One-dimensional diffusion equations for the vertical transport in an oscillating stratified lake of varying cross-section

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ABSTRACT

Lateral uniformity of stratified lakes permits the description of vertical diffusion in them by a one-dimensional equation, with full regard to their three-dimensional nature. Such an equation should not be obtained simply by neglecting the horizontal divergence, because then significant terms are discarded. The complete equation is obtained by integration of the full three-dimensional equation over the volume under isopycnals. Two functions appear in it that do not appear in the simple equation (without horizontal divergence): the horizontal cross-section of the lake, and the surface sources. These correspond to two effects: the variation of the horizontal cross-section of the lake with depth and the existence of sources or sinks on the lake-bottom at different depths. The same equation can be applied to oscillating lakes, by describing temperatures and concentrations as functions of a new independent variable, the volume under isopycnals. The diffusivities and sources in the equation must then be interpreted as weighted averages, influenced by the correlation between the quantities appearing in them.

1. Introduction

Many lakes are stably stratified, through some part of their depth, by vertical density gradients caused by temperature or salinity gradients. The stability of the stratification prevents large-scale vertical convection and mixing, but not usually local turbulence. Any vertical transport of heat and solutes is carried, under these conditions, by eddy- and molecular-diffusion.

Most lakes are also uniform laterally (this condition, to be defined later, should be distinguished from the stricter condition of horizontal uniformity), partly because the external conditions are more or less the same all over the lake, partly because the lateral diffusion and mixing is much stronger than the vertical. Isopycnals (equal density surfaces), isotherms and equal concentration surfaces of any solute would tend then to coincide. Without any external disturbance, hydrostatic equilibrium would require that the isopycnals in the lake would be exactly horizontal. This condition, hereafter called *Horizontal Uniformity*, occurs rarely, if at all, in real lakes. Under the influence of wind, tide and seiches, the isopycnals tilt and oscillate. Nevertheless, the oscillations do not usually cause large-scale vertical mixing, diffusion remaining the mechanism of transport between adjacent isopycnals. Hence the importance of the diffusion equations for the understanding of the distribution and transport of heat and matter in lakes.

Because of their lateral uniformity, one-dimensional vertical diffusion models were often applied to lakes of the above type. The equations for such models were obtained by taking the general diffusion equations, (1) below, and discarding all the terms that involve the two horizontal dimensions. (Recent authors using this procedure are Lerman & Stiller, 1969; Sweers, 1969, and many others.) This procedure is equivalent to a model of the lake, of uniform depth (or flat floor and vertical sides), whose only bottom-surface sources of heat and matter are on the floor, not at the sides. Taking a column of water at the deepest point and neglecting the horizontal terms of the divergence of the fluxes of heat and matter, is also equivalent to this model. The neglect of the horizontal divergence in such simple models is not always justified.

There are two causes for the existence of horizontal divergence in real lakes. One is the

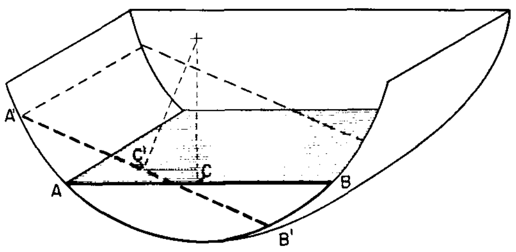

Fig. 1. The illustrated lake has the shape of half a cylinder lying on its side. AB is a horizontal isopycnal; $A'B'$ is the same isopycnal tilted. C', the bisector of $A'B'$, is at the average height of the tilted isopycnal, higher than the horizontal one. The barred area is the horizontal cross section of the lake at the height of C.

variation of horizontal cross-section area of the lake (the plane area bounded by a depth contour) with depth. The second is the existence of sources of heat and matter on the lake bottom at depths above the deepest point. The need to include these effects in the diffusion equations of lakes was already felt. Lerman & Stiller (1969) had to assume such sources to explain discrepancies in H_2S concentrations between observation and a simple model of the types described above. Dutton & Bryson (1962) corrected their heat flux calculations for the variations of horizontal cross-section with depth. Neither formulated a corrected diffusion equation.

Another complication in applying a one-dimensional model to a lake is caused by seiches and internal waves. When these are present, the height of an isopycnal loses its meaning. As Sweers (1968) points out, averaging simultaneous temperature profiles at different points of a lake, or profiles taken during a short time in one or more points, can lead to distortion of the shape of the thermocline. He recommends averaging the depths of isotherms instead. In a lake of constant depth this procedure results indeed in temperature profiles having temperature gradients equal to the average gradient at the isotherms. In many cases it is the best available approximation, but in a lake of varying depth it is not necessarily true, as the following imaginary example shows (Fig. 1). In the illustrated lake, tilting of an isopycnal will change its average depth and the vertical distance between two isopycnals tilted to different angles will depend on the two angles.

A quantity that does not change with oscillations, neglecting the mixing for a moment, is the volume of water under isopycnals (hereafter called *Sub-isopycnal Volume*). This suggests that the choice of this quantity as the independent variable instead of the isopycnal height, will lead to equations, and will enable a representation of measurements, that are applicable in the presence of oscillations.

In this paper I develop a set of one-dimensional diffusion equations which include the horizontal divergence, transform the independent variable to the sub-isopycnal volume, and then I examine the terms obtained by averaging to smooth away oscillations of the lake. In the development of the equations only two assumptions are made. One is the *Assumption of Stratification:* The layer of the lake that is under consideration is stably stratified. This means that there are no large scale convection currents and mixing, and in particular that the oscillations in the lake do not change sub-isopycnal volumes by large-scale mixing. The transport of heat and solute between adjacent isopycnals can be described then as a diffusion process. The second is the *Assumption of Lateral Uniformity:* The isotherms and the equal concentration surfaces of the solutes under consideration coincide with the isopycnals.

2. The horizontally uniform lake

The following equations are taken as the basic balance equations of mass, heat and solute. The last two are named usually the diffusion equations.

$$\frac{\partial \varrho}{\partial t} = -\nabla \cdot \varrho \mathbf{v}$$

$$\frac{\partial}{\partial t}\varrho c T = \nabla \cdot \varrho c K \cdot \nabla T - \nabla \cdot \varrho c T \mathbf{v} + H$$

$$\frac{\partial}{\partial t}\varrho \omega = \nabla \cdot \varrho D \cdot \nabla \omega - \nabla \cdot \varrho \omega \mathbf{v} + R \qquad (1)$$

where

ϱ is the density
t the time
$\mathbf{v}$ the velocity of flow
c the heat capacity
T the temperature
K the heat diffusivity tensor (including turbulent diffusivity)

H the source of heat per unit volume
ω the mass fraction of a solute
D the diffusivity tensor of the solute (including turbulence)
R the source of solute per unit volume

Integrating over the volume of water below height h, measured from the deepest point in the lake, and using the divergence theorem to convert volume integrals to surface integrals, we obtain for a horizontally uniform lake

$$\int_0^h A \frac{\partial \varrho}{\partial t} dz = -A(h)\varrho v_z - \int_0^h \varrho v_n \, dC$$

$$\int_0^h A \frac{\partial}{\partial t} \varrho c T dz = \varrho c \, A(h)\,(K \cdot \nabla T)_z + \int_0^h \varrho_C c_C (K \cdot \nabla T)_n \, dC - \varrho c \, A(h) T v_z - \int_0^h \varrho_C c_C T_C v_n \, dC + \int_0^h AH \, dz$$

$$\int_0^h A \frac{\partial}{\partial t} \varrho \omega \, dz = \varrho \, A(h)\,(D \cdot \nabla \omega)_z + \int_0^h \varrho_C (D \cdot \nabla \omega)_n \, dC - \varrho \, A(h) \omega v_z - \int_0^h \varrho_C \omega_C v_n \, dC + \int_0^h AR \, dz \quad (2)$$

where

z is the vertical coordinate, measured upward from the deepest point of the lake. As a subscript it marks the vertical component of a vector.
C is the surface area of the part of the bottom of the lake that is bounded by the contour at height z
n subscript, marks the component of a vector that is perpendicular to the lake-bottom, positive outwards
$A(z)$ is the horizontal cross-section of the lake at height z

As all the properties of the lake are constant over the horizontal isopycnals, the integration over the isopycnals becomes simple multiplication by their area.

Differentiating with respect to the height, a set of one dimensional equations is obtained.

$$A \frac{\partial \varrho}{\partial t} = -\frac{\partial}{\partial z} A \varrho v_z - \varrho_C v_n \frac{\partial C}{dz}$$

$$A \frac{\partial}{\partial t} \varrho c T = \frac{\partial}{\partial z} \varrho c \, A (K \cdot \nabla T)_z - \frac{\partial}{\partial z} \varrho c \, A T v_z + [\varrho_C c_C (K \cdot \nabla T)_n - \varrho_C c_C T_C v_n] \frac{dC}{dz} + AH$$

$$A \frac{\partial}{\partial t} \varrho \omega = \frac{\partial}{\partial z} \varrho A (D \cdot \nabla \omega)_z - \frac{\partial}{\partial z} \varrho A \omega v_z + [\varrho_C (D \cdot \nabla \omega)_n - \varrho_C \omega_C v_n] \frac{dC}{dz} + AR \quad (3)$$

These equations can be simlified further. Because the horizontal gradients vanish, we get

$$(K \cdot \nabla T)_z = K_z \, \partial T / \partial z$$

$$(D \cdot \nabla \omega)_z = D_z \, \partial \omega / \partial z \quad (4)$$

The second term on the right in (3_1), and the terms in brackets in $(3_{2,\,3})$, are the addition of mass, heat and solute from the bottom-surface at height z, by flow, advection and diffusion. We can treat them as surface sources.

The slope of the bottom of lakes is usually small. Rarely it exceeds a gradient of 0.1. We can write

$$dC/dz = \eta dA/dz \quad (5)$$

where η is the average of cosarctan(gradient) of the bottom-surface at that depth. Because cosarctan $0.1 = 0.995$, η can be equated to one in most cases with a very small error. Alternately, it can be incorporated into the surface sources.

Now the one dimensional equations can be written in a more concise and convenient form

$$A \frac{\partial \varrho}{\partial t} = -\frac{\partial}{\partial z} A \varrho v_z + IA'$$

$$A \frac{\partial}{\partial t} \varrho c T = \frac{\partial}{\partial z} \varrho c \, A K_z \frac{\partial T}{\partial z} - \frac{\partial}{\partial z} \varrho c \, A T v_z + QA' + AH$$

$$A \frac{\partial}{\partial t} \varrho \omega = \frac{\partial}{\partial z} \varrho \, A D_z \frac{\partial \omega}{\partial z} - \frac{\partial}{\partial z} \varrho \, A \omega v_z + SA' + AR \quad (6)$$

where

$A' = dA/dz$
I is the bottom-surface source of mass per unit area
Q the bottom-surface source of heat per unit area
S the bottom-surface source of solute per unit area

In a lake with a rounded bottom, A tends to zero as z goes to zero. A' can tend, at the same time, to a constant, to infinity or to zero, as the case may be. In the last case A' tends to zero more slowly than A. An imaginary example for a constant A' would be a lake that has the shape of a praboloid near its bottom, in which A is proportional to z while A' is constant. In any case the only terms left in (6), when z tends to zero, are the terms containing A'. Equations (6) become then

$$-\varrho v_z + I = 0$$

$$\varrho c K_z \frac{\partial T}{\partial z} - \varrho c T v_z + Q = 0$$

$$\varrho D_z \frac{\partial \omega}{\partial z} - \varrho \omega v_z + S = 0 \qquad (7)$$

In lakes that are stratified down to their deepest point, these equations are the bottom boundary conditions for (6).

The volume of water, V, below height z is a monotonic function of z. Using the relation

$$A = dV/dz \qquad (8)$$

equations (6) can be transformed from equations in z to equations in V. Dividing by A we get

$$\frac{\partial \varrho}{\partial t} = -\frac{\partial}{\partial V}\varrho G + I A'/A$$

$$\frac{\partial}{\partial t}\varrho c T = \frac{\partial}{\partial V}\varrho c A^2 K_z \frac{\partial T}{\partial V} - \frac{\partial}{\partial V}\varrho c T G + Q A'/A + H$$

$$\frac{\partial}{\partial t}\varrho \omega = \frac{\partial}{\partial V}\varrho A^2 D_z \frac{\partial \omega}{\partial V} - \frac{\partial}{\partial V}\varrho \omega G + S A'/A + R \qquad (9)$$

where $G = A v_z$ is the total vertical flow.

Equations (6) and (9) are the general balance equations for a horizontally uniform lake. In treating a real lake as a one dimensional object, diffusion equations of the type of (6) are applied to it. This does not necessarily entail that horizontal uniformity was assumed. The meaning that the terms in the balance equations assume, when applied to an oscillating lake, is examined in the next section.

3. The oscillating lake

The isopycnals in a real stratified lake are never horizontal, therefore no real stratified lake is uniform horizontally. The slope of the isopycnals is small even in extreme oscillations, its gradient is typically less than 10^{-3}. Nevertheless, the height difference between the opposite ends of an isopycnal can amount to a few meters. Therefore, we can, on the one hand, treat the diffusion between layers as vertical, and on the other hand we cannot average temperature or concentration profiles over the whole area of the lake.

Because the sub-isopycnal volume is not changed by the oscillations, a solution to the last difficulty would be to calculate the volume under the isopycnals and use it as the independent space variable. Alternately, an equivalent static height of the isopycnals, which is the height that an isopycnal would attain had it been horizontal, can be used. The concept of this height is based on that of the sub-isopycnal volume, the relation between them being

$$\int_0^{z_s} A(z)\,dz = V(z_s) \qquad (10)$$

where

z_s is the equivalent static height
V the sub-isopycnal volume

The equivalent static height is not an average height of the isopycnal in any sense.

In the previous section it was shown that, in a static lake, the balance equations can be formulated in terms of the volume. Yet, the resulting equations (9) cannot be directly applied to an oscillating lake. There are a few additional complications. One of the complications that arise is that, when tilted, the isopycnals are not parallel, and therefore temperature, concentration and density gradients are not constant over each isopycnals. This prevents the surface integrals over the isopycnals from being simply multiplication by the area, as in (2). In addition, the vertical diffusivities change with the density gradient because they depend on the local stability. In order to obtain one dimensional equations, the fluxes have to be averaged over the isopycnals. The dependence of the diffusivities on the gradients introduces a correlation between them. The averages, therefore, cannot be factored into a multiple of averages. Another complication is that these quantities oscillate with the oscillations of the isopycnals, and therefore have to be averaged with respect to the time over several oscillations.

Time averages are meaningful in the diffusion equations because the periods of oscillations are considerably shorter than the time scale of the variations that concern the lymnologist: typically less than a day against weeks.

In the following only the heat equation is analysed. The analysis of the other equations is similar.

Repeating the steps that led from (1) to (6), we obtain by integration over the volume below an isopycnal, converting volume integrals into surface integrals by the divergence theorem

$$\int_0^V \frac{\partial}{\partial t} \varrho c T\, dV = \varrho c\, A_p\, \overline{K_z \frac{\partial T}{\partial z}} + \int_0^V \overline{\varrho_C c_C (K \cdot \nabla T)_n}\, dC_p - \varrho c\, T A_p \bar{v}_z - \int_0^V \overline{\varrho_C c_C T_C v_n}\, dC_p + \int_0^V \bar{H} dV \tag{11}$$

where

A_p is the surface area of the isopycnal

C_p is the surface area of the lake bottom below the isopycnal

and the bar designates space averages: in the first and third terms on the right—over the isopycnal surface, in the second and fourth terms—around it (over dC_p), and in the last term—over the volume element.

In the expressions $K_z \partial T/\partial z$ and v_z, the slight deviation from the horizontal was neglected. To be exact, the diffusivity, the gradient and the velocity must be expressed locally in coordinates normal and tangent to the isopycnal, and the normal components used in the equation.

Differentiating with respect to the volume, introducing the previous notation for the total vertical flow and the surface sources, we have the momentary balance equation

$$\frac{\partial}{\partial t} \varrho c T = \frac{\partial}{\partial V} \varrho c\, A_p\, \overline{K_z \frac{\partial T}{\partial z}} - \frac{\partial}{\partial V} \varrho c\, TG + \bar{Q} \frac{\partial C_p}{\partial V} + \bar{H} \tag{12}$$

which should be averaged over several oscillations to obtain

$$\frac{\Delta}{\Delta t} \varrho c T = \frac{\partial}{\partial V} \varrho c \left\langle A_p\, \overline{K_z \frac{\partial T}{\partial z}} \right\rangle - \frac{\partial}{\partial V} \varrho c T \langle G \rangle + \left\langle \bar{Q} \frac{\partial C_p}{\partial V} \right\rangle + \langle \bar{H} \rangle \tag{13}$$

where the brackets designate short-time averages, and Δt is the period of time averaging.

Equivalently, the equivalent static height can be used. Multiplying by A we get

$$A \frac{\Delta}{\Delta t} \varrho c T = \frac{\partial}{\partial z_s} \varrho c \left\langle A_p \overline{K_z \frac{\partial T}{\partial z}} \right\rangle - \frac{\partial}{\partial z_s} \varrho c\, T \langle G \rangle + A \left\langle \bar{Q} \frac{\partial C_p}{\partial V} \right\rangle + A \langle \bar{H} \rangle \tag{14}$$

The variation of A_p, K_z and $\partial T/\partial z$, and the correlation between them, is rarely known, if at all. Therefore, equations of the type of (6) are always applied to lakes, but the diffusivities and source terms in them are understood to be apparent "overall average" quantities. Comparing (6) with (14) we obtain expressions for the apparent quantities:

$$K_z^* = \frac{\langle A_p \overline{K_z \partial T/\partial z} \rangle}{A \partial T/\partial z_s}$$

$$Q^* = \frac{A \langle \bar{Q}\, \partial C_p/\partial V \rangle}{A'}$$

$$A \frac{\Delta}{\Delta t} \varrho c T = \frac{\partial}{\partial z_s} \varrho c A\, K_z^* \frac{\partial T}{\partial z_s} - \frac{\partial}{\partial z_s} \varrho c\, T \langle G \rangle + Q^* A' + A \langle \bar{H} \rangle \tag{15}$$

where the last equation is the vertical-diffusion equation in its final form.

The apparent vertical diffusivity differs from the average of the local vertical diffusivity over the isopycnals for two reasons. One is the correlation between it, the temperature gradient and the isopycnal area. The second is the difference between the area of an oscillating isopycnal and its static area.

The apparent bottom-surface source includes not only average of sources that would be present in a static lake, but also sources peculiar to oscillating lakes. The bottom sediments absorb heat when in contact with a hot upper isopycnal and return it to the water when in contact with a colder lower one, as described by Dutton & Bryson (1962).

4. Conclusion

Lateral uniformity of stratified lakes permits the description of vertical diffusion in them by one dimensional equation with full regard to their three dimensional nature. The equations include effects formerly neglected in diffusion equations that were applied to lakes, but this had to be paid for. The resultant equations cannot usually be solved analytically. They depend, for their applicability, on some knowledge of the internal dynamics of the lakes, and the meaning of the parameters in the equations is not simple. Nevertheless, the equation lends itself easily to numerical finite difference solution, and today, when fast digital computers are almost universally available, lack of analytical solution is no objection. The calculation of sub-isopycnal volumes requires either some knowledge of the lake dynamics or a representative set of profiles. This may not be available for every lake, but where it is, the present equations will model the lake better than the simpler ones.

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ОДНОМЕРНОЕ УРАВНЕНИЕ ДИФФУЗИИ ДЛЯ ВЕРТИКАЛЬНОГО ПЕРЕНОСА В ОЗЕРЕ ПЕРЕМЕННОГО ПОПЕРЕЧНОГО СЕЧЕНИЯ С ОСЦИЛЛИРУЮЩЕЙ СТРАТИФИКАЦИЕЙ

Горизонтальная однородность стратифицированных озер позволяет описывать вертикальную диффузию в них одномерном уравнении с полным учетом их трехмерной природы. Такое уравнение не следует получать, просто пренебрегая горизонтальной дивергенцией, потому что тогда не будет учтены существенные члены. Полное уравнение получено интегрированием полного трехмерного уравнения по объему, заключенному между изопикническими поверхностями. В нем появляются две функции, которых нет в упрощенном уравнении (без горизонтальной дивергенции): горизонтальное поперечное сечение озера и поверхностные источники. Это соответствует двум эффектам: вариации с глубиной горизонтального поперечного сечения озера и существованию источников или стоков на дне озера на различных глубинах. То же самое уравнение может быть применено к озерам с осциллирующей стратификацией путем описания температур и концентраций как функций новой независимой переменной, объема между изопикнами. Коэффициенты диффузии и источники в уравнении тогда должны интерпретироваться как взвешенные средние, определяемые корреляцией между величинами, появляющимися в них.