The effect on ocean circulation of a change in the sign of β

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The purpose of this note is to clarify an issue raised originally by Welander (1968) and subsequently discussed by Kamenkovitch & Mitrofanov (1971), Johnson et al. (1971) and Fandry & Leslie (1972). When the depth of the ocean varies, the variation of the parameter f/h, where f is the Coriolis parameter and h is the depth of the (homogeneous) ocean, determines the form of the interior flow. The circulation intensifies to the west if f/h increases northward (as it does in Stommel's (1948) model) and to the east if f/h increases southward. The question is: what happens in the vicinity of the curve for which $(\partial/\partial y)\,(f/h) = 0$ (y is the northward coordinate)?

Fandry & Leslie (1972) discuss the issue and point out the deficiencies of the solutions proposed earlier, but they appear not to have known that Kamenkovitch & Mitrofanov (1971) had solved the problem. They then sketch a plausible solution whose form agrees with the calculated one. In this note we show the results of an experiment which is the laboratory analogue to the oceanic flow. Also, we provide a simple physical argument for the phenomenon and include a calculation for the laboratory flow which serves to confirm the boundary layer solution of Kamenkovitch & Mitrofanov.

The β-effect has been simulated in the laboratory by Stommel et al. (1958). When a pie-shaped basin partially filled with fluid rotates about its apex, the free surface of the fluid has a paraboloidal shape. Now suppose that a constant source or sink of fluid is introduced. The equation for the free surface is

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial r} + \frac{v}{r}\frac{\partial h}{\partial \theta} = 0 \qquad (1)$$

where r, θ are polar coordinates with corresponding velocities (u, v), t is time and h is the depth of the fluid. When the flow is slow, (1) may be approximated by

$$u = -\frac{\partial h/\partial t}{\partial h/\partial r} \qquad (2)$$

where $\partial h/\partial t$ is a constant determined by the rate of filling or draining of the basin and $\partial h/\partial r$ is the radial variation of the undisturbed depth of the fluid. If the tank is being filled ($\partial h/\partial t > 0$), the radial flow is toward the apex (north). Draining ($\partial h/\partial t < 0$) the tank leads to a flow toward the rim (south). This slow (quasi-steady) flow is also geostrophic and it is therefore horizontally non-divergent so that

$$\frac{\partial v}{\partial \theta} = -\frac{\partial}{\partial r}(ru) \qquad (3)$$

When the tank has a flat bottom, h is quadratic in r because of the effect of centripetal acceleration, and since the radial flow is inversely proportional to r, each side of (3) vanishes. Hence, if v vanishes anywhere (as it does at a straight radial wall), it must vanish everywhere. Thus only radial flow is generated and, as Stommel et al. (1958) have shown, a return flow takes place via a western boundary layer. This is the laboratory analogue to the westward intensification in the ocean, with the paraboloidal free surface playing the same role as the variable Coriolis parameter.

Now suppose that the tank has a sloping bottom. The radial flow is still given by (2) but $\partial h/\partial r$ is determined by the paraboloidal free surface *and* the sloping bottom. Since the former

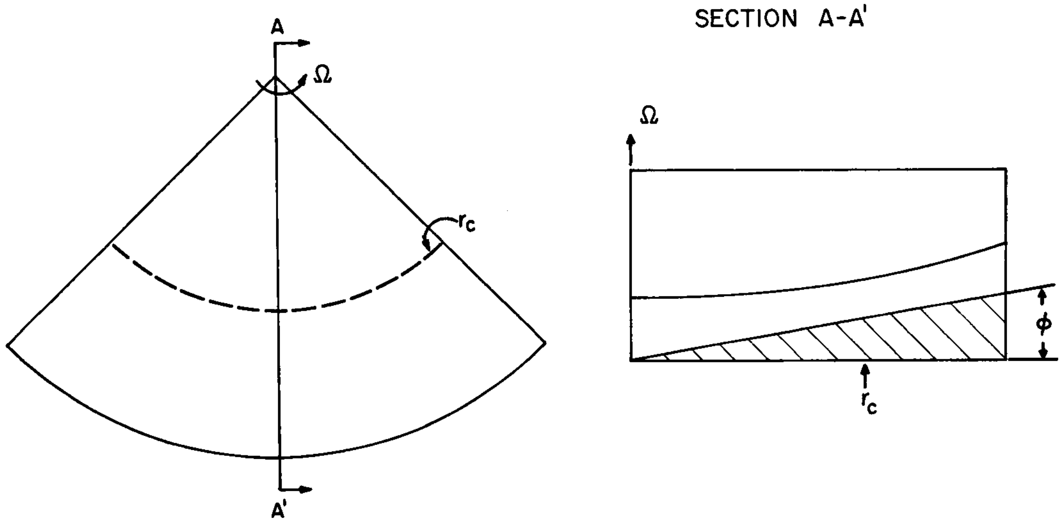

Fig. 1. A view of the top of the pie-shaped basin is shown on the left. The dashed arc, marked r_c, is the critical contour where the depth of the fluid is minimum and the β-effect changes sign. A sketch of a vertical-radial cross section on the right shows that the depth of the fluid is minimum at r_c.

can be looked upon as the β-effect, the latter can be isolated as a topographic effect. Thus even though both effects are due to variable h, the two can be separated in interpretation and the analogue to the oceanic case with variable topography is established.

In particular, we consider in Fig. 1 a linear sloping bottom with the slope such that the depth of the fluid decreases outward to a certain radius, r_c, and then increases. The azimuthal arc through r_c is the critical contour where the "β-effect" changes sign. When the flow is driven by a sink, the radial velocity is toward the apex for $r < r_c$ and away from the apex for $r > r_c$. One can easily see that an eastern boundary layer is generated for $r < r_c$ and a western boundary layer for $r > r_c$, when the rotation is counterclockwise.

For this laboratory flow Kuo & Veronis (1971) have shown that for a certain range of parameters the interior flow is determined by an equation having the same form as that given by Stommel (1948) when the retarding force in the ocean is described by a linear drag law. Thus, in complete analogy to the oceanic case, the nondimensional equation for the stream function ψ (or equivalently, the pressure p) is

$$\varepsilon \nabla^2 p + \alpha \frac{\partial p}{\partial \theta} = 2 \frac{dh}{dt}, \ p = 0 \tag{4}$$

on boundaries where $\varepsilon = 1/2\sqrt{\nu/\Omega a^2}(<1)$; ν is kinematic viscosity; a is the radious of the tank; Ω is the rotation rate; α includes the combined effects of topography and the paraboloidal free surface and is a function of the radius; and $\partial h/\partial t$ is the draining rate. For the linearly sloping bottom, we have

$$\alpha = \frac{\Omega^2 a}{g} - \frac{\tan \phi}{r} \tag{5}$$

where $\Omega^2 a/g$ is the Froude number, whose value is determined by the rotation rate, and ϕ is the angle between the sloping bottom and the horizontal (see Fig. 1). At the critical contour, α vanishes.

The usual boundary layer analysis can be applied to equation (5) and we note that the thickness of the boundary layer along the radial walls is given by

$$\delta = \varepsilon / |\alpha| \tag{6}$$

The boundary layer is on the eastern side for $r < r_c$ and on the western side for $r > r_c$. In the vicinity of $r = r_c$ both boundary layers widen with each thickness approaching infinity as $r \to r_c$. Hence, the boundary layer approximation breaks down and it is necessary to treat the vicinity of r_c as a special region. Kamenkovitch & Mitrofanov (1971) have done so for the

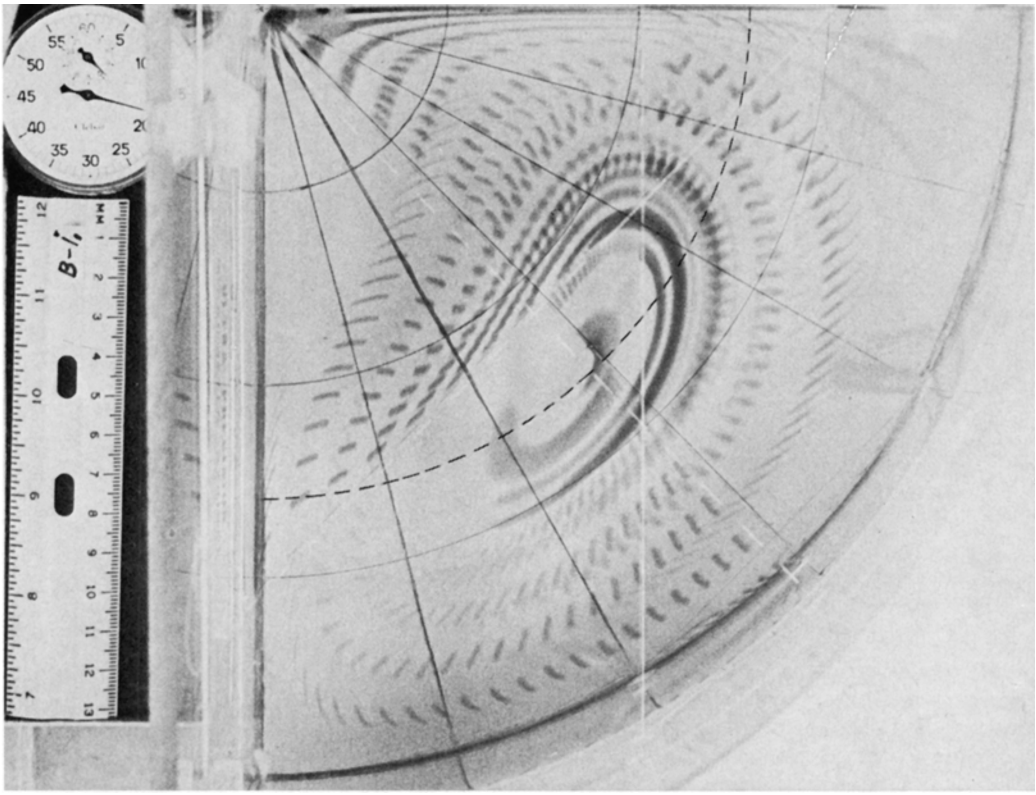

Fig. 2. The flow pattern in the laboratory experiment was made visible by dye streaks generated along the straight wires (faintly visible as dashed white lines). The critical contour, r_c, is marked as the dark, dashed arc. A boundary current is formed on the eastern side for $\alpha < 0 (r < r_c)$ and on the western side for $\alpha > 0 (r > r_c)$. The experimental data are $\Omega = 2.8$ rad sec^{-1}, $\nu = 0.1$ cm^2 sec^{-1}, $a = 20$ cm, mean height of fluid $\simeq 8$ cm, $F = 0.16$, $\phi = 0.1$, $r_c = 12.5$ cm, $dh/dt = 7.8 \times 10^{-4}$ cm sec^{-1}.

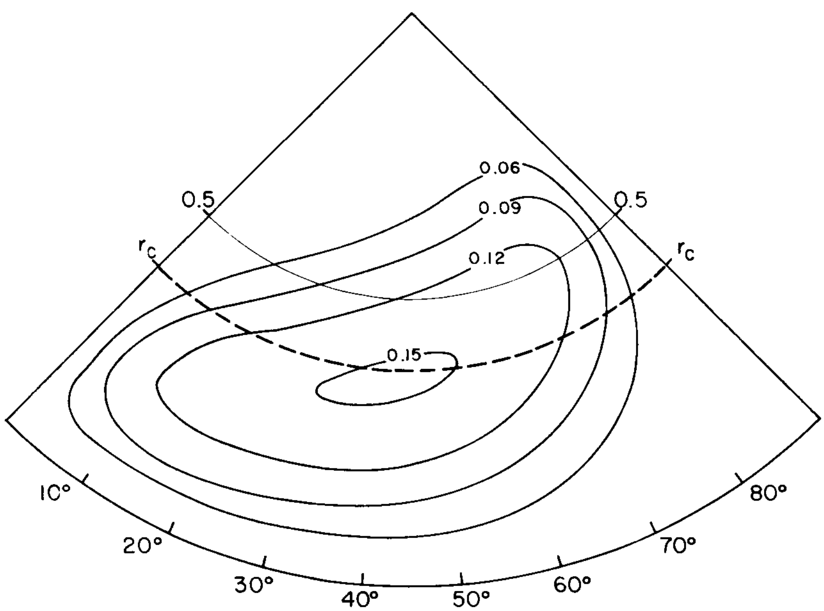

Fig. 3. Calculated contours of p from equation 4 using the experimental parameters listed in Fig. 2.

oceanic case by the use of a boundary layer specifically adapted to this region.

In the laboratory model one can carry out a similar, boundary layer calculation. The situation is complicated here by the existence of additional boundary layers near the apex and the rim. Kuo & Veronis (1971) have shown that the rim boundary layer serves to turn the slow interior flow around to feed the side boundary layer. The boundary layer near the apex distributes the fluid from the side boundary layer to the interior.

Since the foregoing discussion clarifies the physical role of each of the boundary layers, we choose the most expedient method to obtain the overall circulation. Equation (4) has been solved by numerical relaxation for the set of parameters corresponding to the laboratory experiment shown in Fig. 2. The calculated streamlines are shown in Fig. 3. Both patterns show that the flow merges across the critical contour by forming a gyre whose center is near the mid-longitude of the tank. The patterns agree with the one obtained by Kamenkovitch & Mitrofanov (1971) for the oceanic case.

In these flows friction (induced by the bottom Ekman layer for the laboratory case) is important in the boundary layers but not in the interior. As the boundary layers penetrate into the interior, so do the frictional effects and in the vicinity of the critical contour friction is important across the entire basin. In this region the flow resembles that shown by Stommel (1948) where frictional effects were shown to be important in the entire basin when the Coriolis parameter is constant.

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