On the saturation amplitude for unstable baroclinic waves

By J. E. HART, *Department of Astro-Geophysics, University of Colorado, Boulder, Colorado*

(Manuscript received August 20; revised version December 14, 1973)

ABSTRACT

The equilibrium amplitude to which small unstable perturbations of a two-layer baroclinic current grow is determined by analytical theory. The range of validity and accuracy of this theory is investigated by comparing the theoretical predictions with more accurate numerical solutions and with laboratory experiments. Reasonably good results are obtained for fairly large values of the "small parameter" $\varepsilon = (F - F_c)/F_c$, where F is the internal Foude number, equal to $4\Omega^2 L^2/g(\Delta\varrho/\bar{\varrho})H$. The theoretical solution is only applicable for parameters for which the final wave state is steady.

1. Introduction

It is well known that large scale eddies resulting from baroclinic instability play an important part in the general circulation of the earth's atmosphere. The transports of heat and momentum by these eddies make a strong contribution to the overall energy and momentum balances. For many purposes one would like to be able to predict the eddy transports as functions of the externally imposed or mean conditions. One current problem where such a parameterization would be extremely useful is concerned with models of climate. In simple models like those of Budyko (1969) and Sellers (1969, 1973), the northward eddy transport of heat, averaged around latitude circles, is usually parameterized as being proportional to the mean north–south temperature difference or gradient; for example,

$$\overline{vT} = -\beta(\nabla T \cdot \hat{y}) \tag{1}$$

The diffusion constant β is determined by use of current observational data. The criticism can then be made that the use of eq. (1) in a climate model may not be accurate for climatic regimes very much different from our present climate.

It would be desirable to put formulations like those expressed in eq. (1) on a more firm theoretical footing. The theoretical problem is akin to that of Rayleigh-Bénard convection (or of Taylor double-cylinder flow) where one tries to determine the heat flux as a function of the applied temperature difference (or more precisely, the Nusselt number as a function of the Rayleigh number). Success has been attained in cases where the basic or mean state is only weakly unstable (Malkus & Veronis, 1958; Davey, 1962; Stuart, 1962). If the structure of the eddy is similar to that of the eigenfunction determined from linear stability theory, as will be the case if the flow is near critical, then one can compute the desired correlations within an amplitude coefficient A^2 which is not determined by the linear stability theory. Stone (1972*a*, *b*) has used this approach in calculating heat fluxes for a climate model. The eddy shape is given by linear theory and the amplitude is determined by using an *ad hoc* assumption which relates the meridional and zonal velocities. One can, however, consistently determine the equilibrium amplitude for weakly unstable flows by the method introduced by Stuart (1960) and Watson (1960). Pedlosky (1971) and Drazin (1972) have done this for simple models of baroclinic instability. Since baroclinic waves are thought to be of prime importance in the eddy transports it is of interest to establish the range of validity of such theories. This is done here by comparing the finite-amplitude theory with numerical and experimental results, as has been done for convective (Hart, 1972*b*) and Taylor double-cylinder (Stuart, 1960; Snyder & Lambert, 1966) flows.

2. The model

We consider the two-layer quasi-geostrophic model of baroclinic instability. Although this is a crude approximation to the real atmosphere it does have a rather close laboratory analog against which theoretical results can be compared. In particular we consider the motion in a cylinder of radius L, rotating with basic frequency Ω, containing two fluids each of depth H, with kinimatic viscosity ν, density ratio $\Delta\varrho/\bar{\varrho}$, and driven from above by a contact lid rotating with frequency $\omega + \Omega$. If the Rossby number $R_0 = \omega/2\Omega$ is small, and if the Ekman number is not too big $(E^{\frac{1}{2}} \equiv (\nu/\Omega H^2)^{\frac{1}{2}} < R_0)$ the equations governing the motion in the two layers are the potential vorticity equations:

$$\frac{d}{dt}[\nabla^2 P_2 + F(P_2 - P_1)] + \delta[\tfrac{3}{2}\nabla^2 P_1 - \tfrac{1}{2}\nabla^2 P_2 - 2] = 0 \tag{2.1}$$

and

$$\frac{d}{dt}[\nabla^2 P_2 + F(P_1 - P_2)] + \delta[\tfrac{3}{2}\nabla^2 P_2 - \tfrac{1}{2}\nabla^2 P_1] = 0 \tag{2.2}$$

with

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$

and

$$\frac{d}{dt} = \frac{\partial}{\partial t} + J(P, \quad).$$

The P_i are the pressure fluctuations in each layer. The velocities are determined by the geostrophic relations $P_r = v$, and $P_\theta = -ru$. F is the Froude number $4\Omega^2L^2/g\Delta\varrho/\bar{\varrho}H$ and $\delta = E^{\frac{1}{2}}/(\sqrt{2}R_0)$. Further details of the model and the experiment can be found in the paper by the author (Hart, 1972*a*) which we will hereafter call I.

Eqs. (2.1) and (2.2) are to be solved subject to regularity conditions at $r = 0$ and

$$\frac{1}{r}\frac{\partial P_i}{\partial\theta} = 0$$

and

$$\int_0^{2\pi} r\frac{\partial^2 P_i}{\partial r\,\partial t}\,dr = 0 \quad \text{at } r = 1.$$

Solutions of the form

$$P_1 = 0.75\frac{r^2}{2} + |\Delta|^{1/2}\varphi_1^{(1)} + |\Delta|\varphi_1^{(2)} + \dots \tag{2.3}$$

and

$$P_2 = 0.25\frac{r^2}{2} + |\Delta|^{1/2}\varphi_2^{(1)} + |\Delta|\varphi_2^{(2)} + \dots \tag{2.4}$$

are tried. The first terms in (2.3) and (2.4) represent the steady axisymmetric (or zonal) flow. The second terms represent instabilities (non-axisymmetric motions), and subsequent terms the higher order corrections. The small parameter Δ is defined by $\Delta = F - F_c$ where F_c gives the neutral curve for infinitesimal perturbations as a function of δ and $U = \frac{1}{2} - \frac{1}{4}R_0\Delta\varrho/\bar{\varrho}$ a parameter which results from the basic parabolic curvature of a rotating interface. The terms of order $|\Delta|^{\frac{1}{2}}$ are governed by the linear stability equations (see I) which have solutions

$$\varphi_1^{(1)} = \mathrm{Re}\{A(T)\,e^{in(\theta - ct)}J_n(l_m r)\} \tag{2.5}$$

and

$$\varphi_2^{(1)} = \varphi_1^{(1)} \tag{2.6}$$

with $J_n(l_m) = 0$, along the neutral curve $F_c = l^2/4U$ + (viscous corrections involving δ). T is a slow-time variable equal to $|\Delta|^{\frac{1}{2}}t$. For fixed geometry, fluid viscosity, and static stability,

$$\delta = \frac{1}{\sqrt{2}}R_0^{-1}(G_r F)^{-1/4}$$

where $G_r = g\Delta\varrho/\varrho H^5/\nu^2L^2$ = constant ($G_r^{-\frac{1}{4}} = 0.014$ in the experiment). Thus $F_c = F_c(R_0;\ \Delta\varrho/\bar{\varrho},\ G_r)$. Fig. 1 shows a comparison between the theoretical stability boundary and the experimentally observed onset of non-axisymmetric motions (●). It is seen that the agreement is quite good over a fairly wide parameter range. The reader is referred to paper I for further discussion of these results and of the experimental method.

Inside the stability boundary but outside the crosses, the instabilities grow to a constant amplitude. The instabilities are travelling waves with non-dimensional phase speed $c \sim 0.5$, but the envelope of the oscillations (measured by observing the interface height at a fixed point $\theta = 0$, $r^* = 0.7L$) is constant. Inside the crosses, amplitude modulation and turbulence are observed (see Hart, 1973). It is the purpose of this paper to develop and test a theory for the equilibrium or saturation amplitude of instabilities in the steady wave region of Fig. 1.

The amplitude equation of $A(T)$ is determined by solving the order $|\Delta|^{\frac{1}{2}}$ problem and by removing the secularities from 0 $(|\Delta|^{\frac{3}{2}})$ problem. Details for the rectilinear geometry (channel flow) are given by Pedlosky (1971) and modifications for cylindrical geometry and interfacial friction by Hart (1973). For later use we note that

$$\varphi_1^{(2)} = \Phi_1(r, T)$$

$$\varphi_2^{(2)} = \mathrm{Re}\left\{\frac{-16iv}{n}\left(\dot{A} + \frac{\delta}{|\Delta|^{1/2}}A\right)e^{in(\theta - ct)}J_n(lr)\right\} + \Phi_2(r, T). \tag{2.7}$$

The amplitude equations determined by removing secularities from the $|\varphi|^{3/2}$ problem are

$$\ddot{A} + \frac{\delta}{|\Delta|^{1/2}}\dot{A}\left(1 + \frac{2l^2}{2F_c + l^2}\right) - \frac{n^2}{8(l^2 + 2F_c)}A + \frac{n^2 A}{4(2F_c + l^2)\,J_{n+1}^2(l)\,U}\cdot\int_0^1 J_n^2(lr)\frac{\partial}{\partial r} \times \left\{(2F_c - l^2)\,\Phi_2 - \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\Phi_2\right\}dr, \tag{2.8}$$

$$\frac{\partial}{\partial T}[\nabla^2\Phi_2 - 2F\Phi_2] + \frac{2\delta}{|\Delta|^{1/2}}\nabla^2\Phi_2 = \frac{2l^2}{r}J_n(lr)\,J_n'(lr)\left(|\dot{A}|^2 + \frac{2\delta}{|\Delta|^{1/2}}|A|^2\right) \tag{2.9}$$

The qualitative comparison between experiment and various steady and time-dependent (limit-cycle) solutions of eqs. (2.8) and (2.9) was given by Hart (1973). We are here interested in making a quantitative comparison for the steady wave region of Fig. 1, where theory predicts steady wave solutions.

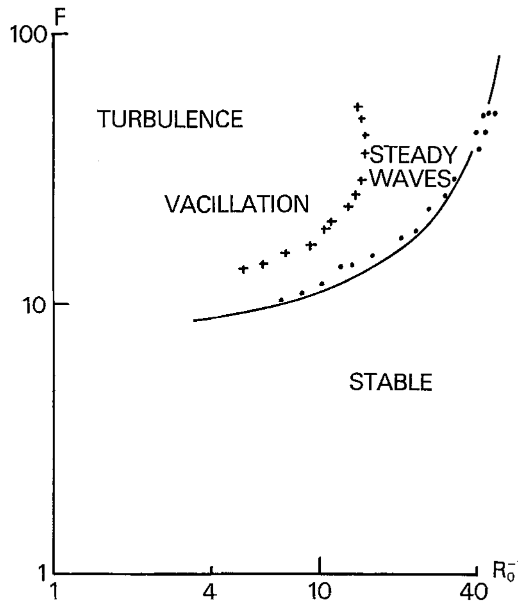

Fig. 1. Regime diagram for baroclinic waves in a cylinder with flat ends. Points and crosses represent the experimental data while the solid curve gives the theoretical neutral curve.

From eqs. (2.8) and (2.9) with $\partial/\partial T \equiv 0$ we can determine the steady amplitude A_s. We find

$$|A_s|^2 = \frac{J_{n+1}^2(l)\,U}{2\int_0^1 J_n^2(lr)\frac{\partial}{\partial r}[(2F_c - l^2)\hat{\Phi}_2 - \nabla^2\hat{\Phi}_2]\,dr} \tag{2.10}$$

where

$$\nabla^2\hat{\Phi}_2 = \frac{2l^2}{r}J_n(lr)\,J_n'(lr), \tag{2.11}$$

must be solved with $(\partial\hat{\Phi}_2/\partial r) = 0$ $er = 1$. If centrifugal effects are negligible $(0.25R_0^{-1}\Delta e/\bar{e} \to 0$, see I) then $U = \frac{1}{2}$ and we can find directly;

$$|A_s|^2 = -\frac{J_{n+1}^2(l)}{16l^2\int_0^1 \frac{1}{r}\frac{\partial}{\partial r}\frac{(J_n J_n')^2}{2}dr} \tag{2.12 a}$$

This can be compared to Pedlosky's equilibrium solution for a channel flow

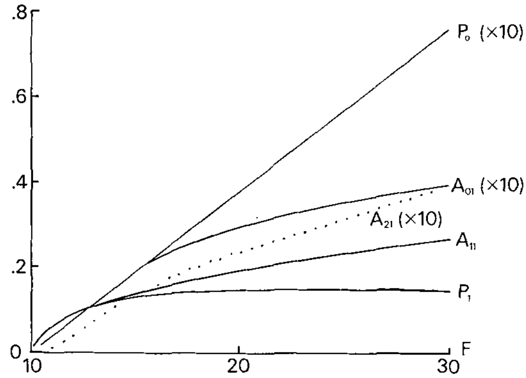

Fig. 2. Comparison of numerical and analytical theories for $R_0 = 0.1$. P_1 and A_{11} are the amplitudes of the fundamental pressure wave in the upper layer (with structure proportional to $e^{i\theta}J_1(l_1r)$) for the asymptotic and numerical calculations respectively. P_0 and A_{01} are the amplitudes of the wave-induced mean correction to the basic state (with structure $J_0(l_1r)$), and A_{21} indicates the importance of the dominant higher azimuthal wave number 2 harmonic which is not included in the asymptotic theory.

$$|A_s|^2 = \frac{(U_1 - U_2)^2}{m^2 a^2 \pi^2} \tag{2.12 b}$$

for wavenumber $a^2 = m^2\pi^2 + k^2$ and vertical shear $(U_1 - U_2)$.[1] Unfortunately the integral in (2.12) is not trivial and if $U \neq \frac{1}{2}$ we must solve eq. (2.11) before we can use (2.10). The solution to (2.11) is obtained by finite Hankel transform. The integrals which result have been evaluated numerically for $n = 1$ in order to make specific comparisons with the experiment, for which the $n = 1$ azimuthal and the $m = 1$ radial mode are always the preferred mode of instability. For the typical values

$$U = 0.4, \; F_c = 10., \quad |A_s|^2 = 0.00098.$$

How accurate is our prediction of the saturation amplitude? The weak-amplitude expansion implies several assumptions which are intimately connected with the smallness of $\Delta = F - F_c$; or since F_c can be of order 10, of $\varepsilon = (F - F_c)/F_c$. We have clearly neglected all higher radial and azimuthal harmonics of the fundamental. This is of course consistent in cases where ε is small enough. But in applying these results in the calculation of fluxes, say, we may wish to know how large ε may be before we lose substantial accuracy. We have obtained numerical solutions of the governing vorticity equations by substituting

$$P_i = \sum_{n=0}^{N} \sum_{m=1}^{M} A^i_{nm} J_n(l_m r)\, e^{in\theta} + \ast + \begin{pmatrix} 0.75 \\ 0.25 \end{pmatrix} r^2/2, \tag{2.13}$$

separating orthogonal components, and integrating the resulting ordinary non-linear differential equations for $A^i_{nm}(t)$. These solutions indicate when higher modes become important.

3. Results

We first compare numerical and analytical theory for $\Delta e/\bar{e} = 0.042$, $R_0 = 0.1$, $G_r^{-\frac{1}{2}} = 0.014$. Since the terms in (2.13) contain the linear eigenfunction, the numerical solution should agree with the analytic theory for small ε. In particular for these parameters with a flat bottom and top and $R_0 > 0$ it is known (Hart, 1973) that the $n = 1$ mode is always the most unstable, a fact reflected in the stability diagram of Fig. 1. Thus we can directly compare $|\Delta|^{\frac{1}{2}}\varphi_1^{(1)}$, and $A_{11}^{(1)}$. Fig. 2 shows this comparison. For $\varepsilon \lesssim 0.6$ the analytic solution is within 10% of the multimode theory which was computed with $N = 4$, $M = 3$. The discrepancy grows for $F > 16$.

One of the quantities measured in the experiment was the peak to peak interface height fluctuation, h, of the steady waves as they travelled past a fixed probe at $\theta = 0$, $r = 0.7$. Theoretically $h = R_0 F(P_2 - P_1)$, the right hand side of which can be evaluated from (2.5)–(2.7) and (2.10) or from the numerical solutions (2.13). Fig. 3 shows the comparison between analytical and numerical theory and the experiment. We note immediately one difficulty which arises from the fact that the theoretical and experimental neutral curves do not agree exactly. Thus the experimental data appears shifted a small amount to the right. Physically this is

[1] One apparent difference between 2.12a and 2.12b is that 2.12b contains the mean shear $U_1 - U_2$. In Pedlosky's model the manner in which U_1 and U_2 were set up was left undetermined. In the experiment with top driving, interfacial friction, and equal viscosities in each layer, $U_1 = 3/4$ and $U_2 = \frac{1}{4}$ and so the factor $U_1 - U_2$ does not appear in 2.12a.

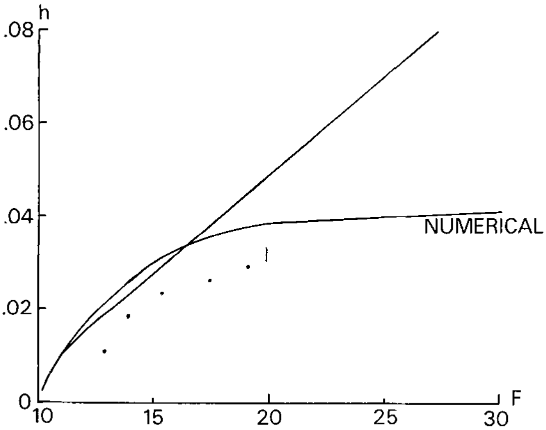

Fig. 3. Peak-to-peak interface fluctuation at $r = 0.7$. Analytical (solid), numerical (marked) theories and the data (points) are shown for $R_0 = 0.1$.

probably due to the presence of rigid side walls in the experiment.[1] If one takes the liberty of correcting (perhaps unjustifiably) for the difference in linear stability criteria by shifting the experimental data to the left until the neutral points coincide, the agreement between the numerical theory and the data is excellent up to $\varepsilon \sim 1$.[2] Above this steady waves are not observed. If this correction is not made errors of order 25% are found away from $F = F_c$ where the error in a meaningless sense, becomes infinite. The analytical theory is also acceptable up to $\varepsilon \sim 1$, above which it diverges considerably from the data. It should be noted that because the interface height is given primarily by the phase shift in the fundamental with height, large changes in h do not require large energy in the high modes—provided these affect the phase of the fundamental. One might wonder why a power series expansion in $|\Delta|^{\frac{1}{2}}$ could be accurate for $|\Delta|^{\frac{1}{2}}$ of order 3. The reason is that because the amplitudes A are so small, successive terms in (2.3)–(2.4) do decrease rapidly even for such large Δ.

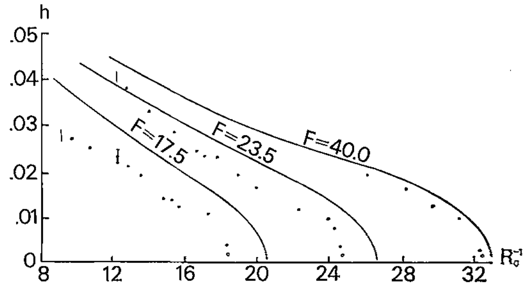

Fig. 4. Comparison of the numerical theory and experimental data for the values of F shown.

Further data are presented in Fig. 4. Here the experiments were run at fixed F, variable R_0, for $\Delta e/\bar{e} = 0.045$, and $G_r^{-\frac{1}{4}} = 0.014$. Here again if a correction is made for the initial onset points (marked with an *o*) the agreement between the multi-mode theory and the experimental results is quite good up to the point where the laboratory waves become non-steady. Even in the vacillating regime it appears that the theory describes the average wave amplitude quite well.

It should be pointed out that although the above results suggest that the equilibrium calculation was found to be accurate over a wide range of F, caution should be exercised in using this result in other systems. It was shown in Hart (1973) that the stability and finite-amplitude behavior of baroclinic waves depend strongly on the experimental geometry; for example the case with the ends flat as in the present study is quite different from that simulating a polar β-plane or a local f-plane. In particular the region where steady waves are observed differs from case to case, especially where 2 wave numbers can be neutrally stable at the same (F, R_0) point. We have studied the case with flat ends because of its experimental simplicity. Nonetheless the results suggest that the saturation amplitude formula can be used with reasonable confidence for fairly supercritical conditions.

4. Conclusions

We have shown that the equilibrium amplitude for baroclinic waves obtained by consider-

[1] Vertical slashes in the figures denote positions where the steady waves break down into amplitude vacillation.

[2] Lateral viscosity effects occur only in a narrow region near $r = 1$. For simplicity the theory was constructed using inviscid conditions there. It is easy to show that the application of no-slip boundary conditions to the linear theory increases the stability of the flow. The effect of the narrow shear zone in the basic flow is less clear, but numerical computations in progress on a slightly different problem indicate that it is small and stabilizing.

ing the interaction between the fundamental unstable wave and its mean-field correction predicts the experimentally observed equilibrium amplitude quite well up to values of F and R_0 roughly twice the critical values. Sources of error are the neglect of higher harmonics and the fact that the steady wave regime is not the observed wave state if the flow is too supercritical. However, previous studies of Bénard convection and Taylor double-cylinder flow indicate that theories which neglect higher harmonics, like those resulting from the assumption that the supercritical flow has the same shape as the neutral eigenfunction, predict certain average quantities (heat or momentum flux) quite well for flows which are as much as 10 times supercritical (Ta = 10 Tac), even though it is known that the higher harmonics become important in the disturbance structure at much smaller supercritical parameter values (Snyder & Lambert, 1966; Hart, 1972*b*). A similar situation might exist here.

From an investigation of the energy equations formed from (2.1) and (2.2) we find that the eddy flux resulting from the baroclinic part of the flow is given by

$$G = \frac{-F}{\pi}\int_0^{2\pi} P_{1\theta} P_2 \, d\theta$$

or

$$G = \frac{-F\lambda}{\pi}\int_0^{2\pi/\lambda} dx \, P_{1x} P_2$$

for the two coordinate systems. This corresponds to the usual longitudinally and vertically averaged north–south heat flux $\overline{vT}$ used in continuous systems. Using (2.5) and (2.7) and the corresponding equations from Pedlosky (1971) we find

$$G = 16F|A_s|^2|\Delta|\delta U J_n^2(lr) \tag{4.1}$$

and

$$G = 4F|A_s|^2|\Delta|\delta(U_1 - U_0)^{-1}\sin^2 m\pi y$$

$$\Rightarrow \frac{2|\Delta|F\delta}{m^2a^2\pi^2}\sin^2 m\pi y, \tag{4.2}$$

using (2.12b) and setting $U_1 - U_2 = \frac{1}{2}$ in the last equation. Note that if $U = \frac{1}{2}$ in (4.1) the results in the two cases are very similar. In each case the eddy flux is maximum in mid-latitudes and zero at the equator and pole (if these are taken as representative latitudinal boundaries).

It is interesting that the equilibrium amplitudes expressed in (2.12) are dependent only on the disturbance scale. The actual magnitude of the fundamental wave is

$$|A_s|^2 = \frac{\Delta(U_1 - U_2)^2}{m^2a^2\pi^2}$$

This is approximately $0.004(F - F_c)(U_1 - U_2)^2$ for scales appropriate to the earth. If we define a Richardson number

$$R_i = \frac{g\dfrac{\Delta e}{\bar{e}}H}{(U_1^* - U_2^*)^2}$$

we find that

$$F = \frac{1}{R_0^2(U_1 - U_2)^2 R_i}$$

Then,

$$A_s^2 = \frac{0.004}{R_0^2}\left(\frac{R_{ic} - R_i}{R_i R_{ic}}\right) \tag{4.3}$$

This can be compared with the amplitude determination in Stone (1972*a*) which assumes that the vertically averaged meridional velocity equals the vertically averaged basic zonal flow. His expression is

$$|A_s|^2 = \frac{1.09}{(1 + R_i)^2} \tag{4.4}$$

The R_i dependence of these two expressions is clearly different, especially if we use the $R_0^2 = 2.5/\pi^2 R_i$ relation of Stone (1972*a*), since then (4.3) becomes

$$A_s^2 \sim 1.6 \times 10^{-2}\left(1 - \frac{R_i}{R_{ic}}\right)$$

for large R_i. However the two equations predict similar magnitudes for R_i near 10, which is Stone's predicted value for Mars. It is of course somewhat unrealistic to compare the two-layer and continuous models, but this may in part explain the similarity between Stone's predicted value for the effective Richardson number for the Martian atmosphere and that predicted by the two-level numerical integration of Leovy & Mintz (1969).

Acknowledgement

This work was supported by the National Science Foundation, Grant GA 34937.

REFERENCES

Budyko, M. I. 1969. The effect of solar radiation variations on the climate of the earth. *Tellus 21*, 611.

Davey, A. 1962. The growth of Taylor vortices in flow between rotating cylinders. *J. Fluid Mech. 14*, 336.

Drazin, P. G. 1972. Non-linear baroclinic instability of a continuous zonal flow of a viscous fluid. *J. Fluid Mech. 55*, 577.

Hart, J. E. 1972*a*. A laboratory study of baroclinic instability. Geophys. *Fluid Dynamics 3*, 181.

Hart, J. E. 1972*b*. Experiments with two-dimensional, rectilinear, convective rolls at supercritical conditions. *Phys. of Fluids 15*, 203.

Hart, J. E. 1973. On the behavior of finite-amplitude baroclinic waves. *J. Atmos. Sci. 30*, 1017.

Leovy, C. & Mintz, Y. 1969. Numerical simulation of the atmospheric circulation and climate of Mars. *J. Atmos. Sci. 26*, 1167.

Malkus, W. V. R. & Veronis, G. 1958. Finite amplitude cellular convection. *J. Fluid Mech. 4*, 225.

Pedlosky, J. 1971. Finite amplitude baroclinic waves with small dissipitation. *J. Atmos. Sci. 28*, 587.

Sellers, W. D. 1969. A global climate model based on the energy balance of the earth-atmosphere system. *J. Appl. Meteor. 8*, 392.

Sellers, W. D. 1973. A new global climate model. *J. Appl. Meteor. 12*, 241.

Snyder, H. A. & Lambert, R. B. 1969. Harmonic generation in Taylor vortices between rotating cylinders. *J. Fluid Mech. 26*, 545.

Stone, P. H. 1972*a*. A simplified radiative-dynamical model for the static stability of rotating atmospheres. *J. Atmos. Sci. 29*, 405.

Stone, P. H. 1972*b*. On non-geostrophic baroclinic stability: Part III. *J. Atmos. Sci. 29*, 419.

Stuart, J. T. 1960. On the non-linear mechanics of wave disturbances in stable and unstable flows. *J. Fluid Mech. 9*, 353.

Stuart, J. T. 1962. Non-linear effects in hydrodynamic stability. *Tenth Int. Congr. of Applied Mech.*, 225.

Watson, J. 1960. On the non-linear mechanics of wave disturbances in stable and unstable flows. *J. Fluid Mech. 9*, 371.

АМПЛИТУДА НАСЫЩЕНИЯ ДЛЯ НЕУСТОЙЧИВЫХ БАРОКЛИННЫХ ВОЛН

С помощью аналитической теории определяется амплитуда равновесия, до которой дорастают малые неустойчивые возмущения двуслойного бароклинного течения. Пределы применимости и точность теории исследуются путем сравнения теоретических предсказаний с более точными численными решениями и лабораторными экспериментами. Довольно хорошие результаты получаются для весьма больших значений «малого параметра» $\varepsilon = (F - F_c)/F_c$, где F — внутренне число Фруда, равное $4\Omega^2 L^2/g(\Delta e/\bar{e})H$. Теоретическое решение применимо только для параметров, для которых окончательное состояние волны устойчиво.