On the question of non-uniqueness of internal hydraulic jumps and drops in a two-fluid system

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(Manuscript received February 20, 1973

ABSTRACT

Conjugate states to a given two-layer flow of stable configuration confied on top and bottom by horizontal walls are studied in detail. There are at most two such states possible from energy considerations. Uniqueness is established by studying the total problem in which the agency responsible for causing shocks is also included. By establishing a suitable correspondence between the rigid top and free surface cases for purely internal shocks, uniqueness of the conjugate state for the latter case is also established.

1. Introduction

Many fluid flows in nature occur in conjugate states when two different flow regimes are joined to each other by a stationary shock. In the classical example of a hydraulic jump, a supercritical regime is joined to a subcritical regime by means of a shock which is in the form of an abrupt jump in the free surface level. The conjugate state to a given supercritical flow is uniquely determined in the classical case. The results of Yih & Guha (1955) for two-layer flows, with the upper layer having a free surface, indicate that multiple conjugate states to a given flow are possible. Yih & Guha felt that downstream control would decide uniqueness. In the study of purely internal shocks, the free surface can be profitably dispensed with. This eliminates the complexities that arise from the free surface wave mode, without in any way obscuring the phenomena under investigation. It is for this reason that the problem of two-layer flow confined on top and bottom by horizontal walls is studied here. We refer to the problem of investigating the conjugate states joined by a stationary shock as the isolated problem. This problem is studied in Section 2. When an agency (such as a hump on the floor or a contraction) that causes the shock is also included in the formulation of the problem, it is referred to as the total problem. We note that in the total problem the shock, in general, is a moving one. The total problem is studied in Section 3, and the results show that only one of the conjugate states is physically realizable.

In Section 4, a suitable correspondence between the rigid top and free-surface cases is established in order to prove the uniqueness of conjugate state to a given flow for the latter case also.

2. Isolated problem

The definition sketch for the isolated problem is given in Fig. 1. h and h' are the depths of the lower layer upstream and downstream from the shock, respectively, H is the distance between the walls, ϱ_1 and ϱ_2 are the densities, and q_1^* and q_2^* are the dimensional layer discharges. Suffices 1 and 2 stand for lower and upper fluids respectively. The pressure at the upstream interface is taken as zero, while that at the downstream interface is p^*. The pressure at the shock front is assumed hydrostatic, and shear stresses are neglected. Normalizing lengths by H, densities

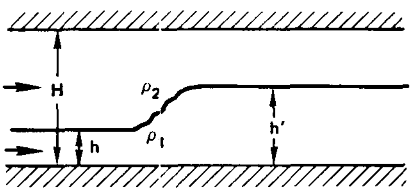

Fig. 1. Definition sketch for the isolated problem.

by ϱ_1, discharges by $g^{1/2}H^{3/2}$, pressures by $\varrho_1 gH$, use of the momentum theorem applied in turn to the lower and upper layers yields, upon elimination of pressure, the following equation for $\beta = h'/H$:

$$\begin{aligned}\mathcal{K}(\beta;\alpha,q_1,q_2,r) \equiv{} & \beta^4-(3-2\alpha)\beta^3 \\ & +\left(2-\frac{2q_1^2}{s\alpha}-4\alpha+\alpha^2-\frac{2q_2^2 r}{s(1-\alpha)}\right) \\ & \times\beta^2-\left(\frac{2r\alpha q_2^2}{s(1-\alpha)}+\alpha^2-2\alpha-\frac{6q_1^2}{s\alpha}+\frac{2q_1^2}{s}\right) \\ & \times\beta+\frac{2q_1^2}{s}-\frac{4q_1^2}{\alpha s}=0; \quad 0<\alpha<1, 0<\beta<1\end{aligned} \tag{1}$$

where $\alpha = h/H$, $q_i = q_i^*/g^{1/2}H^{3/2}$ $(i=1, 2)$, $r = \varrho_2/\varrho_1$, $s = 1-r$, and the trivial solution $\beta=\alpha$ has been factored out.

The solution for the roots of the quartic $\mathcal{K}=0$ represents the conjugate depths to the given flow. For a root to be acceptable, the requirement that a shock must be accompanied by a loss of energy should be satisfied. This requirement for the present problem is

$$\begin{aligned}\Delta E \equiv{} & q_1\left(\frac{q_1^2}{2\alpha^2}+\alpha\right)+rq_2\left(\frac{q_2^2}{2(1-\alpha)^2}+\alpha\right)-q_1 \\ & \times\left\{\frac{1}{2}\frac{q_1^2}{\beta^2}+\beta+\frac{2\left[(\beta-\alpha)\left(\frac{q_1^2}{\alpha}-\frac{1}{2}\beta^2-\frac{1}{2}\alpha\beta\right)\right]}{\beta(\alpha+\beta)}\right\} \\ & -rq_2\left\{\frac{1}{2}\frac{q_2^2}{(1-\beta)^2}+\beta\right. \\ & \left.+\frac{2\left[(\beta-\alpha)\left(\frac{q_1^2}{\alpha}-\frac{1}{2}\beta^2-\frac{1}{2}\alpha\beta\right)\right]}{r\beta(\alpha+\beta)}\right\}>0\end{aligned} \tag{2}$$

where β are roots of $\mathcal{K}=0$.

If the critical state is defined as one which is conjugate ot itself, we get the criticality criterion from (1) by setting $\beta=\alpha$, i.e.,

$$J(\alpha;q_1,q_2,r)\equiv\frac{q_1^2}{\alpha^3}+\frac{rq_2^2}{(1-\alpha)^3}-s=0 \tag{3}$$

In terms of the modified layer Froude numbers

$$\mathcal{F}_1^2 = u_1^{*2}/sgh, \; \mathcal{F}_2^2 = u_2^{*2}/sg(H-h)$$

where u_i^*, $i=1, 2$ are the dimensional layer velocities, (3) can be written in the simple form

$$\mathcal{F}_1^2 + r\mathcal{F}_2^2 = 1 \tag{4}$$

One can show that the criticality criterion given by (3) or (4) is precisely the condition for no upstream propagation of long infinitesimal waves. The supercritical and subcritical flows are given by $J>0$ and $J<0$, respectively. For prescribed layer discharges, the regions of values of α for which the flow is either supercritical or subcritical are shown in Fig. 2, where J is plotted as a function of α.

In order to find whether a shock represents a jump or a drop in the interface level, we look upon jumps as the steepening of crests and drops as the steepening of troughs of a finite wave. From (3), we obtain

$$F_1^2 = \frac{\alpha s - s\alpha^2}{1-\alpha+r\alpha\gamma} \tag{5}$$

where $F_i^2 = u_i^{*2}/gH$, $i=1, 2$ and $\gamma = F_2^2/F_1^2$. If we move with the lower fluid (5) gives the square of the speed of propagation of a long infinitesimal wave by definition of the critical state. We now replace α by $\alpha+\varepsilon$, $\varepsilon>0$ and compare the resulting value of F_1^2 with that given by (5). Crests steepen if the former is greater than the

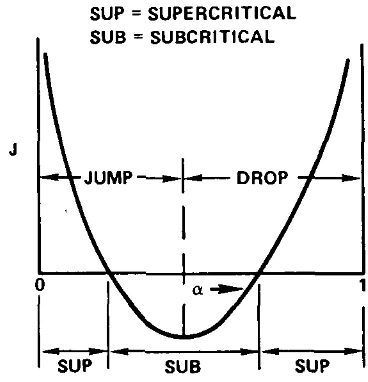

Fig. 2. The function $J(\alpha)$.

latter, while they flatten if it is less (i.e., troughs steepen). This yields the following criteria:

Jump: $\mathcal{M} > 0$ (6)

Drop: $\mathcal{M} < 0$ (7)

where

$$\mathcal{M}(\alpha; q_1, q_2, r) \equiv (q_1^2/\alpha^4) - (rq_2^2)/(1-\alpha)^4 \quad (8)$$

For equal layer discharges and Boussinesq fluids ($r \approx 1$), (6) and (7) yield $\alpha < 1/2$ for a jump and $\alpha > 1/2$ for a drop. Criteria (6) and (7) are precisely those obtained by Long (1956) for solitary waves of elevation and depression in a two-fluid system.

It is interesting to note that $\mathcal{M} \propto -\partial J/\partial\alpha$ and, in view of this, the ranges of values of α for prescribed layer discharges for which a jump or drop solution may be expected are as shown in Fig. 2 (only supercritical values being relevant).

2.1. *Investigation of roots of* $\mathcal{K} = 0$

We first consider the cases when one of the two layers is queiscent (*a*) $q_2 = 0$.

Equation (1) can be written as

$$\mathcal{K} \equiv (\beta - 1)\left(\frac{s\beta}{4} - \frac{s}{2} + \frac{\alpha s}{4}\right)\left(\beta^2 + \alpha\beta - \frac{2q_1^2}{\alpha s}\right) = 0$$

For $0 < \beta < 1$, the only admissible root is given by

$$\beta = (-\alpha + \sqrt{\alpha^2 + 8q_1^2/\alpha s})/2 \quad (10)$$

or

$$\frac{h'}{h} = \frac{-1 + \sqrt{1 + 8\mathcal{F}_1^2}}{2} \quad (11)$$

(*b*) $q_1 = 0$

For this case

$$\mathcal{K} \equiv \beta(\beta + \alpha)\left\{\frac{s\beta^2 - \beta(3s - s\alpha)}{4} - \left(\frac{rq_2^2}{2(1-\alpha)} + \frac{\alpha s}{4} + \frac{s}{2}\right)\right\} = 0 \quad (12)$$

The only admissible root is given by

$$\beta = \frac{\dfrac{3s - s\alpha}{4} - \sqrt{\left(\dfrac{s - \alpha s}{4}\right)^2 + \dfrac{rsq_2^2}{(1-\alpha)}}}{s/2} \quad (13)$$

or, upon rearranging,

$$\frac{H - h'}{H - h} = \frac{-1 + \sqrt{1 + 8r\mathcal{F}_2^2}}{2} \quad (14)$$

Comparing (11) to (14), a certain symmetry between jumps and drops is at once apparent, especially for Boussinesq fluids ($r \approx 1$). We also note that the shock strengths for jumps and drops in Boussinesq fluids, with one of the layers quiescent, are given by expressions identical to that for the classical jump, only with gravity being replaced by the modified gravity.

The plots for $\mathcal{K}(\beta)$ for cases (*a*) and (*b*) are as shown in Fig. 3.

It can be shown that for $0 < \beta < 1$

$$\frac{\partial \mathcal{K}}{\partial q_1} < 0, \quad \frac{\partial \mathcal{K}}{\partial q_2} < 0 \quad (15)$$

In view of the inequalities (15) and the nature of curves $\mathcal{K}(\beta)$ for cases (*a*) and (*b*), it is clear that there are at most two roots of the quartic $\mathcal{K} = 0$ in $0 < \beta < 1$, as shown by the dashed line. When both roots satisfy the enrgy loss criterion given by (2), as they do in some cases, the conjugate state to a given upstream state is not uniquely determinable by the consideration of energy loss alone. In order to resolve the question of non-uniqueness of the conjugate state, a further study of the two roots is necessary.

In addition to the inequalities (15) it can be shown that $\partial\mathcal{K}/\partial r < 0$, $0 < \beta < 1$. This inequality together with those given by (15) are very meaningful, for they reveal that the behavior of the two roots of $\mathcal{K} = 0$ with variations in the flow parameters is quite opposed to each other. Calling the root closer to the given state as the

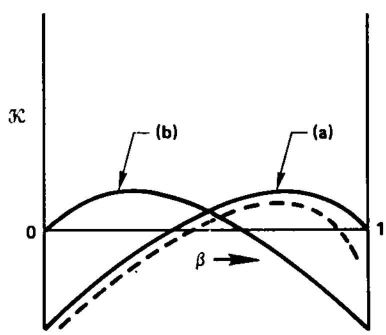

Fig. 3. The function $\mathcal{K}(\phi)$.

first root and the other the second root, the former can be shown to behave in a manner that would be predicted on the basis of an analogy between conjugate states of one- and two-fluid systems, while the behavior of the other is quite the opposite. It is well-known that the shock strength increases with increase in the degree of supercriticality of the flow in the classical case of a hydraulic jump. For two-fluid systems, the degree of supercriticality of the flow increases with an increase of q_1, q_2, and/or r. In view of the inequalities (15), along with $\partial \mathcal{K}/\partial r < 0$, the strength of the shock as given by the first root increases with increase in the degree of supercriticality of the upsteam flow, whereas it actually decreases with the second root is considered. In particular, when J approaches zero, i.e., when the flow approaches the critical condition, the first root approaches the given state, whereas the second root departs further away from it. Physical arguments of this type strongly suggest that it is the first root only that is physically meaningful but do not give a conclusive proof of the nonrealizability of the second root, nor do they throw any light on how the second root comes about. Solution of the total problem answers these objections adequately.

It is interesting to note that, in view of the above inequalities, a degree of supercriticality of the given flow may be reached, above which no shock solutions will be possible. This result is in distinct contrast to that of the classical free-surface problem, in which shock solutions to *any* given supercritical state exist. Such limitations on shock solutions in a two-fluid system are discussed in detail by Mehrotra (1973)

3. The total problem

The definition sketch for the total problem is given in Fig. 4. A two-layer flow confined on top and bottom by walls which are horizontal except for a laterally finite, single crested hump on the lower wall is considered on an inviscid, hydrostatic basis. $\phi^*(x, t)$ is the depth of the lower layer in the neighborhood of the hump which joins the given upstream depth, h, of the lower fluid either continuously or discontinuously. $\hat{q}_i^*(x, t)$, $i = 1, 2$, are the dimensional layer discharges, and $y^* = b^*(x)$ is the equation

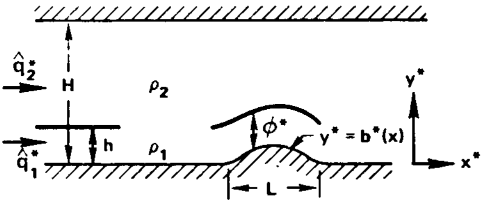

Fig. 4. Definition sketch for flow over a hump.

of the hump. The coordinate axes are as shown in Fig. 4, and the various quantities are made nondimensional in the manner of Section 2.

For flows which are steady and continuous throughout, the constancy of Bernoulli head and mass in each layer yields the following equality:

$$\frac{1}{2}\left(\frac{\hat{q}_1}{\phi}\right)^2 + s(\phi + b) - \frac{1}{2}r\left(\frac{\hat{q}_2}{1-\phi-b}\right)^2 = \left(\frac{\hat{q}_1}{\alpha}\right)^2 + s\alpha - \frac{1}{2}r\left(\frac{1-\alpha}{\hat{q}_2^2}\right)^2 \qquad (16)$$

Let the left-hand side $\equiv f(\phi, b)$ and the right-hand side $\equiv j(\alpha)$. The functions $f(\phi, b)$ and $j(\alpha)$ are plotted in Fig. 5, the former being plotted for values of ϕ and b at the crest. The curves $f(\phi, b)$ for sections between the crest and horizontal floor will lie between these curves. The extrema of these curves occur at the local critical levels, because they are defined by the equations

$$-\frac{\partial f}{\partial \phi} = \frac{\hat{q}_1^2}{\phi^3} + \frac{r\hat{q}_2^2}{(1-\phi-b)^3} - s = 0$$

$$-\frac{\partial j}{\partial \alpha} = \frac{\hat{q}_1^2}{\alpha^3} + \frac{r\hat{q}_2^2}{(1-\alpha)^3} - s = 0 \qquad (17)$$

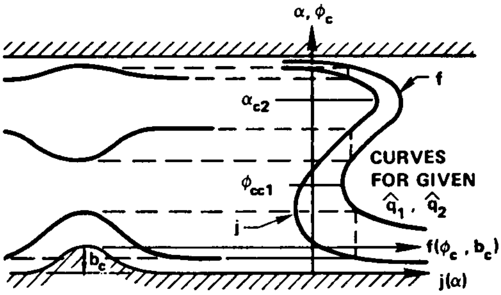

Fig. 5. Some absolutely supercritical and subcritical flow configurations, along with the curves $f(\phi_c, b_c)$ and $j(\alpha)$.

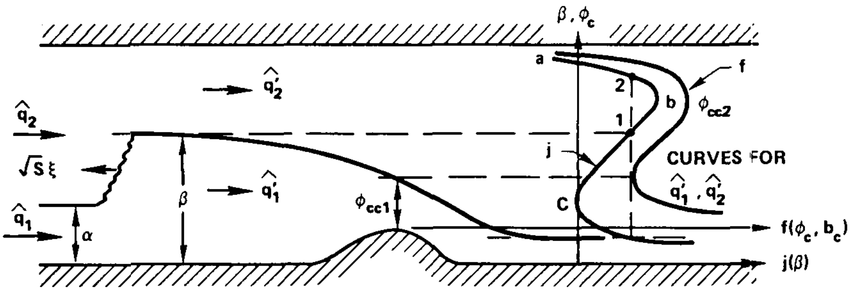

Fig. 6. The total problem for a moving surge.

which represent precisely the local criticality criterion (compare (17) with (3) or (4)). Some steady flow configurations for absolutely subcritical and supercritical flows are sketched in Fig. 5. The lower and upper curves of the interface correspond to supercritical flows, whereas the intermediate one corresponds to a subcritical flow. The curves $j(\alpha)$, $f(\phi, b)$ are analogous to the specific energy curves for the classical free-surface case. Since in the two-fluid case there are two critical levels, these curves bend back after the second higher critical level in contrast to the classical free surface case, where there is no such bending back of the specific energy curve. We shall see later that it is owing to this feature that two conjugate states are obtained for the two-fluid case.

It is clear that for steady continuous solutions to exist, $0 < \alpha < \alpha_{c2}$

$$\frac{1}{2}\left(\frac{\hat{q}_1}{\alpha}\right)^2 + s\alpha - \frac{1}{2}r\left(\frac{\hat{q}_2}{1-\alpha}\right)^2 > \frac{1}{2}\left(\frac{\hat{q}_1}{\phi_{cc_1}}\right)^2 + (s\phi_{cc_1} + b_c) - \frac{1}{2}r\left(\frac{\hat{q}_2}{1-\phi_{cc_1}-b_c}\right)^2 \tag{18}$$

where suffices c and cc_1 refer to values at the crest and at the lower critical level above the crest, respectively. α_{c2} is the value of a α corresponding to the upper critical level.

When the critierion given by (18) is not met the flow may adjust by being controlled by the crest with a critical condition occurring there, and subcritical and supercritical flow occurring upstream and downstream from it, respectively. A steadily moving shock then propagates upstream. In a frame of reference moving with the shock, the isolated problem is obtained. The total problem is sketched in Fig. 6. We note that the functions j and f, defined by (16), now relate to the layer discharges $\hat{q}_1'$ and $\hat{q}_2'$ behind the moving shock, and α in (16) should be replaced by β.

Two important observations must be made here. First, the critical condition at the crest can correspond only to the lower critical level of $f(\phi_c, b_c)$ and, second, state 2 on leg $a-b$ cannot be realized behind the surge. It is easy to see from the curves $f(\phi_c, b_c)$ and $j(\alpha)$ that this must be so if the interface configuration is to be continuous between the shock and the crest.

The complete formulation of the total problem is done for a somewhat simplified case, viz., Boussinesq fluids are assumed and the velocity in each layer ahead of the surge is the same and is u^* (Long, 1970) Such simplification does not in any way alter the arguments to be presented for the uniqueness of the conjugate state. We have the following equations.

Momentum balance for the isolated problem:

$$\frac{(\hat{\mathfrak{J}}+\xi)^2}{\alpha}\left\{\left(\frac{(1-\alpha)^2}{\alpha(1-\beta)} - \frac{(1-\alpha)}{\alpha}\right)\left(\frac{\beta+\alpha}{2\alpha}\right) - \left(\frac{\alpha-\beta}{\beta}\right)\left(\frac{2-\alpha-\beta}{2\alpha}\right)\right\} + \frac{1}{2}\left(\frac{\alpha^2-\beta^2}{\alpha^3}\right)\left(\frac{2-\alpha-\beta}{2\alpha}\right) = 0 \tag{19}$$

Criticality criterion at the crest:

$$\left\{\frac{(\hat{\mathfrak{J}}+\xi)(1-\alpha)}{(1-\beta)} - \xi\right\}^2\left\{\frac{(1-\beta)^2}{(1-\phi_{cc}-b_c)^3}\right\} - \left\{\frac{(\hat{\mathfrak{J}}+\xi)\,\alpha}{\beta} - \xi\right\}^2\frac{\beta^2}{\phi_{cc}^3} - 1 = 0 \tag{20}$$

Constancy of Bernoulli head behind the shock,

$$\frac{1}{2}\left\{(\hat{\mathfrak{J}}+\xi)\left(\frac{1-\alpha}{1-\beta}\right)-\xi\right\}^2\left\{\frac{(1-\beta)^2}{(1-\phi_{cc}-b_c)^2}-1\right\}$$

$$-\frac{1}{2}\left\{(\hat{\mathfrak{J}}+\xi)\frac{\alpha}{\beta}-\xi\right\}^2\left\{\frac{\beta^2}{\phi_{cc}}-1\right\}-(\phi_{cc}+b_c-\beta)=0 \tag{21}$$

Requirement of energy loss trough the shock;

$$\left(\frac{\alpha-\beta}{\alpha}\right)\left\{\frac{\alpha^3}{\beta^2(\alpha+\beta)}-\frac{r(1-\alpha)\alpha^2}{(1-\beta^2(2-\alpha-\beta))}\right\}<0 \tag{22}$$

In the above equations, we have defined

$$\hat{\mathfrak{J}}^2=u^{*2}/sgH \quad \text{and} \quad \xi^2=c^2/sgH, \tag{23}$$

c being the dimensional speed of propagation of the shock. Equations (19) to (21) are to be solved for the three unknowns ξ, β, and ϕ_{cc} subject to (22), given $\hat{\mathfrak{J}}$, α, and b_c. Long (1972) has recently solved this problem numerically. Under some conditions, he found two sets of acceptable solutions corresponding to the two roots of the isolated problem. Long felt that this nonuniqueness should be resolved by experiments. In an attempt to resolve the question of nonuniqueness theoretically, the present authors solved these equations independently in order to study further the nature of the two solutions. Our results are summarized in the following paragraph.

There is a value of $\hat{\mathfrak{J}}=\hat{\mathfrak{J}}_{cr}$ for which the flow is just critical and a value of $\hat{\mathfrak{J}}=\hat{\mathfrak{J}}_{\max}$ beyond which no shock solution for the isolated problem exists. Typical plots of ξ versus b_c are shown in Figs. 7*a* and 7*b*. In Fig. 7*a*, $\hat{\mathfrak{J}}<\hat{\mathfrak{J}}_{cr}$ and, in Fig. 7*b*, $\hat{\mathfrak{J}}_{cr}<\hat{\mathfrak{J}}<\hat{\mathfrak{J}}_{\max}$. When $\hat{\mathfrak{J}}<\hat{\mathfrak{J}}_{cr}$, i.e., the oncoming flow is subcritical, the values of ξ for which $\hat{\mathfrak{J}}+\xi=\hat{\mathfrak{J}}_{cr}$ and $\hat{\mathfrak{J}}+\xi=\hat{\mathfrak{J}}_{\max}$ are ξ_{cr} and $\xi_{\max}$, respectively. Curves 1–2 and 3–4 represent two sets of solutions of Equations (19) to (21). The values of β increases along the cruve 1–2, while they decrease along 3–4. Once again the anomalous character of the second solution is at once apparent; the shock strength actually decreases with an increase in the crest height. We also note that, along cruve 3–4, the shock speed increases with a decrease in shock strength, whereas it increases with an increase in shock strength along curve 1–2, which is the expected behavior. The value of β at point 1 in Fig. 7*a* is equal to α. For the isolated problem with layer velocity equal to $s(\hat{\mathfrak{J}}+\xi_1)$, points A and B represent the two jump solutions corresponding to two different hump heights but with the same upstream flow. Our numerical computations show that points on the cruve 1–2 travel upwards along the leg $c-b$ of Fig. 6, while the points on the curve 3–4 travel downwards along the leg $a-b$. We have already argued that states on leg $a-b$ cannot join the critical state at the crest continuously. Hence the second conjugate state cannot be realized physically by any downstream control. When $\hat{\mathfrak{J}}_{cr}<\hat{\mathfrak{J}}<\hat{\mathfrak{J}}_{\max}$, i.e., the oncoming flow is supercritical (cf. Fig. 7*b*), the above remarks concerning the anamalous character of the second

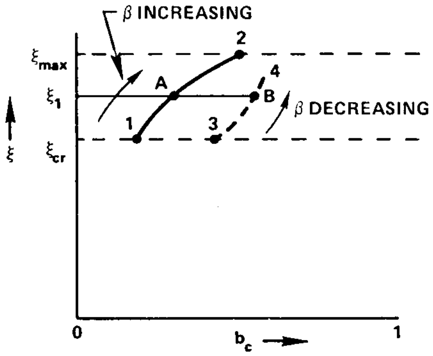

Fig. 7a. Plot of nondimensional velocity (ξ) of the surge versus obstacle height (b_c) for a subcritical oncoming stream ($\hat{\mathfrak{J}}<\hat{\mathfrak{J}}_{cr}$).

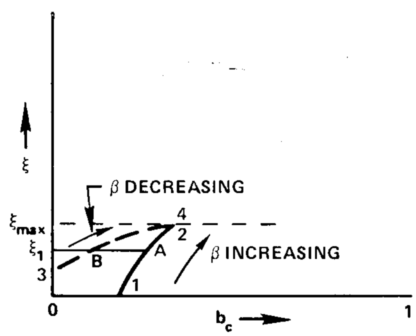

Fig. 7b. Plot of nondimensional velocity (ξ) of the surge versus obstacle height (b_c) for a subcritical oncoming stream when a surge is possible ($\hat{\mathfrak{J}}_{cr}<\hat{\mathfrak{J}}<\hat{\mathfrak{J}}_{\max}$).

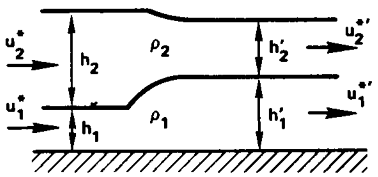

Fig. 8. Definition sketch for the isolated problem when the upper layer has a free surface.

set of solutions (represented by curve 3–4) and its nonrealizability also apply. We also note that the first set allows the existence of a stationary shock, namely, point 1, for a supercritical oncoming flow.

We now establish a suitable correspondence between the isolated problem considered in Section 2 and the one in which the upper layer has a free surface (Yih & Guha, 1955).

4. Correspondence between the rigid top and free surface cases for internal shocks

The definition sketch for the free surface case is given in Fig. 8. The following equations are obtained with the use of the momentum theorem

$$K(a-1) = (b-1)[\{2\hat{F}_2^2/b(b+1)\} - 1] \tag{24}$$

$$(r/K)(b-1) = (a-1)[\{2\hat{F}_1^2/a(a+1)\} - 1] \tag{25}$$

where $a = h_1'/h_1$, $b = h_2'/h_2$, $K = h_1/h_2$,, $\hat{F}_1^2 = u_2^{*1}/gh_1$, $\hat{F}_2^2 = u_2^{*2}/gh_2$, $r = \varrho_2/\varrho_1$.

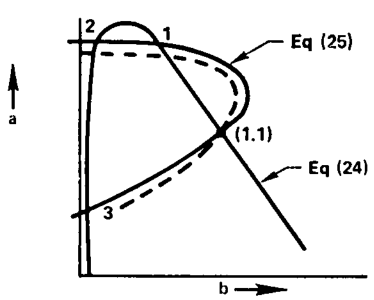

Fig. 9a. Conjugate states for the case of an internal jump when the upper layer has a free surface.

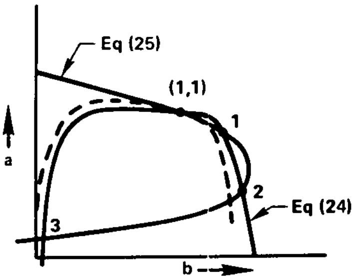

Fig. 9b. Conjugate states for the case of an internal drop when the upper layer has a free surface.

Equations (24) and (25) are to be solved simultaneously for a and b. Equations (24) and (25) are plotted for typical internal jump and drop cases in Figs. 9*a* and 9*b*. Excluding the solution $a = 1$, $b = 1$ (the trivial solution), there are three solutions, one of which is characterized by $a < 1$, $b < 1$. Hayakawa (1970) has demonstrated that this solution is always unacceptable from energy consideration for purely internal shocks. Thus, there are at most two solutions, and we propose to study their behavior with variations in the parameters of the problem. For the jump case, we may reduce $\hat{F}_1$, keeping $\hat{F}_2$ constant, and thus approach the critical condition. Equation (25) behaves as shown by the dotted lines. We note that whereas root 1 approaches the point (1, 1), root 2 does not, in precisely the same way as the roots for the rigid top case do. Similar remarks apply for the drop case when $\hat{F}_2$ is reduced keeping $\hat{F}_1$ constant. Root 1 is the one closer to the given state. We can therefore state that, of the two admissible conjugate states to a given upstream state, the one which is closer to the given state only is physically realizable for both the rigid top and free surface cases, purely internal shocks being considered in the latter case.

5. Acknowledgement

This research has been supported by the National Science Foundation through Grant GA-31247. The results were presented at the International Symposium on Stratified Flows (Novosibirsk, 1972).

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К ВОПРОСУ О НЕЕДИНСТВЕННОСТИ ВНУТРЕННИХ ГИДРАВЛИЧЕСКИХ СКАЧКОВ И ПЕРЕПАДОВ В ДВУХСЛОЙНОЙ СИСТЕМЕ

Детально изучаются сопряженные состояния потока двухслойной устойчивой жидкости, ограниченной сверху и со дна горизонтальными стенками. Из энергетических соображений существует максимум два таких возможных состояния. Единственность устанавливается путем изучения полной задачи, в которой также учитывается причина, вызывающая разрывы. Установлением подходящего соответствия между случаями с жесткой верхней стенкой и со свободной поверхностью в случае чисто внутренних разрывов найдена единственность сопряженного состояния и для последнего случая.