Limitations on the existence of shock solutions in a two-fluid system

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ABSTRACT

It is explained why shock solutions exist for only a limited range of values of the parameters of the problem of an immiscible, inviscid and incompressible two-layer supercritical flow of stable stratification confined on top and bottom by horizontal walls. For a given α, the non-dimensional depth of the lower layer, and r, the ratio of the density of the upper fluid to that of the lower fluid, the regions where shock solutions exist can be demarcated in $q_1 - q_2$ space, where q_1 and q_2 are the non-dimensional discharges in the lower and upper layers respectively.

1. Introduction

In the classical case of a homogeneous fluid with a free surface flowing supercritically on a horizontal floor, shock solutions exist for all values of the upstream Froude number greater than one, i.e. it is possible to find a downstream subcritical state which is conjugate to *any* given upstream supercritical state. However, a supercritical two-layer flow admits of internal shock solutions only within a restricted range of supercriticality of the flow. This is true for both the cases when the upper fluid has a free surface and when it is bounded by a rigid top, purely internal shocks being considered in the former case and supercriticality being referred to the internal wave mode of fastest celerity. In studying purely internal shocks it is profitable to dispense with the free surface. This eliminates complexities that arise from the free surface wave mode without in any way obscuring the phenomena under investigation. A two-layer flow confined on top and bottom by horizontal walls is, therefore, considered here on an immiscible, inviscid and incompressible basis.

The existence of shock solutions is studied first as an isolated problem, in which the conjugate states are joined to each other by a stationary shock, without reference to an agency that is responsible for the shock. This is done in section 2. In section 3 the results of the isolated problem are related to the so-called total problem in which an agency that causes the shock (when they exist) is also included. The agency is taken to be a hump on the floor. It is to be noted that the shock in the total problem is, in general, a moving one for prescribed upstream state and the height of the hump. In a frame of reference moving with the shock, the isolated problem can be retrieved.

2. Existence of shocks: isolated problem

The definition sketch for the isolated problem is given in Fig. 1. The depths of the lower layer ahead of, and behind, the shock are h and h' respectively; q_i^* (starred symbols, when they appear, refer to dimensional quantities) and ϱ_i are the layer discharges and densities respectively, ($i = 1, 2$ throughout the text). Suffices 1 and 2 refer to lower and upper layers respectively. The distance between the walls is H. Normalizing lengths by H, densities by ϱ_1, and discharges by $g^{1/2} H^{3/2}$, using momentum theo-

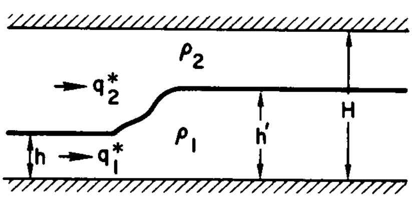

Fig. 1. Definition sketch: Isolated problem.

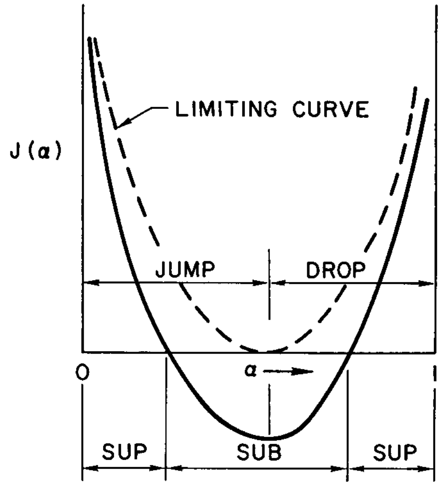

Fig. 2. The function $J(\alpha)$. *Sup*, Supercritical; *Sub*, subcritical.

rem in each layer by turns, and eliminating the unknown pressure at the downstream interface relative to that at the upstream interface, one gets the following equation for $\beta \equiv h'/H$,

$$\begin{aligned}\mathcal{K}(\beta; q_1, q_2, r, \alpha) \equiv \beta^4 - (3-2\alpha)\beta^3 \\ + \left(2 - \frac{2q_1}{s\alpha} - 4\alpha + \alpha^2 - \frac{2q_2^2 r}{s(1-\alpha)}\right)\beta^2 \\ - \left(\frac{2r\alpha q_2^2}{s(1-\alpha)} + \alpha^2 - 2\alpha - \frac{6q_1^2}{s\alpha} + \frac{2q_1^2}{s}\right)\beta \\ + \frac{2q_1^2}{s} - \frac{4q_1^2}{\alpha s} = 0; \quad \begin{matrix} 0<\alpha<1 \\ 0<\beta<1 \end{matrix} \end{aligned} \tag{1}$$

where $\alpha = h/H$, $q_i = q_i^*/g^{1/2}H^{3/2}$, $r = \varrho 2/\varrho_1$ and $s = 1-r$.

In the derivation of (1) the velocities in the two layers are assumed to be uniform, the pressure at the shock front is hydrostatic and shear stresses are neglected. The trivial solution $\beta = \alpha$ has been factored out. The criteria for critical condition and whether the shock represents a jump or drop in the interface level have been found by Mehrotra & Kelly (1972), among others, and are given by,

Critical condition:

$$J(\alpha; q_1, q_2, r) \equiv \frac{q_1^2}{\alpha^3} + \frac{rq_2^2}{(1-\alpha)^3} - s = 0 \tag{2}$$

Jump: $\mathcal{M} > 0$

Drop: $\mathcal{M} < 0$

where

$$\mathcal{M}(\alpha; q_1, q_2, r) \equiv \frac{q_1^2}{\alpha^4} - \frac{rq_2^2}{(1-\alpha)^4} \tag{3}$$

For a given α, supercritical and subcritical flows are defined by $J>0$ and $J<0$ respectively. We note that

$$\mathcal{M} = \frac{-\partial J}{\partial \alpha} \tag{4}$$

In Fig. 2, a typical plot of J as a function of α is shown for prescribed q_1, q_2 and r. The ranges of values of α for which the flow is supercritical or subcritical are shown. In view of (4) the ranges of values of α for which jump or drop solutions may be expected will be as shown in Fig. 2. Only the supercritical values are of relevance. In supercritical ranges the value of J represents the degree of supercriticality of the flow. In the jump range this value increases with decreasing α, while in the drop range the behavior is just the opposite. Increase in q_1, q_2 and/or r increases the supercritical ranges until the subcritical range vanishes altogether. Then $J>0$ for $0<\alpha<1$.

The roots of the quartic $\mathcal{K} = 0$ represent the conjugate depths to a given flow. It has been shown by Mehrotra & Kelly (1972) that in $0<\beta<1$ there are at most two roots which can

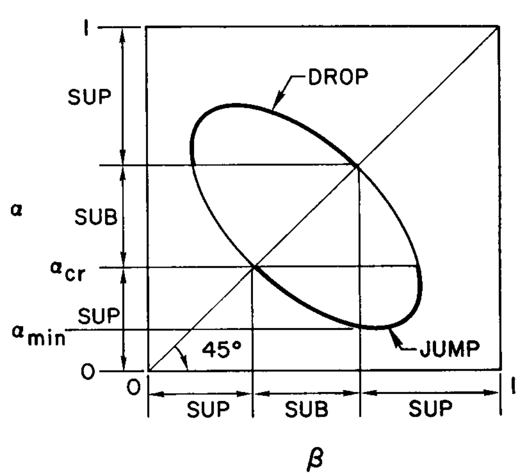

Fig. 3. The $\alpha - \beta$ plot. *Sup*, Supercritical; *Sub*, subcritical.

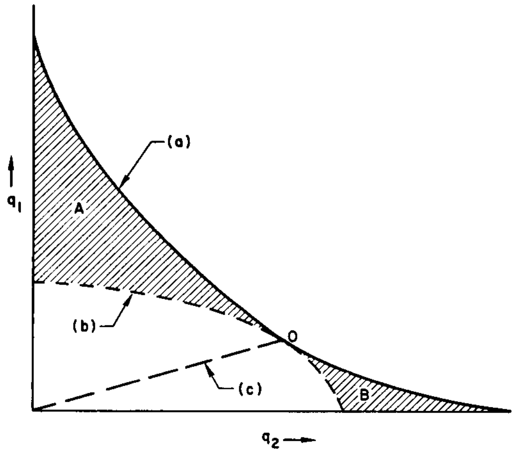

Fig. 4. Regions of existence of shock solutions in $q_1 - q_2$ space.

be denoted by $\beta_i (i = 1, 2)$. A typical plot of α versus β will be as shown in Fig. 3. The upstream and downstream states can be shown to be mutually conjugate. Owing to this property the $\alpha - \beta$ plot is symmetrical about a line inclined at 45° to either axis. A remarkable symmetry between the jump and drop solutions is at once apparent. Subsequent discussion will, therefore, be limited to jump solutions, corresponding interpretations for the drop solutions being straightforward. The ranges of values of α and β for which the flow upstream or downstream is supercritical or subcritical are shown. When q_1, q_2 and r are such that $J > 0$, for $0 < \alpha < 1$ there is no subcritical range and the $\alpha - \beta$ plot vanishes.

The limiting curve J is shown in Fig. 2. Thus when $J > 0$, for $0 < \alpha < 1$ no shock solutions are possible. When $J < 0$ for some range of values of α there is a minimum value of $\alpha = \alpha_{min}$ below which no shock solutions are possible (Fig. 3). If $\alpha = \alpha_0$ and $r = r_0$ are prescribed the relationship between q_1 and q_2 which separates regions in $q_1 - q_2$ space where shock solutions exist and where they do not, is given by

$$\beta_1 = \beta_2 \tag{5}$$

In Fig. 4, $J = 0$ and $\mathcal{M} = 0$ are plotted in $q_1 - q_2$ space for $\alpha = \alpha_0$ and $r = r_0$ and are marked (b) and (c) respectively. The region above (b) represents supercritical flows while below it represents subcritical flows. The region above (c) represents jump solutions while below it represents drop solutions. Curve (a) can be obtained from (5). Shock solutions exist within the hatched regions marked A and B. Curve (a) is tangential to (b) at 0, the point of intersection of (b) and (c). This can be proved as follows. Let (q_{10}, q_{20}) be the coordinates of 0. Point 0 is critical since it is on (b) ($J = 0$) and is also the demarcation point of jump-drop regions since it is on (c) ($\mathcal{M} = 0$). Curve $J(\alpha)$ plotted as a function of α for values of q_1, q_2 and r equal to q_{10}, q_{20} and r_0 respectively will be as shown in Fig. 5. In view of (4) and the above observations the curve is tangential to the α-axis at $\alpha = \alpha_0$. This is precisely the limiting curve; any increase in q_1 and/or q_2 beyond q_{10} and/or q_{20} will result in $J(\alpha) > 0$, for $0 < \alpha < 1$ and, as we have seen, no shock solutions will be possible. Hence curve (a) must pass

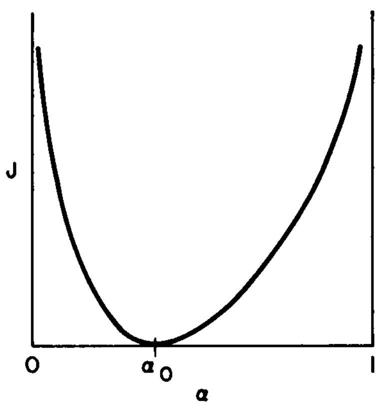

Fig. 5. The limiting curve $J(\alpha)$ for $q_1 = q_{10}$, $q_2 = q_{20}$ and $r = r_0$.

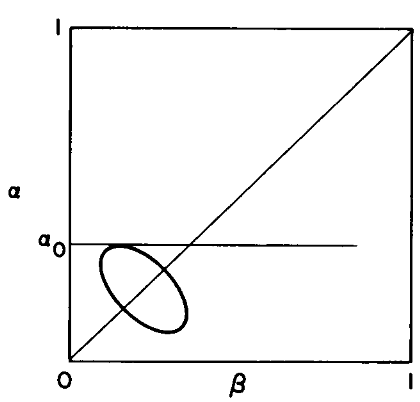

Fig. 6. The $\alpha - \beta$ plot for a point on curve (a) bordering region B.

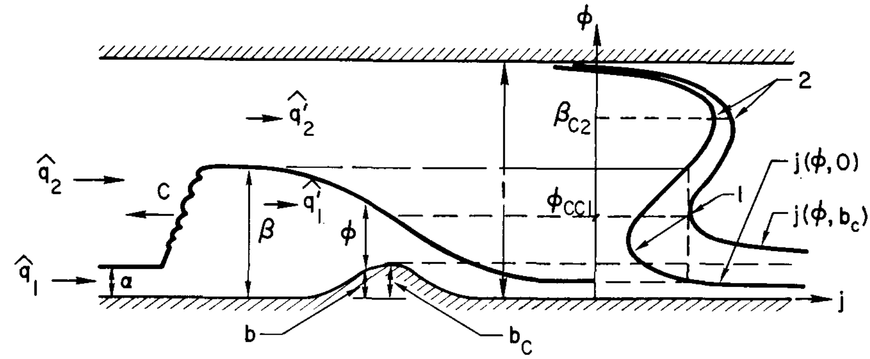

Fig. 7. Definition sketch: Total problem.

through 0. Indeed, it must be tangential to (b) at 0, since, if it is not, pairs of values (q_1, q_2) on (a) exist such that $J(\alpha_0) < 0$ (subcritical). In view of Fig. 3, eq. (5) which describes the curve (a) cannot be satisfied.

Data in region A will give shock solutions which are jumps, while those in region B will give drops. It is interesting to note that, though in the example given here, jumps are favored, there is a small region where drops are possible. A limit point from this region on (a) will be characterized by an $\alpha - \beta$ plot shown in Fig. 6.

With increasing r, for a given α, the regions of existence of shock solutions in $q_1 - q_2$ space diminish until when $r \to 1$, regions A and B tend to vanish altogether (no internal gravity waves).

With increasing α, for a given r, region B increases at the expense of A. This is to be expected, since it is well-known that a narrower upper layer favors drops, while a narrower lower layer favors jumps.

We now relate the existence and nonexistence of shock solutions to the total problem in which an agency in the form of a single crested hump is included.

3. Existence of shock solutions related to the total problem

The definition sketch for the total problem is given in Fig. 7. All quantities have been non-dimensionalized in the manner of section 2. The layer discharges upstream and downstream from the steadily propagating shock are $\hat{q}_i$ and $\hat{q}'_i$ respectively and c is the speed of propagation of the shock. The constancy of Bernoulli head behind the shock yields, upon simplification, the following equation,

$$\frac{1}{2}\left(\frac{\hat{q}'_1}{\phi}\right)^2 + s(b+\phi) - \frac{1}{2}r\left(\frac{\hat{q}'_2}{1-\phi-b}\right)^2 = \text{constant} \quad (6)$$

where ϕ and b are the depth of the lower layer and the height of the hump respectively. Denoting the left hand side of (6) by $j(\phi, b)$ it can be easily shown that $\partial j/\partial\phi = 0$ yields the local critical condition, viz.

$$-\frac{\partial j}{\partial \phi} = \frac{\hat{q}_1'^2}{\phi^3} + \frac{r\hat{q}_2'^2}{(1-\phi-b)^3} - s = 0 \quad (7)$$

The function j is plotted against ϕ for $b = b_c$ and $b = 0$ in Fig. 7, the curves corresponding to other values of b, $0 \leq b \leq b_c$, can be shown to be between these two curves. Points 1 and 2 represent critical levels. When for a given height of the crest, the speed of propagation of the shock is such that the pair of layer discharges,

$$\{\hat{q}_1 + c\alpha, \hat{q}_2 + c(1-\alpha)\} \quad (8)$$

lies on the curve (a) of Fig. 4, i.e. when the isolated problem with layer discharges given by (8) has reached the limiting condition $\beta_1 = \beta_2$, numerical computations for the total problem show that $j(\beta_{c2}, 0) = j(\phi_{cc1}, b_c)$, provided the total problem yields both the roots (Mehrotra & Kelly 1972). Here β_{c2} is the critical depth where $b = 0$ corresponding to point 2 and ϕ_{cc1} is the critical depth where $b = b_c$ corresponding to point 1 (Fig. 7). This is the limiting condition under which the requirement that the flow be critical at the crest can be met, since any increase in the layer discharges of the oncoming flow over the limiting value will make $j(\beta_{c2}, 0) < j(\phi_{cc1}, b_c)$ and critical condition cannot be obtained at the crest. When the oncoming flow is such that $J(\alpha) > 0$ for $0 < \alpha < 1$, the function j is monotonic for all values of b, and the flow cannot be critical at the crest.

4. Conclusions and discussions

The nonexistence of shock solutions in some supercritical two-layer flows is related to the impossibility of finding a critical condition at a control such as the crest of a hump on the floor. Since a shock must necessarily be caused by a disturbance in a supercritical flow, the disturbance must control the flow in its vicinity. It is perhaps worth emphasizing that the questions of control and critical condition at the control are intimately connected. A disturbance cannot act as a control in its vicinity unless the condition is critical at the control. When critical condition obtains at the control, the flow upstream from it is subcritical and downstream from it is supercritical. It is well-known that subcritical flows depend essentially on the conditions imposed downstream, while supercritical flows depend on conditions imposed upstream. This should be quite evident if the mechanism of information propagation and the definitions of subcritical and supercritical flows are clearly understood. Thus a crest can act as a control only when the condition is critical there. When the supercriticality of the flow is such that the flow cannot be controlled by *any* disturbance, shocks cannot form and the disturbance is negotiated purely locally in the manner of an absolutely supercritical flow. One should be careful to distinguish between absolutely supercritical flows in general and those flows for which shock solutions do not exist. Any supercritical flow can negotiate some disturbances purely locally and, yet, may admit shock solutions with the proper strength of the disturbance. (*All* supercritical flows of a homogeneous fluid with a free surface belong to this category.) Such supercritical flows belong in the regions A and B of Fig. 4. The flows for which shock solutions do not exist are, on the other hand, absolutely supercritical *no matter what* the strength of the disturbance is.

Ranges of values of the parameters of the problem can be found within which shock solutions exist. There are two cases when shock solutions cease to exist. In one, the flow is such that there is no range of values of the lower layer depth for which it is subcritical for prescribed layer discharges and ratio of the densities of the two layers. No shock solution exists for *any* value of lower layer depth. In the other, a subcritical range exists and shock solutions cease to exist when the lower layer depth falls below a minimum (in the jump region) or exceeds a maximum (in the drop region).

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REFERENCES

Mehrotra, S. C. & Kelly, R. E. 1972. On the question of nonuniqueness of internal hydraulic jumps and drops in a two-fluid system. Presented at *International Symposium on Stratified Flows*, Novosibirsk.

ОГРАНИЧЕНИЯ НА СУЩЕСТВОВАНИЕ УДАРНЫХ РЕШЕНИЙ В ДВУХСЛОЙНОЙ ЖИДКОСТИ

Объясняется, почему ударные решения существуют только для ограниченного интервала величин параметров в задаче о сверхкритическом движении двухслойной невязкой несжимаемой несмешивающейся устойчиво стратифицированной жидкости, ограниченной сверху и снизу горизонтальными стенками. Для данных α, безразмерной глубины нижнего слоя, и τ — отношения плотностей жидкости верхнего и нижнего слоев, области, где существуют ударные решения, могут быть очерчены в пространстве q_1, q_2, где q_1 и q_2 — безразмерные расходы жидкости в верхнем и нижнем слоях соответственно.