SHORTER CONTRIBUTION

A note on the methods of analyzing traveling waves

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1. Introduction

In recent years, methods have been presented to resolve large-scale atmospheric disturbances into traveling wave components. Eliasen (1958) and Eliasen & Machenhauer (1965) computed the amplitude and phase angle of the atmospheric wave as functions of time. To resolve disturbances into progressive and retrogressive wave components, Kao (1968) used the two-dimensional Fourier analysis in longitude and time while Hayashi (1971) applied the Fourier analysis in longitude and cross spectral analysis in time. Deland (1964, 1972), on the other hand calculated the quadrature spectrum between the amplitudes of the sine- and cosine- longitudinal Fourier components to estimate the amplitude of "pure" traveling waves.[3] The purpose of this article is to clear up the puzzling variety and give a unique picture of the spectral techniques for resolving traveling atmospheric waves. In Section 2 I compare Kao's and Hayashi's methods. In fact Hayashi's method is equivalent to Kao's plus a special kind of spectral window in the frequency domain. I suggest that a better spectral window be used in order to have a less distorted spectral estimate. Section 3 shows that Deland's method gives incorrect results whenever progressive and retrogressive waves exist with the same wavenumber and frequency but different magnitude.

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[3] The term "pure traveling wave" is used by Deland (1972) to distinguish the traveling wave resolved by Kao's and Hayashi's methods.

2. The wavenumber-frequency spectral techniques for estimating the amplitudes of large-scale traveling waves

In this section I present the computational form for the two-dimensional Fourier analysis in longitude and time and then compare Kao's and Hayashi's methods. A discussion will also be made of the application of spectral windows to the wavenumber-frequency spectrum.

Let $Q(k, \pm\omega)$ be the complex Fourier coefficient of an atmospheric disturbance $q(\lambda, t)$ and $Q(k, \pm\omega) = Q_r(k, \pm\omega) + iQ_i(k, \pm\omega)$; then the real and imaginary parts of $Q(k, \pm\omega)$ are as follows:

$$Q_r(k, \pm\omega) = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} q(\lambda, t)\cos(k\lambda \pm \omega t)\,d\lambda\,dt$$

$$Q_i(k, \pm\omega) = -\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} q(\lambda, t)\sin(k\lambda \pm \omega t)\,d\lambda\,dt \tag{1}$$

where λ and t stand for longitude and time; k and ω are wavenumber and frequency. Time t is in normalized form $t = (2\pi/T)t'$ with observational period T and real time t'. It is seen in Eq. (1) that the positive sign of frequency represents the wave traveling westward and the negative sign of frequency means the wave traveling eastward.

Then the wavenumber-frequency spectrum may be computed as follows (Kao 1968; Kao & Wendell 1970):

$$E_{q'q}(k, \pm\omega) = 2[Q_r'(k, \pm\omega)Q_r(k, \pm\omega) + Q_i'(k, \pm\omega)Q_i(k, \pm\omega)] \tag{2}$$

where $Q'(k, \pm\omega) = Q'_r(k, \pm\omega) + iQ'_i(k, \pm\omega)$ is the Fourier coefficient for another variable $q'(\lambda, t)$.

Equation (1) is rearranged as follows so that the fast Fourier transform could be applied:

$$Q_r(k, \pm\omega) = \tfrac{1}{2}\frac{1}{2\pi}\left[\int_0^{2\pi} C_k(t)\cos(\omega t)\,dt \pm \int_0^{2\pi} S_k(t)\sin(\omega t)\,dt\right]$$

$$Q_i(k, \pm\omega) = \tfrac{1}{2}\frac{1}{2\pi}\left[\int_0^{2\pi} S_k(t)\cos(\omega t)\,dt \mp \int_0^{2\pi} C_k(t)\sin(\omega t)\,dt\right] \quad (3)$$

where

$$\tfrac{1}{2}C_k(t) = \frac{1}{2\pi}\int_0^{2\pi} q(\lambda, t)\cos(k\lambda)\,d\lambda$$

$$\tfrac{1}{2}S_k(t) = -\frac{1}{2\pi}\int_0^{2\pi} q(\lambda, t)\sin(k\lambda)\,d\lambda \quad (4)$$

Equations (4) and (3), then (2), are the computational procedures for two-dimensional Fourier analysis in longitude and time.

For the purpose of subsequent discussion, the following definitions are made:

$$\tfrac{1}{2}A_{k,\omega} = \frac{1}{2\pi}\int_0^{2\pi} C_k(t)\cos(\omega t)\,dt$$

$$\tfrac{1}{2}B_{k,\omega} = -\frac{1}{2\pi}\int_0^{2\pi} C_k(t)\sin(\omega t)\,dt$$

$$\tfrac{1}{2}a_{k,\omega} = \frac{1}{2\pi}\int_0^{2\pi} S_k(t)\cos(\omega t)\,dt$$

$$\tfrac{1}{2}b_{k,\omega} = -\frac{1}{2\pi}\int_0^{2\pi} S_k(t)\sin(\omega t)\,dt \quad (5)$$

Then Eq. (3) becomes

$$Q_r(k, \pm\omega) = \tfrac{1}{4}(A \mp b)$$

$$Q_i(k, \pm\omega) = \tfrac{1}{4}(a \pm B) \quad (6)$$

where the suffix is omitted for A, B, a, b, for simplicity.

By using Eq. (6), we will be able to show that Hayashi's generalized formula for power spectrum (his Eq. (4–4)) is the same as Kao's formula (Eq. (2)) for $q'(\lambda, t) \equiv q(\lambda, t)$, as follows:

$$\begin{aligned}4E_{qq}(k, \pm\omega) &= 8[Q_r^2(k, \pm\omega) + Q_i^2(k, \pm\omega)]\\ &= \tfrac{1}{2}(A \mp b)^2 + \tfrac{1}{2}(a \pm B)^2\\ &= \tfrac{1}{2}(A^2 + B^2) + \tfrac{1}{2}(a^2 + b^2) \pm (Ba - Ab)\\ &= P_\omega(C_k) + P_\omega(S_k) \pm 2Q_\omega(C_k, S_k) \quad (7)\end{aligned}$$

is where P_w and Q_w are power and quadrature spectrum respectively. Generalized formula for cospectrum (his Eq. (4–11)) is the same as Eq. (2) for $q'(\lambda, t) \not\equiv q(\lambda, t)$, as follows:

$$\begin{aligned}4E_{q'q}(k, \pm\omega) &= 8[Q'_r(k, \pm\omega)\,Q_r(k, \pm\omega) + Q'_i(k, \pm\omega)\,Q_i(k, \pm\omega)]\\ &= \tfrac{1}{2}[(A \mp b)(A' \mp b') + (a \pm B)(a' \pm B')]\\ &= \tfrac{1}{2}(AA' + BB') + \tfrac{1}{2}(aa' + bb') \pm \tfrac{1}{2}(Ba' - Ab') \pm \tfrac{1}{2}(B'a - A'b)\\ &= K_\omega(C_k, C'_k) + K_\omega(S_k, S'_k) \pm Q_\omega(C_k, S'_k) \mp Q_\omega(S_k, C'_k) \quad (8)\end{aligned}$$

where K_w is cospectrum and prime refers to the values obtained for the other variable $q'(\lambda, t)$. Hayashi used notation $W(x, t)$ instead of $q(\lambda, t)$ and $W'(x, t)$ for $q'(\lambda, t)$.

The distinction here is that Hayashi's method allows time-cross spectra to be computed either through the lag-correlation or through direct Fourier transform method, while Kao's cannot employ the lag-method. In time domain, it is known that the application of frequency window is implicit in the method of lag-correlations. Therefore, the lag-method gives more statistical confidence in the results than does the direct transform method. However, some spectral weights resulting from the lag-method are negative and will greatly distort the spectral estimates whenever there are rapid fluctuations in the true spectrum.

It is clear that the application of the spectral window to each individual time-cross spectrum on the right-hand sides of Eqs. (7) and (8) is the same as the application of the frequency spectral window to the wavenumber frequency spectrum on the left-hand sides of these equations. Therefore, the shortcomings of both Kao's and the lag-methods could be overcome by the application of a suitable explicit frequency spectral window to the spectral estimates by the two-dimensional Fourier analysis. Another advantage of this method is that the fast Fourier transform technique can be used in time domain as well as in space domain.

3. The quadrate spectral technique for resolving traveling waves

To resolve traveling waves, Dland (1964, 1972) calculated the quadrature spectrum between the amplitudes of the sine- and cosine-longitudinal Fourier components. Hayashi (1971) pointed out that Deland's method cannot resolve a pure standing wave into progressive and retrogressive components. The reason is that the quadrature spectrum equals zero when the progressive and retrogressive waves have the same magnitude and the same wavenumber and frequency.

In this section, I will further show that the use of the quadrature spectrum alone to estimate the amplitude of the pure traveling wave is not correct whenever the progressive and retrogressive waves have the same wavenumber and frequency but different magnitude.

To show this, I will use notations similar to Hayashi's. The disturbances $W(x, t)$ at a particular latitude could be decomposed into wavenumber-frequency domain as follows:

$$W(x, t) = \sum_k W_k(x, t)$$

$$W_k(x, t) = C_k(t) \cos kx + S_k(t) \sin kx$$

$$C_k(t) = \sum_\omega (A_{k,\omega} \cos \omega t + B_{k,\omega} \sin \omega t)$$

$$S_k(t) = \sum_\omega (a_{k,\omega} \cos \omega t + b_{k,\omega} \sin \omega t) \tag{9}$$

where x and t stand for longitude and time, k and ω are wavenumber and frequency, respectively. Assuming both standing waves and traveling waves exist, we may express $W_k(x, t)$ as follows:

$$W_k(x, t) = \sum_\omega R^0_{k,\omega} \cos(kx + \phi^0_{k,\omega}) \cos(\omega t + \phi^{0'}_{k,\omega}) + \sum_\omega R_{k,\pm\omega} \cos(kx \pm \omega t + \phi_{k,\pm\omega}) \qquad (10)^1$$

where $R^0_{k,\omega}$ and $\phi^0_{k,\omega}$, $\phi^{0'}_{k,\omega}$ are the amplitude and phases of the standing wave, $R_{k,\pm\omega}$ and $\phi_{k,\pm\omega}$ are the amplitude and phase of the pure traveling wave (+for westward-moving or retrogressive wave, − for eastward-moving or progressive wave), respectively.

It can be shown by comparing equation (9) and (10) that

$$\begin{aligned} A &= R^0_{k,\omega} \cos \phi^0_{k,\omega} \cos \phi^{0'}_{k,\omega} + R_{k,\pm\omega} \cos \phi_{k,\pm\omega} \\ B &= -R_{k,\omega} \cos \phi^0_{k,\omega} \sin \phi^{0'}_{k,\omega} \mp R_{k,\pm\omega} \sin \phi_{k,\pm\omega} \\ a &= -R^0_{k,\omega} \sin \phi^0_{k,\omega} \cos \phi^{0'}_{k,\omega} - R_{k,\pm\omega} \sin \phi_{k,\pm\omega} \\ b &= R^0_{k,\omega} \sin \phi^0_{k,\omega} \sin \phi^{0'}_{k,\omega} \mp R_{k,\pm\omega} \cos \phi_{k,\pm\omega} \end{aligned} \tag{11}$$

where the suffix is omitted for A, B, a, b. Therefore, the quadrate-spectrum is given by the following form

$$\begin{vmatrix} A & B \\ a & b \end{vmatrix} = \mp (R_{k,\pm\omega})^2 \mp R^0_{k,\omega} R_{k,\pm\omega} \tag{12}$$

This shows that the use of quadrature-spectrum to estimate the amplitude of a pure traveling wave will be accurate only when the amplitude of the standing wave is negligible at that wavenumber and frequency. By trigonometry, the standing wave may be viewed as two waves, progressive and retrogressive, with the same magnitude, wavenumber and frequency. Therefore, it is equivalent to say that the use of Deland's method to estimate the amplitude of a pure traveling wave will not be correct whenever the opposite traveling wave exists at that wavenumber and frequency.

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[1] Note that there is only one traveling wave (eastward or westward). In Eq. (10) and remaining equations in the text, $\pm$sign gives one alternative equation instead of two equations.

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