Eddy diffusion coefficients in the stable atmospheric surface layer

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ABSTRACT

The dependence of the ratio of the eddy heat diffusivity to the eddy momentum diffusivity on the Richardson number is examined in terms of the relationship between the Deacon numbers of paired wind and temperature profiles in the Antarctic surface layer. It is found empirically that for $\mathrm{Ri} \geqslant +0.1$, the diffusivity ratio varies as the minus two-thirds power of the Richardson number, while for $0 < \mathrm{Ri} < +0.1$ no clear empirical relationship exists. It is further shown that the representation of wind and temperature profiles by specific analytical expressions may involve implicit assumptions about the diffusivity ratio, and requires specific relationships between the wind and temperature profile Deacon numbers. Such relationships are not observed in this empirical study.

Introduction

Recently a number of papers have examined the variation of the relationship between wind and temperature profiles with increasing stability in the atmospheric surface layer. While such a relationship involves fluxes of momentum and sensible heat, vertical gradients of wind and temperature, and the ratio of eddy diffusion coefficients, these studies have generally limited themselves to determining the validity of the log-linear law (Monin & Obukhov, 1954) in representing the variation of gradients with a stability parameter such as the Richardson number, and have subsequently inferred the structure both of the diffusion coefficients and of the fluxes.

McVehil (1964) found a reasonable fit between the log-linear law and observed profiles at moderate Richardson numbers although he recognized departures from similarity in averaged data obtained in the Antarctic. O'Brien (1965) examined the statistics of wind and temperature profiles averaged by stability classes, and found that the log-linear law was in adequate agreement with the observations. Oke (1970) and Webb (1970) found that the log-linear law represented the observed condition quite well to relatively high Richardson numbers, although Webb found that the profiles reverted to a simple logarithmic form at Richardson numbers above 0.16. Businger et al. (1971) determined that the stable boundary layer had a quite simple wind and temperature structure which could be represented by linear variations of the non-dimensional gradients with a stability parameter. The last three papers deal explicitly with the problem of the diffusion coefficient ratio and conclude that over the range of stabilities considered, the ratio does not differ significantly from unity.

The common conclusion of these studies is that with suitably chosen parameter values, the log-linear law—which had been derived for small deviations of the Richardson number about a neutrally stable stratification—does in fact generate profiles which do not differ significantly from observed profiles. In using such a technique, however, it is possible that implicit assumptions about the diffusion coefficient ratio in the analytic definitions of the wind and temperature profiles may affect conclusions drawn about the variation of the ratio with stability, hence it is necessary to separate results that follow directly from the data and results that follow from the mathematical treatment of the data.

The present study deals with two related aspects of the surface layer problem. The first is to examine specific relationships between the Deacon numbers of individual wind and temperature profiles which would be expected as analytical consequences of mathematical techniques and model assumptions, both outside the context of a specific model of the surface layer, and in the context of the log-linear law.

The second is to examine the actual relationship between the Deacon numbers of paired wind and temperature profiles under stable surface conditions, to define their variation with the Richardson number, and based upon that variation attempt a definition of the ratio of the eddy diffusion coefficients.

The advantage of the Deacon number for this analysis lies in the fact that it depends only upon the profile curvature, not upon an assumed relationship between the vertical momentum and heat fluxes. It also does not introduce *a priori* assumptions about the eddy diffusion coefficients.

The data used to derive the Deacon numbers was extracted from the Antarctic micrometeorological summary of Dalrymple (1961) for the South Pole which has the particular advantage of combining relatively high values of both wind and wind shear with stable temperature stratifications. This same effect, however, tends to keep the magnitude of the computed Richardson numbers low, since extremely high values occur generally at mid-latitudes under radiatively produced inversions, when both the wind and the wind shear are going to zero.

The Deacon number and the log-linear law

Deacon (1949) found that the vertical wind shear in the diabatic surface layer could be represented by a function of the form

$$\frac{du}{dz} = az^{-\beta} \tag{1}$$

with β greater than unity for unstable conditions, equal to unity for neutral conditions, and less than unity for stable conditions. The constant term a is given by $u^*/kz_0^{1-\beta}$ where u^* is the friction velocity, defined as the square root of the shearing stress divided by the density of the air, k is von Karman's constant, and z_0 is the surface roughness length. The parameter β is now generally called the Deacon number (usually given as De) and may be solved for in equation (1) if $d\beta/dz$ is sufficiently small to be neglected.

$$De \equiv \beta = -\frac{d\ln u'}{d\ln z} = -\frac{zu''}{u'} \tag{2}$$

Primes are used to indicate a vertical derivative, and u is the horizontal wind speed.

A relation similar to (1) may be defined for the vertical temperature gradient which also may be expressed in terms of its Deacon number

$$DE = -\frac{d\ln \theta'}{d\ln z} = -\frac{z\theta''}{\theta'} \tag{3}$$

where θ is the potential temperature.

While the Deacon number is a convenient parameter for expressing the vertical curvature of the wind and the temperature profiles, its variation with the Richardson number (as an example of a stability parameter) is not explicitly defined, and its use was generally supplanted by the mathematically more appealing log-linear law, which is essentially a Maclaurin's series expansion of a universal function $\phi(z/L)$ as defined by Monin & Obukhov (1954).

The log-linear law is given by

$$u' = \frac{u^*}{kz}\phi_m(z/L) = \frac{u^*}{kz}\left(1+\frac{\alpha z}{L}\right) \tag{4}$$

where z/L is a stability parameter, and α is an arbitrary constant.

The analogous form of equation (4) for the temperature profile raises the question of the diffusion coefficient ratio. The sensible heat flux may be defined in terms of an arbitrary parameter T^*, unambiguously as

$$\frac{Q_0}{\varrho c_p} = u^*T^* \tag{5}$$

from which it follows, from the general flux-gradient relationship, that

$$\theta' = -\frac{1}{K_h}\frac{Q_0}{\varrho c_p} = -\frac{u^*T^*}{K_h} = -\frac{K_m\phi_m}{K_h}\frac{T^*}{kz} \tag{6}$$

The last relation follows from the substitution of $K_m\phi_m = kzu^*$. In equation (6) K_h and K_m

are the eddy diffusivities for sensible heat and momentum.

The term $K_m \phi_m / K_h$ which appears in equation (6) is normally replaced by a function $\phi_h(z/L)$ to yield similar forms of the expressions for the wind shear and temperature gradient, however, this definition introduces a specific dependence of the diffusivity ratio on the ratio of the universal functions. If, for example, ϕ_m is assumed to equal ϕ_h, the ratio of the diffusion coefficients is equal to one, independent of the stability of the surface layer.

For any given pair of wind and temperature profiles, it is possible to compute the stability in terms of the Richardson number defined as

$$\mathrm{Ri} \equiv \frac{g\theta'}{T(u')^2} \tag{7}$$

For conditions close to neutral stability the parameter z/L can be equated to Ri, and the constant α in equation (4) can be computed unambiguously from observed profiles, see for example Pandolfo, 1966. However, assuming that the temperature profiles are given by (4) and (6), the relationship between z/L and Ri is given by

$$\frac{z}{L} = \frac{K_h}{K_m} \mathrm{Ri}\, \phi_m = \frac{K_h}{K_m} \mathrm{Ri} \left(1 + \alpha \frac{7}{L}\right) \tag{8}$$

which includes both α and K_h/K_m as unknowns in a single equation. While α is normally assumed to be a constant, it is reasonable to suppose that K_h/K_m is a function of stability in the general diabatic case (cf. Munn, 1966), although the precise relationship is not defined analytically.

There are, of course, a number of ways of side-stepping this problem. O'Brien (1965) avoids it by using z/L, rather than Ri, as a stability index. Oke (1970) arbitrarily sets K_h/K_m equal to unity and $\alpha = 5$ in fitting the Monin-Obukhov function to the observations, while Webb (1970) estimates α and L independently, using the log-linear laws for both the wind and the temperature profile, and the definitions of Ri and L. His analysis leads to an expression of the Richardson number in the form

$$\mathrm{Ri} = \frac{z}{L}\left(1 + \frac{\alpha z}{L}\right)^{-1} \tag{9}$$

which, however, amounts to an assumption of $K_h/K_m = 1$ in equation (8) above, and having the effect of isolating α as the one constant to be defined empirically.

Thus, it appears that the reduction of the information content of paired wind and temperature profiles to the four parameters, u^*, T^*, α and L introduces a rigidity into the subsequent analysis of these profiles, which may then bias the conclusions about the variation of α, K_h and K_m with increasing stability. It is clear that any relationship between the wind and temperature profiles must come from parameters that are not as dependent upon assumptions about the structure of the surface layer.

The Deacon numbers as functions of Ri

It was shown by H. Lettau (1962) that the Richardson number and the Deacon numbers can be related if Ri is expressed in natural logarithms and differentiated with respect to the logarithm of the height to give

$$\frac{d \ln \mathrm{Ri}}{d \ln z} = -2\frac{d \ln u'}{d \ln z} + \frac{d \ln \theta'}{d \ln z} = 2\,\mathrm{De} - \mathrm{DE} \tag{10}$$

Since the Deacon numbers are not initially assumed to be functionally dependent upon one another at any given height, the logarithmic height derivative of the logarithm of the Richardson number is not fixed in advance, although in any practical situation, either (or both) the left-hand side or the right-hand side of (10) are known.

Consequences of the log-linear law

If the vertical wind and temperature profiles in the surface layer are specified in an analytically similar way as for example, in (4) and (6), then at any given height,

$$\theta' = -\frac{T^*}{u^*}\frac{K_m}{K_h} u' \equiv \gamma \frac{K_m}{K_h} u' \tag{11}$$

where γ is defined as $-T^*/u^*$. Equation (11) is in fact true whenever the wind shear is proportional to the temperature gradient, and does not depend upon the specific properties of the log-linear law, however, since this law is thought to best represent conditions in a stable atmosphere it has been taken to illustrate this point.

When $\gamma u'$ is substituted into the defining equation for DE, it follows that

$$\delta \equiv \mathrm{De} - \mathrm{DE} = \frac{z}{\gamma}\frac{\partial \gamma}{\partial z} + \frac{zK_h}{K_m}\frac{\partial}{\partial z}\left(\frac{K_m}{K_h}\right) \quad (12)$$

The difference of the two Deacon numbers has been defined as δ. Since γ is constant for any given experimental situation in which the flux ratio is constant, the value of δ is a measure of the vertical variation of K_m/K_h. If this ratio is constant with height, then De must equal DE in all cases. This, of course, follows directly and intuitively from the assumption under which (11) was derived, however, the magnitude of the variability of the diffusivity ratio with height can be assessed through (12) when actual values of De and DE are computed from the observed individual profiles.

The logarithmic height variation of the Richardson number defines a second parameter, $\Delta \equiv 2\mathrm{De} - \mathrm{DE}$.

This vertical variation of the Richardson number becomes fixed if the individual Deacon numbers are known, or if a specific relation between the Deacon numbers is assumed. If the profiles are similar, that is, if De = DE (see above) it follows that

$$\Delta = \mathrm{De} = \mathrm{DE} = \frac{d \ln \mathrm{Ri}}{d \ln z} \quad (13)$$

The precise variation of Ri with height then depends upon the specific value of the Deacon numbers. For example, if De = DE = 1, the Richardson number varies linearly with height.

In the context of the log-linear law, and assuming similarity of the profiles, the Richardson number is given by equation (8). Differentiating the logarithm of this expression with respect to ln z produces

$$\Delta = \mathrm{De} = \mathrm{DE} = \phi^{-1}(z/L) \quad (14)$$

which implies an inverse variation of the Deacon number with the Monin–Obukhov universal function as necessary in the atmospheric surface layer.

Testing the mathematical consequences

The results of the previous sections indicate that specific assumptions about the form of the wind and temperature profiles lead to specific values of the two parameters δ and Δ. Since these may be computed directly from the vertical curvature of paired wind and temperature profiles, it is possible to compare direct values of the parameters with what may be called the mandated values.

Profile data for this purpose was taken from the wind and temperature tabulations of Dalrymple (1961) in the atmospheric surface layer at Amundsen–Scott base in the Antarctic. Austral winter values were used to insure a stable thermal stratification and positive Richardson numbers, although the relatively high wind shear values precluded extreme Richardson numbers, which in the data sample never exceeded +0.6.

Deacon numbers and parameter values were computed at nominal heights of 50 cm, 100 cm, 200 cm, and 400 cm and are presented in Fig. 1. The open circles refer to ambient Richardson numbers greater than +0.1, while the dots refer to Richardson numbers less than +0.1. The parameters are graphed along these two axes in order to enhance the possible clustering of data points about specific values expected from equations (12) and (14). As a result the independently computed Deacon numbers do not appear in an orthogonal coordinate system and the entire ensemble of data points shows an order that is somewhat spurious., however, the exact dependence of De upon DE for example, is not critical to this development.

The distribution of data points in Fig. 1 shows that the observed range of values of both De and DE is roughly the same from approximately +2 to −1, however, that there is little correlation between paired values of De and DE.

A certain qualitative association between the Deacon numbers and the Richardson number also appears in the data. High values of the Richardson number are generally confined to the upper right quadrant of the diagram and are associated with negative temperature profile Deacon numbers, DE. The wind profile Deacon number, De, seems, however, to be unrelated to variation in the Richardson number.

There is some tendency for points to fall along the line DE = 1, which indicates that the logarithmic temperature gradient is constant with height. With an assumption of zero heat flux divergence it follows that the turbulent heat diffusion coefficient increases linearly with height, which gives the appearance of an adiabatic surface layer. A unit Deacon number, however, does not necessarily imply an adiabatic temperature stratification.

Conclusions about specific parameter values in Fig. 1 are largely negative. There is no apparent tendency for the points to cluster either about the line De = DE, about the line 2De − DE = 1 (cf. Elliott, 1957), or about the line 3De − 2DE = 1 (cf. Stearns, 1968). This last relationship, however, which runs approximately normal to the trend of the points in the Figure and is not shown, was obtained from examining limiting cases and should not be applied directly to the ensemble of points.

Thus, the empirical results, based on this particular data sample, do not point to specific values of δ and Δ as appropriate for satisfying the requirements of a mathematical model. While it is true that the data sample represents extreme conditions, it is also true that an explanation of the structure of the stable surface layer, if it is to have general validity, must be applicable to all situations.

K_h/K_m as a function of the Richardson number

The variation with the Richardson number of the ratio of the turbulent exchange coefficient of heat to that of momentum is of fundamental importance in defining the structure of the stable surface layer, since on the one hand it limits the straightforward applicability of the log-linear law, while on the other hand it determines the point at which the random turbulence of the surface layer passes into a more orderly wave regime.

If the vertical fluxes are assumed nondivergent in the surface layer,

$$\frac{-Q_0}{\varrho c_p \dfrac{\tau}{\varrho}} \equiv F_r = \frac{K_h}{K_m}\frac{\theta'}{u'} \simeq \frac{K_m \Delta u}{K_h \Delta\theta} \tag{15}$$

where F_r is defined as the flux ratio, and $\Delta\theta$ and Δu are temperature and wind speed differences over the same vertical height difference. The ratio K_h/K_m will, therefore, vary inversely with the ratio of $\Delta\theta/\Delta u$, which is an easily determined quantity for a suitable range of Richardson numbers. Webb (1970) found that the ratio $\Delta u/\Delta\theta$ did not vary appreciably with Ri in observations in Australia, however, McVehil (1964) found that a parameter P, defined as

$$P \equiv \left(\frac{\Delta\mu}{\Delta\theta}\right)_1 \bigg/ \left(\frac{\Delta\mu}{\Delta\theta}\right)_2 \tag{16}$$

where the subscripts 1 and 2 refer to an upper and lower height interval, showed a definite decrease with Ri in Antarctic data, but remained constant in the O'Neill data. He recognized a clear difference in profile behavior at the two sites, but felt the evidence only called for not rejecting a possible decrease in K_h/K_m with increasing Richardson number.

An alternate relationship between K_h/K_m and the gradients of temperature and wind may be derived from the Deacon numbers. From the definitions of the vertical heat and momentum fluxes, and equation (1), the flux ratio may be expressed as

$$F_r = \frac{u^* T^*}{(u^*)^2} \cdot \frac{K_h}{K_m}\left(\frac{z}{z_0}\right)^{\delta} \tag{17}$$

Since a zero value for δ is a consequence of any assumption of proportionality between the vertical temperature gradient and the wind shear, it normally follows that the diffusivity ratio is independent of height if the flux ratio is constant with height. However, $\delta = 0$ is not a good assumption for the Antarctic profiles, and in fact since δ can be either positive or negative it is not even possible to specify *a priori* the direction of variation of the diffusivity ratio.

Making the somewhat better assumption that F_r is independent of height, equation (17) shows that K_h/K_m must vary as $(z/z_0)^{-\delta}$. Expressing this in logarithms and differentiating with respect to the logarithm of the Richardson number, yields

$$\frac{\partial \ln(K_h/K_m)}{\partial \ln \mathrm{Ri}} = \frac{\partial \ln(K_h/K_m)}{\partial \ln z}\,\frac{\partial \ln z}{\partial \ln \mathrm{Ri}} = -\frac{\delta}{\Delta} \tag{18}$$

The last relationship follows from the definitions of δ and Δ. At high Richardson numbers

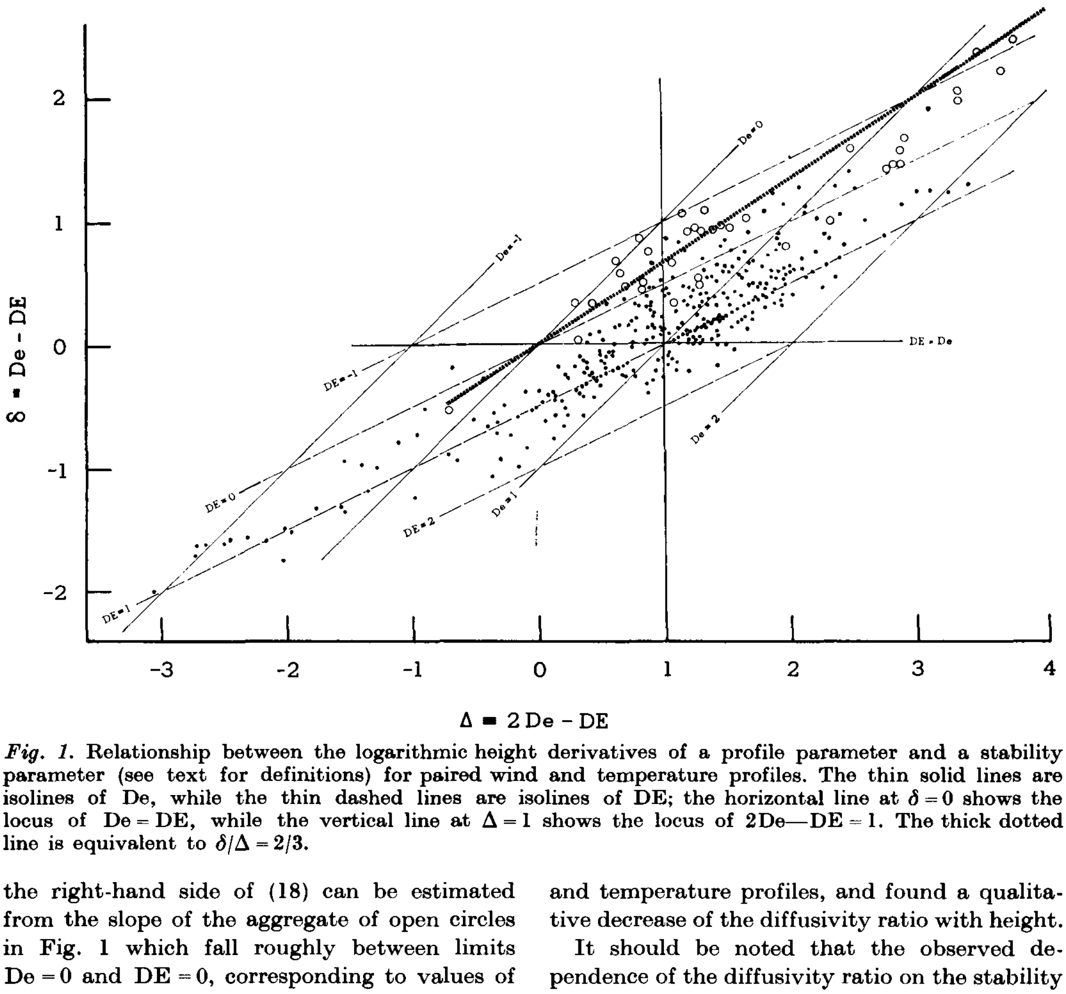

Fig. 1. Relationship between the logarithmic height derivatives of a profile parameter and a stability parameter (see text for definitions) for paired wind and temperature profiles. The thin solid lines are isolines of De, while the thin dashed lines are isolines of DE; the horizontal line at $\delta = 0$ shows the locus of De = DE, while the vertical line at $\Delta = 1$ shows the locus of 2De—DE = 1. The thick dotted line is equivalent to $\delta/\Delta = 2/3$.

the right-hand side of (18) can be estimated from the slope of the aggregate of open circles in Fig. 1 which fall roughly between limits De = 0 and DE = 0, corresponding to values of δ/Δ of 1 and 0.5, and are in fact represented reasonably well by the line $\delta/\Delta = \frac{2}{3}$ which has been included in the diagram. The minus two thirds power dependence of the diffusivity ratio on the Richardson number is in qualitative agreement with Ellison's (1957) contention that the ratio decreases with increasing Richardson number such that their product remained finite, and with Stearns' (1968) discussion of the diffusivity ratio varying as $(\phi_m)^{-\frac{1}{2}}$.

Various individual determinations of the diffusivity ratio under stable conditions (e.g. Munn, 1966, H. Lettau & Davidson, 1957) show that it is generally less than one, indicating qualitatively at least a decrease from its neutrally stable value with increasing Richardson number. McVehil (1964) examined the characteristics of averaged Antarctic wind and temperature profiles, and found a qualitative decrease of the diffusivity ratio with height.

It should be noted that the observed dependence of the diffusivity ratio on the stability appears to be limited to Richardson numbers exceeding $+0.1$, since the values for Ri $< .1$ in Fig. 1 do not show the same degree of organization and in fact do not appear to pass through the origin in the (δ, Δ) coordinate system. It is also limited by the necessary assumption of a constant flux ratio, which, however, according to McVehil (1964), is quite reasonable even for the Antarctic surface layer.

Discussion

The essential conclusion that may be drawn from this analysis is that the eddy diffusivities of heat and momentum are not very well specified in any analytical sense; that in any given experimental situation they represent simply the factors relating ambient flux and gradient values, and as such they have very little extrapolative value to other experimental situations. A large portion of the thrust of surface-layer meteorology has in fact been to find the fixed parameter to which the flux-gradient relationship may be anchored.

It is possible to design separate self-consistent systems for the vertical structure of both the wind and temperature which involve specific values of the vertical flux, the eddy diffusion coefficients, and a measure of diabatic effects. The only rule that must be observed is that if the flux is non-divergent, whatever the variation of the gradient may be, the eddy diffusion coefficient must vary inversely with it. Since the information content of such a system is quite low, it is desirable to increase it by consolidating the two systems and reducing the number, of variables involved however, since there are neither analytical nor empirical rules for such a consolidation, any expression relating ratios of either flux values, eddy diffusion coefficients, or measures of diabatic effects must originate in some arbitrary assumption concerning the relative structure of the wind and temperature profile. The degree of arbitrariness may be assessed by noting that there is essentially no correlation between De and DE in Fig. 1. In view of this it appears that the question of the structure of the stable atmospheric surface layer will not be completely resolved without a substantial body of precise observations of heat and momentum flux values and wind and temperature gradients.

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КОЭФФИЦИЕНТЫ ТУРБУЛЕНТНОЙ ДИФФУЗИИ В УСТОЙЧИВОМ ПРИЗЕМНОМ СЛОЕ АТМОСФЕРЫ

Зависимость отношения коэффициентов турбулентного обмена для тепла и количества движения от числа Ричардсона исследуется с помощью соотношений между числами Дикона для профилей ветра и температуры, одновременно измеренных в приземном слое воздуха в Антарктике. Эмпирически найдено, что отношение коэффициентов обмена изменяется как степень — 2/3 числа Ричардсона для $Ri \geqslant +0,1$, в то время как при $0 < Ri < +0,1$ не существует простого эмпирического соотношения. Далее показано, что представление профилей ветра и температуры специфическими аналитическими выражениями может неявно подразумевать предположения об отношении коэффициентов обмена, а также требует специальных соотношений между числами Дикона для профилей температуры и ветра. Такие соотношения не обнаружены в данной эмпирической работе.