distribution of $[\bar{M}]$ for the northern hemisphere given in Fig. 2, calculated approximately for a spherical earth.

The main point to be brought out is that the mean meridional circulations, responsible for the first term on the right in (8), cannot produce a net transport of M across any one of these surfaces, as indicated by eq. (10). Such transports must then be due to the eddy terms as shown by (11) and (12) (unless molecular effects enter the picture). See also Starr (1974).

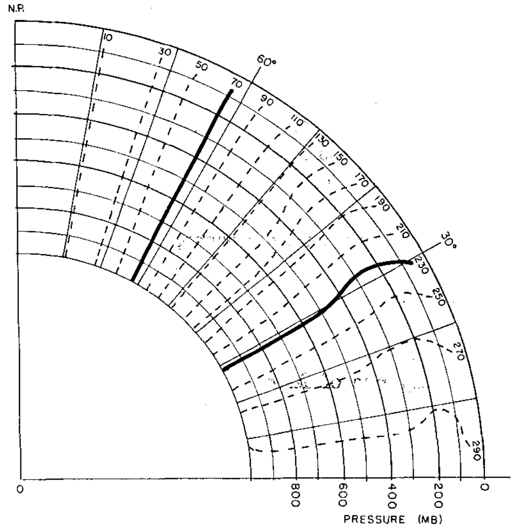

Fig. 2. Meridional cross-section through the atmosphere of the field of $[\bar{M}]$ in units of 10^{11} cm^2 sec^{-1}, based on 60 months of northern hemispheric upper air data covering the period May 1958 through April 1963.

An illuminating case is obtained by choosing the two $[\bar{M}]$ surfaces which at the bottom boundary separate the mean easterlies from the mean westerlies at about 30 and 60° N. These particular surfaces are shown by heavy lines in Fig. 2. The mean westerlies at the ground abstract angular momentum from the portion of the atmosphere between these two surfaces, necessitating a convergence of $\bar{T}$. We see that in accordance with (11) or (12) this must be due to eddy processes, any transport by molecular effects being small. The mean meridional circulations have no direct contribution to this balance.

We know that the atmosphere is essentially a thin layer in the sense that roughly 95% of its mass lies below 20 km. It is then true that although the surfaces of $[\bar{M}]$ are not latitude surfaces, their spread in latitude is not great—a few degrees perhaps. It is then to be expected that the transport of angular momentum across latitude surfaces by the mean meridional circulation should be relatively small, though not necessarily zero. This is apparently true (see Starr, 1968).

At low latitudes, say around 20° N, there still is a necessity for an eddy transport northward across $[\overline{M}]$ surfaces, since the mean meridional circulations cannot accomplish this, whatever be their direction or magnitude. The same is true as we proceed northward of 30° N.

As a further feature of the present subject let us imagine for the moment that the eddy terms are completely absent, so that only steady symmetric motions are present. With reference to Fig. 1, no transport of angular momentum can then take place across S if AB is a line of constant M. Since the state is steady, no convergence of M can be taking place into the toroidal volume defined by ABC. It follows that no transport can take place across the (arbitrary) curve AC. We may choose this latter curve to define a latitude surface. It follows further then, that if mean meridional circulations are to cause an angular momentum transport in the more general mean state across latitude surfaces, such a transport must hinge on the presence of eddies, since it would be impossible without them, under our general assumptions.

General comments

It is necessary to point to certain limitations and further interpretations when the concepts derived above are applied to both the ideal and the actual atmosphere. Likewise mention needs to be made of the possible use of these concepts for the interpretation of corresponding processes in systems other than the earth's atmosphere. It perhaps hardly requires pointing out that each of these systems offers great intricacies of its own, the adequate treatment of which needs much more of an effort than can be entered upon presently. Accordingly what is said below in these instances can only serve the purpose of reminding readers who are interested of some of the problems involved.

(1) The condition that exists in the atmosphere, namely that the mean differential rotation is small compared with the basic rotation Ω, results in the fact that the relative motions do not alter too greatly the locations of the surfaces of constant absolute angular momentum. Stated more precisely we have that, in terms of $[\overline{M}]$,

$$r|[\bar{u}]| << r^2\Omega \tag{14}$$

from eq. (13). This suggests a nondimensional quantity N_R such that

$$N_R = \frac{|[\bar{u}]|}{r\Omega} \tag{15}$$

The ratio N_R is then akin to, but not identical with, the Rossby number. Thus we might conceive a situation in which the latter is large while N_R is small. It might therefore be useful to designate N_R by the term rotational Rossby number.

(2) Let us consider the dishpan experiment in the form without the central core, so that the space occupied by the fluid is simply connected. The heat sink at the center may then be conceived of as a flush cold source at the bottom, thus simulating atmospheric conditions at more polar latitudes in this respect. For a medium like water under ordinary conditions of the experiment the presence of a much lower Reynolds number as compared with that in the atmosphere becomes important.

The low rotation and high heating case (the upper left region in Fig. 2 of Hide, 1969) might be expected to yield a symmetric circulation without eddies, due to a mean meridional convection cell. The question is how the angular momentum can be transferred from the outer regions to the bottom surface westerlies (assuming these exist) nearer the axis of rotation. This cannot be due to eddies since there are none, and from what has been said no angular momentum can be fed across surfaces of constant values of this quantity by the mean meridional circulation. All that is left is the action of molecular viscosity which, however, can only cause an angular momentum flux normal to the angular velocity surfaces toward regions of lower rotation.

To complete the picture it is therefore necessary to consider that the rotational Rossby number is high enough so that the shapes of the M surfaces depart grossly from that of coaxial cylinders. Thus near to the center these surfaces might be rather horizontal, so that there exists a vertical decrease of angular velocity downward enabling molecular viscosity to feed angular momentum across them to the bottom. Actually the attainment of a true symmetric regime of this kind may be difficult. Thus Starr & Long (1953) in trying to do this managed to obtain only a beautiful wave number one picture (the pertinent streak photographs are reproduced also in Lorenz, 1967). However, with no doubt better controlled experimental conditions, Hide & Mason (1970) seem to have had no difficulty in attaining complete symmetry.

(3) According to (11) a nonzero value of $\bar{T}$ may result from the oscillation of a zonally symmetric flow as a consequence of the first integral on the right. However the needed nonzero time correlation required probably cannot be obtained if $[L]$ is conserved for each of the oscillating rings of particles. Thus if we consider $[L]$ to be the angular momentum $[M]$, proper torques would be needed to change $[M]$ for such rings. Since we now omit molecular effects, the simultaneous presence of eddy pressure torques would have to enter the picture. Thus the second integral in (11) would probably not be zero in actual cases.

Although there is some evidence that symmetrical oscillations contribute to an angular momentum transport in the atmosphere, the subject has as yet not been studied properly. Another probable instance of such action might turn out to be the vacillation mechanism in certain well known dishpan experiments.

(4) In the actual terrestrial atmosphere the presence of topographical irregularities deranges to some extent the actions here discussed. Mountains and other barriers prevent the strict definition of zonal averages in the simple manner indicated above. Probably the total effect is to introduce certain approximations rather than major changes in the results. More specific studies are needed to demonstrate these influences of topography.

(5) The equations used above may also be applied to other types of surfaces which divide the atmosphere into two parts. Thus there may exist surfaces of some quantity $[\bar{L}]$ that form polar domes intersecting the surface at some latitude. The flux of L may then be examined in the same manner as before, and with similar results.

(6) A clear distinction should be made between the present results concerning possible modes of transport of angular momentum and the subject of the generation of kinetic energy of the zonally averaged zonal flow. This kinetic energy generation is given by a different dynamic equation as discussed, e.g., by Starr & Gaut (1969). Hence although the mean meridional circulations cannot cause a net transport of angular momentum across complete $[\bar{M}]$ surfaces, they can and do affect the generation of zonal kinetic energy significantly.

(7) Much cannot be said here concerning the conditions in the atmospheres of other celestial bodies. There is however little doubt but that the ideas here expressed must find fulfillment in these systems as well. For example the atmosphere of Jupiter is in such a state of gross motion as to place it in the category of systems with a low rotational Rossby number, being in this respect similar to the terrestrial case. Nevertheless, due to differences in the mode of heating and other conditions, the differential rotation is not the same (see Starr, 1973). In the solar case hydromagnetic effects would no doubt have to be included, at least in the final analysis (see Fischer, 1971).

(8) What in effect has been accomplished in this paper is the following demonstration. Given (*a*) that the atmosphere possesses a low rotational Rossby number and (*b*) that the surface torques are more or less as observed, it then follows that the circulation regime cannot be of the axisymmetric kind but must instead be of the eddy type. In this theoretical demonstration it is to be noted that the rotational Rossby number is not an external but an internal parameter, and that likewise the presence of surface torques is an internally determined factor. Some disturbance of the ideal situation due to a mountainous lower boundary must be recognized, as already stated, but would favor an axisymmetric flow regime only if the mountains were themselves axisymmetric.

Were the rotational Rossby number considerably higher it might be possible to have an ideal symmetrical regime, but even this may be doubted. Aside from mountain effects, a true zonally uniform circulation might be difficult to achieve just as it is in the dishpan with a flush cold source.

Acknowledgements

The writer wishes to express his appreciation to Dr John Whitehead of the Woods Hole Oceanographic Institution for helpful discussions of the material in this paper. Thanks are due to Mr Richard Rosen for discussions and for aid in preparing the present article, and also to Mrs Catherine Colclough for typing the manuscript. This research was supported by the Atmospheric Sciences Section, National Science Foundation, NSF Grant No. GA-36021X.

Note added in proof: My colleague Dr J. Van Mieghem of the University of Brussels kindly brought to my attention (private communication) the fact that my arguments do not prohibit a transport of the total enthalpy of moist air across latitude walls by the mean meridional circulation. This is because mean surfaces of this quantity are quasi-horizontal over a large spread of latitude. I might add, however, that moisture transport provides a small net mean meridional mass flux which, although of no practical concern for momentum, would require a special kind of treatment in the case of latent heat.

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НЕКОТОРЫЕ КИНЕМАТИЧЕСКИЕ И ДИНАМИЧЕСКИЕ СВОЙСТВА АТМОСФЕРЫ

Исходя только из самых общих условий, имеющих место в атмосфере, теоретически показано, что циркуляционный режим должен иметь вихревой тип, а не осесимметричный. Доказательство основано на том, что, как показывается, средняя меридиональная циркуляция не может переносить угловой момент через поверхности постоянного среднего абсолютного углового момента. В ходе доказательства дается определение вращательному числу Россби — безразмерному параметру, несколько отличному от обычного числа Россби. Кинематическая часть обсуждения приложима ко многим количественным характеристикам помимо углового момента. Указывается на возможное применение результатов к системам помимо земной атмосферы.