

Computationally efficient schemes and boundary conditions for a fine-mesh barotropic model based on the shallow-water equations

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ABSTRACT

Two types of computationally efficient semi-implicit and leap-frog schemes for integrating the shallow-water equations are presented. Their accuracy and stability are investigated, both for the pure initial value problem and for limited-area forecasts. The proper formulation of boundary conditions for the latter case is also discussed. Two sets of stable and sufficiently accurate boundary conditions are given. The practical usefulness of these conditions is also supported by computer experiments.

1. Introduction

Already in the famous monograph "Weather Prediction by Numerical Process", Lewis F. Richardson (1922) suggested the centered-difference or "leap-frog" method (which he called the "step-over" method) for the solution of the forecast equations, being the simplest explicit finite-difference scheme of second order accuracy. The size of such a computational problem was however much too large for even a single 24 h forecast during this pre-computer era, but Richardson's motives for choosing the "leap-frog" time integration scheme were kept in mind and found to be sound when computers later made numerical forecast computations possible. The dissipative Lax-Wendroff method (and the semi-dissipative two-step variant proposed by Richtmyer) has also been used but does not seem to have any direct advantages over the original leap-frog scheme. All these schemes do however have a major drawback. For stability reasons, the time-step may not exceed a maximum value, set by the spatial mesh size times the inverse velocity of the most rapid wave solution to the equations. For the quasi-geostrophic type of solution we are interested in, this upper limit is far less than the most favourable time-step (giving the highest "accuracy per arithmetic operation" ratio).

To improve this ratio, we must either eliminate the fast gravity-wave type of solutions or find a time integration method with less severe stability conditions.

The first alternative is carried out in "balanced" forecast schemes. In the derivation of such schemes, a number of simplifying approximations are however necessary, and the second alternative is more attractive. An integration method of this type is the implicit mid-point method, which for linear equations coincides with the trapezoidal or Crank-Nicholson rule. In fact, this scheme was already suggested in Richardson's book as a means of obtaining starting values for the leap-frog scheme but was rejected due to its inherent non-linearity. In a simple ADI-form, it was again considered by Gustafsson (1971). He found that the extra time required for solving the implicit system meant that it could not compete with the leap-frog scheme until the time-step was 3–4 times larger. Due to the time-averaging of all terms the accuracy of the scheme is then reduced.

During the last years, a so-called "semi-implicit" method has gained a considerable interest. The basic idea of this method is a separation of the space derivative and zero order terms into two groups: the terms respon-

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sible for the severe time-step restrictions are treated as in the implicit mid-point rule over a double time-step, while the remaining ones are kept explicit as in the explicit leap-frog scheme.

For the meteorological forecast equations, the semi-implicit scheme was first used by Kwizak (1970) and by Kwizak & Robert (1971), but the same idea was suggested by H.-O. Kreiss for a heat flow problem about 10 years earlier and analyzed by Johansson & Kreiss (1963). Although the scheme is not unconditionally stable, its stability condition does not pose any severe restrictions on the magnitude of the time-step. Moreover, with the larger time-step the accuracy is actually improved. The "semi-implicit" method can therefore be superior to the explicit leap-frog scheme, if only the separation of the terms can be properly done and the extra work needed for inverting the implicit operator does not exceed the gains from the use of a larger time-step. In this paper, we shall analyze the stability and truncation error of the "semi-implicit" approximation to the shallow-water equations on a staggered grid, discuss possible boundary conditions for integration in a closed region, analyze the stability of various corresponding numerical boundary conditions, and finally show results from limited-area integrations.

2. Finite-difference formulation of the shallow-water equations

The shallow-water equations are

$$\left. \begin{aligned} u_t &= -\varphi_x - u u_x - v u_y + f v \\ v_t &= -\varphi_y - u v_x - v v_y - f u \\ \varphi_t &= -\varphi(u_x + v_y) - u \varphi_x - v \varphi_y \end{aligned} \right\} \quad (2.1)$$

where, as usual, u and v are the velocity components, φ is the geopotential and $f = f(x, y)$ is the Coriolis parameter. We shall integrate these equations for either an unlimited, periodic or closed rectangular region by a finite-difference method. Thus we introduce grid point values $\varphi_{j,k}^n$ etc. corresponding to the values of φ etc. on a rectangular grid $(x, y, t) = (j\Delta x, k\Delta y, n\Delta t)$, where j, k, n are integers. (For notation simplicity, we shall often omit the j, k -indices in a formula valid for all j, k). We consider here only the case $\Delta x = \Delta y = \Delta s$. In order to simplify the

finite-difference notation, we further use the symbols $D_{0x}, D_{0y}, \bar{}^x$ and $\bar{}^y$ for the centered difference operator and mean operator in the x - and y -directions ($(D_{0x}\varphi)_{j,k} \equiv (\varphi_{j+1,k} - \varphi_{j-1,k})/2\Delta s$, $\bar{\varphi}_{j,k}^x \equiv \frac{1}{2}(\varphi_{j+1,k} + \varphi_{j-1,k})$ etc.) and also the abbreviation $\bar{}^x \equiv \bar{}^y$.

2.1. Semi-implicit schemes

For the problems we are interested in, φ is always positive and approximately constant in time and space, the deviations being less than 5%. We also have $\varphi \gg u^2 + v^2$, and the terms giving the most rapid wave solution to the (linearized) equations are consequently the first ones on the right hand side of (2.1). The semi-implicit scheme should then have the form

$$\left. \begin{aligned} \frac{u^{n+1} - u^{n-1}}{2\Delta t} &= -D_{0x} \frac{\varphi^{n+1} + \varphi^{n-1}}{2} \\ &\quad + \text{remaining terms, taken at time } t_n \\ \frac{v^{n+1} - v^{n-1}}{2\Delta t} &= -D_{0y} \frac{\varphi^{n+1} + \varphi^{n-1}}{2} \\ &\quad + \text{remaining terms, taken at time } t_n \\ \frac{\varphi^{n+1} - \varphi^{n-1}}{2\Delta t} &= -\phi \left(D_{0x} \frac{u^{n+1} + u^{n-1}}{2} \right. \\ &\quad \left. + D_{0y} \frac{v^{n+1} + v^{n-1}}{2} \right) \\ &\quad + \text{remaining terms, taken at time } t_n \end{aligned} \right\} \quad (2.2)$$

where ϕ e.g. can be chosen as φ^n .

By eliminating u^{n+1} and v^{n+1} , we obtain the following equation for $\varphi^{n+1} - \varphi^{n-1}$:

$$(1 - \Delta t^2 \phi (D_{0x}^2 + D_{0y}^2)) (\varphi^{n+1} - \varphi^{n-1}) = \text{terms at } t_n \text{ and } t_{n-1} \quad (2.3)$$

After having solved (2.3) for φ^{n+1} we obtain u^{n+1} and v^{n+1} from the first two equations of (2.2).

If φ is defined at all points of the rectangular grid, this equation actually consists of four disconnected systems, indicating that the use of a staggered grid should be advantageous. To obtain the most accurate representation of the remaining terms, the grid should be staggered in time also. (In the semi-implicit schemes suggested by Kwizak (1970), this was not accomplished, with the result that the deriva-

tives in the advective terms had to be approximated by much less accurate formulae than those of the implicit terms.)

We give here two possible staggered-grid configurations and difference equations:

Type A

φ	u, v	φ	u, v	φ	u, v
u, v	φ	u, v	φ	u, v	φ
φ	u, v	φ	u, v	φ	u, v

at even time-steps at odd time-steps

$$\left. \begin{aligned} \frac{u^{n+1} - u^{n-1}}{2\Delta t} &= -D_{0x} \frac{\varphi^{n+1} + \varphi^{n-1}}{2} \\ &\quad - \frac{\bar{u}^{n,x} + \bar{u}^{n,y}}{2} D_{0x} u^n \\ &\quad - \frac{\bar{v}^{n,x} + \bar{v}^{n,y}}{2} D_{0y} u^n \\ &\quad + f \frac{\bar{v}^{n,x} + \bar{v}^{n,y}}{2} \\ \frac{v^{n+1} - v^{n-1}}{2\Delta t} &= -D_{0y} \frac{\varphi^{n+1} + \varphi^{n-1}}{2} \\ &\quad - \frac{\bar{u}^{n,x} + \bar{u}^{n,y}}{2} D_{0x} v^n \\ &\quad - \frac{\bar{v}^{n,x} + \bar{v}^{n,y}}{2} D_{0y} v^n \\ &\quad - f \frac{\bar{u}^{n,x} + \bar{u}^{n,y}}{2} \\ \frac{\varphi^{n+1} - \varphi^{n-1}}{2\Delta t} &= -\frac{\bar{\varphi}^{n,x} + \bar{\varphi}^{n,y}}{2} \left(D_{0x} \frac{u^{n+1} + u^{n-1}}{2} \right. \\ &\quad \left. + D_{0y} \frac{v^{n+1} + v^{n-1}}{2} \right) \\ &\quad - u^n D_{0x} \varphi^n - v^n D_{0y} \varphi^n \end{aligned} \right\} (2.4)$$

It is easily seen that the inversion of the implicit system corresponds to the solution of two disconnected equations of type (2.3).

Type B is still more compact. It is based on the configuration

φ	u	φ	v
v	v	u	φ
φ	u	φ	v

at even time-steps at odd time-steps

and the equations

$$\left. \begin{aligned} \frac{u^{n+1} - u^{n-1}}{2\Delta t} &= -D_{0x} \frac{\varphi^{n+1} + \varphi^{n-1}}{2} \\ &\quad - \bar{u}^{n,x} D_{0x} \bar{u}^{n,y} - v^n D_{0y} \bar{u}^{n,x} \\ &\quad + f v^n \\ \frac{v^{n+1} - v^{n-1}}{2\Delta t} &= -D_{0y} \frac{\varphi^{n+1} + \varphi^{n-1}}{2} \\ &\quad - u^n D_{0x} \bar{v}^{n,y} - \bar{v}^{n,x} D_{0y} \bar{v}^{n,x} \\ &\quad - f u^n \\ \frac{\varphi^{n+1} - \varphi^{n-1}}{2\Delta t} &= -\bar{\varphi}^{n,x} \left(D_{0x} \frac{u^{n+1} + u^{n-1}}{2} \right. \\ &\quad \left. + D_{0y} \frac{v^{n+1} + v^{n-1}}{2} \right) \\ &\quad - \bar{u}^{n,y} D_{0x} \bar{\varphi}^{n,y} - \bar{v}^{n,x} D_{0y} \bar{\varphi}^{n,x} \end{aligned} \right\} (2.5)$$

giving only one equation of type (2.3).

In the formulation of these schemes, we have tried to minimize the use of averaging-operators in the advective derivative approximations and use the same amount of averaging in all three equations.

The solution of (2.3) may be found by any of the standard methods (e.g. successive over-relaxation or alternating-direction implicit) for elliptic partial differential equations.

In the experiments, we used a "splitting" method, in which equation (2.3) is modified into

$$\begin{aligned} (1 - \Delta t^2 \hat{\phi} D_{0x}^2) (1 - \Delta t^2 \hat{\phi} D_{0y}^2) (\varphi^{n+1} - \varphi^{n-1}) \\ = \text{terms at } t_n \text{ and } t_{n-1} \end{aligned} \quad (2.3')$$

which can be solved directly by two elimination steps:

$$\left. \begin{aligned} (1 - \Delta t^2 \hat{\phi} D_{0x}^2) \bar{\varphi}^{n+1} &= \text{terms at } t_n \text{ and } t_{n-1} \\ (1 - \Delta t^2 \hat{\phi} D_{0y}^2) (\varphi^{n+1} - \varphi^{n-1}) &= \bar{\varphi}^{n+1} \end{aligned} \right\}$$

2.2. Leap-frog schemes

As a comparison, we give here also two corresponding explicit leap-frog schemes:

Type A uses the same grid as semi-implicit scheme A and the equations

$$\left. \begin{aligned}
 \frac{u^{n+1} - u^{n-1}}{2\Delta t} &= -D_{0x} \bar{\varphi}^{n\gamma} - \frac{\bar{u}^{n^x} + \bar{u}^{n^y}}{2} D_{0x} u^n \\
 &\quad - \frac{\bar{v}^{n^x} + \bar{v}^{n^y}}{2} D_{0y} u^n \\
 &\quad + f \frac{\bar{v}^{n^x} + \bar{v}^{n^y}}{2} \\
 \frac{v^{n+1} - v^{n-1}}{2\Delta t} &= -D_{0y} \bar{\varphi}^{n^x} - \frac{\bar{u}^{n^x} + \bar{u}^{n^y}}{2} D_{0x} v^n \\
 &\quad - \frac{\bar{v}^{n^x} + \bar{v}^{n^y}}{2} D_{0y} v^n \\
 &\quad - f \frac{\bar{u}^{n^x} + \bar{u}^{n^y}}{2} \\
 \frac{\varphi^{n+1} - \varphi^{n-1}}{2\Delta t} &= -\frac{\bar{\varphi}^{n^x} + \bar{\varphi}^{n^y}}{2} (D_{0x} \bar{u}^{n^y} \\
 &\quad + D_{0y} \bar{v}^{n^x}) - u^n D_{0x} \varphi^n \\
 &\quad - v^n D_{0y} \varphi^n
 \end{aligned} \right\} (2.6)$$

The more compact *type B* uses the grid

$$\begin{array}{ccc}
 \varphi & v & \varphi \\
 u & & u \\
 \varphi & v & \varphi
 \end{array}
 \quad
 \begin{array}{ccc}
 & & u \\
 & v & \varphi & v \\
 & & u
 \end{array}$$

at even time-steps

at odd time-steps

and the equations

$$\left. \begin{aligned}
 \frac{u^{n+1} - u^{n-1}}{2\Delta t} &= -D_{0x} \varphi^n - \bar{u}^{n^x y} D_{0x} \bar{u}^{n^y} \\
 &\quad - v^n D_{0y} \bar{u}^{n^x} + f v^n \\
 \frac{v^{n+1} - v^{n-1}}{2\Delta t} &= -D_{0y} \varphi^n - u^n D_{0x} \bar{v}^{n^y} \\
 &\quad - \bar{v}^{n^x y} D_{0y} \bar{v}^{n^x} - f u^n \\
 \frac{\varphi^{n+1} - \varphi^{n-1}}{2\Delta t} &= -\bar{\varphi}^{n^x y} (D_{0x} u^n + D_{0y} v^n) \\
 &\quad - \bar{u}^{n^x} D_{0x} \bar{\varphi}^{n^y} - \bar{v}^{n^y} D_{0y} \bar{\varphi}^{n^x}
 \end{aligned} \right\} (2.7)$$

3. Stability conditions for the Cauchy problem

Although no general theory about the existence and uniqueness of solutions to non-linear hyperbolic systems has been developed, Sund-

ström (1973) has tried to find sufficient conditions for the well-posedness of the pure initial-value problem for the shallow-water equations (2.1). For certain boundary conditions, the results could be extended to the limited-area integration problem. The conditions are however of an *a-posteriori* nature: they do not tell in advance for how long time given initial conditions (assumed continuous and differentiable) produce a continuous and differentiable solution. In other words, the predictability of the forecast equations is not known beforehand, but we have all reasons to believe that short-range integrations are always meaningful.

Far less is known about the stability of the corresponding finite-difference approximations, e.g. (2.4) or (2.5). By the term "stability" we mean as usual that small errors in the initial values or the inhomogeneous terms should not give any unbounded error growth in the solution. The size of the errors should be measured by a suitable norm, equivalent to the R.M.S. value. For linear homogeneous problems, the error components satisfy the same equations as the undisturbed solution. We may then choose the norm as the square-root of the energy of the disturbance. If the finite-difference equations are now formulated in an energy-conserving way, the stability follows directly. This method is called "the energy method" of proving stability (see e.g. Kreiss & Widlund, 1967).

Also for non-linear equations describing physical processes, some form of energy conservation law usually exists. It is also often possible to find difference approximations satisfying a corresponding expression, at least for $\Delta t \rightarrow 0$ (semi-conservation).

It is often assumed that the satisfaction of such a conservation law for the total energy (which may be desirable for other reasons) should imply stability of the difference scheme. This is however not true in general. The simple system (2.1) has e.g. the energy conservation law

$$\int \frac{u^2 + v^2}{2} \varphi + \frac{\varphi^2}{2} dx dy = \text{constant}$$

but the conservation of the corresponding quantity for a finite-difference approximation to (2.1) does not guarantee stability in the R.M.S. sense unless $\varphi > \text{constant}$ for all times.

A *necessary* condition for the stability of a non-linear scheme is however that the linearized version of the disturbance equations is stable. If we also assume that the undisturbed solution is continuous and has bounded derivatives, it may be shown that the stability or instability of these equations follows from the stability or instability of the set of corresponding equations with constant coefficients and no zero-order terms, letting the coefficient values vary through the range of values of the actual solution.

For the finite-difference approximation (2.4) to the shallow-water equations, we must thus ascertain that the constant coefficient system

$$\left. \begin{aligned} \frac{u^{n+1} - u^{n-1}}{2\Delta t} &= -D_{0x} \frac{\varphi^{n+1} + \varphi^{n-1}}{2} \\ &\quad - UD_{0x} u^n - VD_{0y} u^n \\ \frac{v^{n+1} - v^{n-1}}{2\Delta t} &= -D_{0y} \frac{\varphi^{n+1} + \varphi^{n-1}}{2} \\ &\quad - UD_{0x} v^n - VD_{0y} v^n \\ \frac{\varphi^{n+1} - \varphi^{n-1}}{2\Delta t} &= -\Phi \left(D_{0x} \frac{u^{n+1} + u^{n-1}}{2} \right. \\ &\quad \left. + D_{0y} \frac{v^{n+1} + v^{n-1}}{2} \right) \\ &\quad - UD_{0x} \varphi^n - VD_{0y} \varphi^n \end{aligned} \right\} \quad (3.1)$$

is stable for all reasonable values of U , V , and Φ . For (2.5) we get instead

$$\left. \begin{aligned} \frac{u^{n+1} - u^{n-1}}{2\Delta t} &= -D_{0x} \frac{\varphi^{n+1} + \varphi^{n-1}}{2} \\ &\quad - UD_{0x} \overline{u^n} - VD_{0y} \overline{u^n} \\ \frac{v^{n+1} - v^{n-1}}{2\Delta t} &= -D_{0y} \frac{\varphi^{n+1} + \varphi^{n-1}}{2} \\ &\quad - UD_{0x} \overline{v^n} - VD_{0y} \overline{v^n} \\ \frac{\varphi^{n+1} - \varphi^{n-1}}{2\Delta t} &= -\Phi \left(D_{0x} \frac{u^{n+1} + u^{n-1}}{2} \right. \\ &\quad \left. + D_{0y} \frac{v^{n+1} + v^{n-1}}{2} \right) \\ &\quad - UD_{0x} \overline{\varphi^n} - VD_{0y} \overline{\varphi^n} \end{aligned} \right\} \quad (3.2)$$

In the "split" variant (2.5'), the last equation is changed into

$$(1 + \Delta t^4 \Phi^2 D_{0x}^2 D_{0y}^2) \frac{\varphi^{n+1} - \varphi^{n-1}}{2\Delta t} = \dots \quad (3.2')$$

For these linear constant-coefficient equations, the construction of suitable R.M.S. norms for the "energy method" stability proof is easy. For (3.1), we get

$$\begin{aligned} &\Phi(u^{n+1} + u^n + v^{n+1} + v^n) + \varphi^{n+1} + \varphi^n \\ &\quad + 2\Delta t U[\Phi(u^{n+1} D_{0x} u^n + v^{n+1} D_{0x} v^n) \\ &\quad + \varphi^{n+1} D_{0x} \varphi^n] + 2\Delta t V[\Phi(u^{n+1} D_{0y} u^n \\ &\quad + v^{n+1} D_{0y} v^n) + \varphi^{n+1} D_{0y} \varphi^n] \\ &= \Phi(u^n + u^{n-1} + v^n + v^{n-1}) + \varphi^n + \varphi^{n-1} \\ &\quad + 2\Delta t U[\Phi(u^n D_{0x} u^{n-1} + v^n D_{0x} v^{n-1}) \\ &\quad + \varphi^n D_{0x} \varphi^{n-1}] + 2\Delta t V[\Phi(u^n D_{0y} u^{n-1} \\ &\quad + v^n D_{0y} v^{n-1}) + \varphi^n D_{0y} \varphi^{n-1}] \\ &\quad - 2\Delta t U[\Phi(u^{n-1} D_{0x} u^n + u^n D_{0x} u^{n-1} \\ &\quad + v^{n-1} D_{0x} v^n + v^n D_{0x} v^{n-1}) \\ &\quad + \varphi^{n-1} D_{0x} \varphi^n + \varphi^n D_{0x} \varphi^{n-1}] \\ &\quad - 2\Delta t V[\Phi(u^{n-1} D_{0y} u^n + u^n D_{0y} u^{n-1} \\ &\quad + v^{n-1} D_{0y} v^n + v^n D_{0y} v^{n-1}) \\ &\quad + \varphi^{n-1} D_{0y} \varphi^n + \varphi^n D_{0y} \varphi^{n-1}] \end{aligned}$$

and summation over all grid point values gives

$$L_I^{n+1} = L_I^n = \dots = L_I^1$$

where

$$\begin{aligned} L_I^{n+1} &= \sum \{ \Phi(u^{n+1} + u^n + v^{n+1} + v^n) + \varphi^{n+1} \\ &\quad + \varphi^n + 2\Delta t U[\Phi(u^{n+1} D_{0x} u^n + v^{n+1} D_{0x} v^n) \\ &\quad + \varphi^{n+1} D_{0x} \varphi^n] + 2\Delta t V[\Phi(u^{n+1} D_{0y} u^n \\ &\quad + v^{n+1} D_{0y} v^n) + \varphi^{n+1} D_{0y} \varphi^n] \}. \end{aligned}$$

With the notation $\lambda = \Delta t / \Delta s$, we obtain the inequality

$$\begin{aligned} &\frac{L_I^{n+1}}{1 + \lambda(|U| + |V|)} \\ &\leq \sum \{ \Phi(u^{n+1} + u^n + v^{n+1} + v^n) + \varphi^{n+1} + \varphi^n \} \\ &\leq \frac{L_I^{n+1}}{1 - \lambda(|U| + |V|)} \end{aligned}$$

so that

$$\begin{aligned} & \sum \sum \{ \Phi(u^{n+1} + u^n + v^{n+1} + v^n) + \varphi^{n+1} + \varphi^n \} \\ & \leq \frac{1 + \lambda(|U| + |V|)}{1 - \lambda(|U| + |V|)} \sum \sum \{ \Phi(u^{n+1} + u^n + v^{n+1} + v^n) \\ & \quad + \varphi^{n+1} + \varphi^n \} \end{aligned}$$

if

$$\lambda(|U| + |V|) < 1 \quad (3.3)$$

showing that (3.1) is stable if this condition is satisfied.

For (3.2), we get instead

$$L_{II}^{n+1} = L_{II}^n = \dots = L_{II}^1$$

where

$$\begin{aligned} L_{II}^{n+1} = & \sum \sum \{ \Phi(u^{n+1} + u^n + v^{n+1} + v^n) + \varphi^{n+1} \\ & + \varphi^n + 2\Delta t U [\Phi(u^{n+1} D_{0x} \bar{u}^n + v^{n+1} D_{0x} \bar{v}^n) \\ & + \varphi^{n+1} D_{0x} \bar{\varphi}^n + \varphi^n D_{0x} \bar{\varphi}^n] \\ & + 2\Delta t V [\Phi(u^{n+1} D_{0y} \bar{u}^n + v^{n+1} D_{0y} \bar{v}^n) \\ & + \varphi^{n+1} D_{0y} \bar{\varphi}^n + \varphi^n D_{0y} \bar{\varphi}^n] \} \end{aligned}$$

which satisfies

$$\begin{aligned} & \left(1 + \frac{\lambda}{2} (|U + V| + |U - V|) \right)^{-1} L_{II}^{n+1} \\ & \leq \sum \sum \{ \Phi(u^{n+1} + u^n + v^{n+1} + v^n) + \varphi^{n+1} + \varphi^n \} \\ & \leq \left(1 - \frac{\lambda}{2} (|U + V| + |U - V|) \right)^{-1} L_{II}^{n+1} \end{aligned}$$

and has the stability condition

$$\lambda \cdot \max(|U|, |V|) < 1. \quad (3.4)$$

With the splitting modification (3.2'), we get from the third equation

$$\begin{aligned} & \sum \sum \varphi^{n+1} - \varphi^{n-1} + \Delta t^4 \Phi^2 [(D_{0x} D_{0y} \varphi^{n+1})^2 \\ & \quad - (D_{0x} D_{0y} \varphi^{n-1})^2] \end{aligned}$$

instead of

$$\sum \sum \varphi^{n+1} - \varphi^{n-1},$$

but the stability condition remains the same.

For the constant-coefficient explicit scheme

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$$\left. \begin{aligned} \frac{u^{n+1} - u^{n-1}}{2\Delta t} &= -D_{0x} \bar{u}^n - U D_{0x} u^n - V D_{0y} u^n \\ \frac{v^{n+1} - v^{n-1}}{2\Delta t} &= -D_{0y} \bar{v}^n - U D_{0x} v^n - V D_{0y} v^n \\ \frac{\varphi^{n+1} - \varphi^{n-1}}{2\Delta t} &= -\Phi(D_{0x} \bar{u}^n + D_{0y} \bar{v}^n) \\ &\quad - U D_{0x} \varphi^n - V D_{0y} \varphi^n \end{aligned} \right\} \quad (3.5)$$

corresponding to the leap-frog scheme A (2.6) we get in the same way the stability condition

$$\lambda(\sqrt{\Phi} + |U| + |V|) < 1, \quad (3.6)$$

and for the scheme B

$$\left. \begin{aligned} \frac{u^{n+1} - u^{n-1}}{2\Delta t} &= -D_{0x} \bar{u}^n - U D_{0x} \bar{u}^n - V D_{0y} \bar{u}^n \\ \frac{v^{n+1} - v^{n-1}}{2\Delta t} &= -D_{0y} \bar{v}^n - U D_{0x} \bar{v}^n - V D_{0y} \bar{v}^n \\ \frac{\varphi^{n+1} - \varphi^{n-1}}{2\Delta t} &= -\Phi(D_{0x} \bar{u}^n + D_{0y} \bar{v}^n) \\ &\quad - U D_{0x} \bar{\varphi}^n - V D_{0y} \bar{\varphi}^n \end{aligned} \right\} \quad (3.7)$$

we get stability if

$$\lambda(\sqrt{\Phi} + \max(|U|, |V|)) < 1 \quad (3.8)$$

Since $\sqrt{\Phi} > |U|, |V|$, the stability conditions of the semi-implicit schemes allow us to use a much larger Δt than is possible in the leap-frog schemes.

4. The accuracy of the difference schemes

4.1. Introductory remarks

Due to the non-linearity of the shallow-water equations, we considered in our stability investigation only a linearized constant-coefficient simplification of the equations, since the stability conditions for that problem do at least give us necessary conditions for the

stability of the complete equations. Instability is also a short-wave phenomenon, and for a sufficiently dense grid, the agreement between the stability conditions for the two problems should be close. The accuracy of a scheme concerns on the other side its possibility to describe rather long-wave features of the solution, and it is not immediately obvious to what extent this could be studied with a constant-coefficient model. As long as the errors are small, their growth is however given by the linearized error equations. Sundström (1973) derived an upper bound for the possible growth and decay rates of eigensolutions to the equations. It is given by one half the sum of the divergence and deformation of the flow. For short-range forecasts of a smooth solution, this growth may be neglected and the error equations can thus be approximated by a constant coefficient simplification. The main source of error is then the truncation errors committed during the integration, giving a linear increase of error in the forecast.

We may thus consider the linear equations

$$\left. \begin{aligned} u_t &= -\varphi_x - Uu_x - Vu_y + fv \\ v_t &= -\varphi_y - Uv_x - Vv_y - fu \\ \varphi_t &= -\Phi(u_x + v_y) - U\varphi_x - V\varphi_y \end{aligned} \right\} \quad (4.1)$$

with constant U , V , Φ and f , which may also be written

$$\begin{aligned} \frac{\partial}{\partial t} \begin{pmatrix} uc \\ vc \\ \varphi \end{pmatrix} &= - \begin{pmatrix} U & 0 & c \\ 0 & U & 0 \\ c & 0 & U \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} uc \\ vc \\ \varphi \end{pmatrix} - \begin{pmatrix} V & 0 & 0 \\ 0 & V & c \\ 0 & c & V \end{pmatrix} \frac{\partial}{\partial y} \begin{pmatrix} uc \\ vc \\ \varphi \end{pmatrix} \\ &+ \begin{pmatrix} 0 & f & 0 \\ -f & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} uc \\ vc \\ \varphi \end{pmatrix} \end{aligned} \quad (4.2)$$

where we have introduced the notation $c = \sqrt{f\Phi}$. The solution to this system can be expressed in terms of Fourier components. For each spatial wave-number (ω_x, ω_y) , (4.2) has solutions of the form

$$\begin{pmatrix} uc \\ vc \\ \varphi \end{pmatrix} = \sum_{k=1}^3 a_k e_k e^{i\nu_k t} \exp \{i(\omega_x x + \omega_y y)\} \quad (4.3)$$

where

$$\begin{aligned} \nu_1 &= -(U\omega_x + V\omega_y) \\ \nu_2 &= -(U\omega_x + V\omega_y) + \sqrt{c^2(\omega_x^2 + \omega_y^2) + f^2} \\ \nu_3 &= -(U\omega_x + V\omega_y) - \sqrt{c^2(\omega_x^2 + \omega_y^2) + f^2} \end{aligned}$$

are the three eigenfrequencies of the system, and

$$\begin{aligned} e_1 &= \begin{pmatrix} -i\omega_y c \\ i\omega_x c \\ f \end{pmatrix} \\ e_2 &= \begin{pmatrix} i\omega_y f - \omega_x \sqrt{c^2(\omega_x^2 + \omega_y^2) + f^2} \\ -i\omega_x f - \omega_y \sqrt{c^2(\omega_x^2 + \omega_y^2) + f^2} \\ (\omega_x^2 + \omega_y^2) c \end{pmatrix} \\ e_3 &= \begin{pmatrix} i\omega_y f + \omega_x \sqrt{c^2(\omega_x^2 + \omega_y^2) + f^2} \\ -i\omega_x f + \omega_y \sqrt{c^2(\omega_x^2 + \omega_y^2) + f^2} \\ (\omega_x^2 + \omega_y^2) c \end{pmatrix} \end{aligned}$$

the corresponding eigenvectors.

We observe here that the eigenvector e_1 is quasi-nondivergent and geostrophically balanced, and that it has a much lower eigenfrequency than the other two solutions. It is thus the meteorologically interesting solution, and the other two components, "gravity waves", are assumed to be "noise".

4.2. Error estimates

4.2.1. Eigensolutions to the difference equations.

For the simplified problem (4.1), the finite-difference approximations corresponding to (2.4) and (2.5) are

$$\left. \begin{aligned} \frac{u^{n+1} - u^{n-1}}{2\Delta t} &= -D_{0x} \frac{\varphi^{n+1} + \varphi^{n-1}}{2} - U D_{0x} u^n \\ &\quad - V D_{0y} u^n + f \frac{\overline{v^{n,x}} + \overline{v^{n,y}}}{2} \\ \frac{v^{n+1} - v^{n-1}}{2\Delta t} &= -D_{0y} \frac{\varphi^{n+1} + \varphi^{n-1}}{2} - U D_{0x} v^n \\ &\quad - V D_{0y} v^n - f \frac{\overline{u^{n,x}} + \overline{u^{n,y}}}{2} \end{aligned} \right\} \quad (4.4)$$

$$\left. \begin{aligned}
 \frac{\varphi^{n+1} - \varphi^{n-1}}{2\Delta t} &= -\Phi \left(D_{0x} \frac{u^{n+1} + u^{n-1}}{2} \right. \\
 &\quad \left. + D_{0y} \frac{v^{n+1} + v^{n-1}}{2} \right) \\
 &\quad - U D_{0x} \varphi^n - V D_{0y} \varphi^n
 \end{aligned} \right\} \left. \begin{aligned}
 \frac{v^{n+1} - v^{n-1}}{2\Delta t} &= -D_{0y} \varphi^n - U D_{0x} \overline{v^{ny}} \\
 &\quad - V D_{0y} \overline{v^{nx}} - f u^n \\
 \frac{\varphi^{n+1} - \varphi^{n-1}}{2\Delta t} &= -\Phi(D_{0x} u^n + D_{0y} v^n) \\
 &\quad - U D_{0x} \overline{\varphi^{ny}} - V D_{0y} \overline{\varphi^{nx}}
 \end{aligned} \right\} \quad (4.7)$$

here called semi-implicit scheme A, and

$$\left. \begin{aligned}
 \frac{u^{n+1} - u^{n-1}}{2\Delta t} &= -D_{0x} \frac{\varphi^{n+1} + \varphi^{n-1}}{2} \\
 &\quad - U D_{0x} \overline{u^{ny}} - V D_{0y} \overline{u^{nx}} + f v^n \\
 \frac{v^{n+1} - v^{n-1}}{2\Delta t} &= -D_{0y} \frac{\varphi^{n+1} + \varphi^{n-1}}{2} \\
 &\quad - U D_{0x} \overline{v^{ny}} - V D_{0y} \overline{v^{nx}} - f u^n \\
 \frac{\varphi^{n+1} - \varphi^{n-1}}{2\Delta t} &= -\Phi \left(D_{0x} \frac{u^{n+1} + u^{n-1}}{2} \right. \\
 &\quad \left. + D_{0y} \frac{v^{n+1} + v^{n-1}}{2} \right) \\
 &\quad - U D_{0x} \overline{\varphi^{ny}} - V D_{0y} \overline{\varphi^{nx}}
 \end{aligned} \right\} \quad (4.5)$$

called semi-implicit scheme B. They differ from (3.1), (3.2) only through the added Coriolis terms.

The leap-frog scheme A corresponding to (2.6) is

$$\left. \begin{aligned}
 \frac{u^{n+1} - u^{n-1}}{2\Delta t} &= -D_{0x} \overline{\varphi^{ny}} - U D_{0x} u^n \\
 &\quad - V D_{0y} u^n + f \frac{\overline{v^{nx}} + \overline{v^{ny}}}{2} \\
 \frac{v^{n+1} - v^{n-1}}{2\Delta t} &= -D_{0y} \overline{\varphi^{nx}} - U D_{0x} v^n \\
 &\quad - V D_{0y} v^n - f \frac{\overline{u^{nx}} + \overline{u^{ny}}}{2} \\
 \frac{\varphi^{n+1} - \varphi^{n-1}}{2\Delta t} &= -\Phi(D_{0x} \overline{u^{ny}} + D_{0y} \overline{v^{nx}}) \\
 &\quad - U D_{0x} \varphi^n - V D_{0y} \varphi^n
 \end{aligned} \right\} \quad (4.6)$$

and the more compact leap-frog scheme B becomes

$$\left. \begin{aligned}
 \frac{u^{n+1} - u^{n-1}}{2\Delta t} &= -D_{0x} \varphi^n - U D_{0x} \overline{u^{ny}} \\
 &\quad - V D_{0y} \overline{u^{nx}} + f v^n
 \end{aligned} \right\}$$

The periodic solutions to the difference schemes have the form

$$\begin{pmatrix} u^n \\ v^n \\ \varphi^n \end{pmatrix} c = \sum_{k=1}^6 b_k f_k z_k^n \exp \{i(\omega_x x + \omega_y y)\} \quad (4.8)$$

which may be compared with the exact solution (4.3) at $t_n = n\Delta t$

$$\begin{pmatrix} uc \\ vc \\ \varphi \end{pmatrix}_{t_n} = \sum_{k=1}^3 a_k e_k (e^{i\nu_k \Delta t})^n \exp \{i(\omega_x x + \omega_y y)\}$$

Notice that we except the three eigensolutions corresponding to the solutions (4.3) of the differential equations now also have three "computational modes" and that z_k takes the place of $e^{i\nu_k \Delta t}$. We shall here mainly study the eigenvector f_1 and the eigenfrequency $(1/(i\Delta t)) \log z_1$ of the quasi-geostrophic solution. (For a one-dimensional problem, we could instead of the eigenfrequency consider the phase velocity $(1/(i\omega \Delta t)) \log z_1$. For two-dimensional problems, the eigenfrequency is a more convenient quantity.)

4.2.2. Errors in the quasi-geostrophic solution to type A schemes. For the semi-implicit scheme A and the leap-frog scheme A, the solution corresponding to the geostrophic mode

$$\begin{aligned}
 &a_1 e_1 \exp \{i(\omega_x x + \omega_y y - (U\omega_x + V\omega_y)t)\} \\
 &\text{is} \\
 &b_1 f_1 \exp \{i(\omega_x x + \omega_y y - \frac{t}{\Delta t} \arcsin(\lambda(U \sin \xi \\
 &\quad + V \sin \eta)))\}
 \end{aligned}$$

where we have introduced the notation $\xi = \omega_x \Delta s$, $\eta = \omega_y \Delta s$.

The eigenvector is for the semi-implicit scheme A

$$f_1 = \begin{pmatrix} -i\omega_y c \frac{\sin \eta}{\eta} 2(1 - \lambda^2(U \sin \xi + V \sin \eta)^2)^{\frac{1}{2}} / (\cos \xi + \cos \eta) \\ i\omega_x c \frac{\sin \xi}{\xi} 2(1 - \lambda^2(U \sin \xi + V \sin \eta)^2)^{\frac{1}{2}} / (\cos \xi + \cos \eta) \end{pmatrix} \quad f$$

and for the leap-frog scheme A

$$f_1 = \begin{pmatrix} -i\omega_y c \frac{\sin \eta}{\eta} 2 \cos \xi / (\cos \xi + \cos \eta) \\ i\omega_x c \frac{\sin \xi}{\xi} 2 \cos \eta / (\cos \xi + \cos \eta) \end{pmatrix} \quad f$$

The ratio between the eigenfrequency of the schemes and the correct value is for small ξ and η

$$R_A = \frac{\arcsin(\lambda(U \sin \xi + V \sin \eta))}{\lambda(U \xi + V \eta)} \\ \approx 1 - \frac{1}{6} \left(\frac{U \xi^3 + V \eta^3}{U \xi + V \eta} - \lambda^2 (U \xi + V \eta)^2 \right) \dots$$

If $V = 0$, $R_A \approx 1 - \frac{1}{6} \xi^2 (1 - \lambda^2 U^2) \dots$ and if $U = V$ and $\xi = \eta$, $R_A \approx 1 - \frac{1}{6} \xi^2 (1 - 4\lambda^2 U^2) \dots$. In these and most other cases, R_A is less than one and increases with λ . Since the weaker stability conditions of the semi-implicit scheme allows us to use a much larger λ than for the leap-frog scheme, it gives a smaller eigenfrequency error.

If $U = V = 0.1 c$, then $c\lambda_{\max}(\text{semi-impl. A}) = 5$, $c\lambda_{\max}(\text{leap-frog A}) = 5/6$, and for $\xi = \eta$ we get the following values for R_A :

	π/ξ					
	2	3	4	6	9	12
$\lambda c = 0.5$	0.63	0.83	0.90	0.96	0.98	0.99
1	0.64	0.83	0.90	0.96	0.98	0.99
2.5	0.67	0.86	0.92	0.97	0.98	0.99
5	1	1	1	1	1	1

For small ξ and η , the eigenvector is for the semi-implicit scheme A,

$$f_1 = \begin{pmatrix} -i\omega_y c \left(1 + \frac{1}{12} (3\xi^2 + \eta^2) - \frac{1}{2} \lambda^2 (U\xi + V\eta)^2 \dots \right) \\ i\omega_x c \left(1 + \frac{1}{12} (\xi^2 + 3\eta^2) - \frac{1}{2} \lambda^2 (U\xi + V\eta)^2 \dots \right) \end{pmatrix} \quad f$$

and for the explicit leap-frog scheme A,

$$f_1 = \begin{pmatrix} -i\omega_y c \left(1 + \frac{1}{12} (-3\xi^2 + \eta^2) \dots \right) \\ i\omega_x c \left(1 + \frac{1}{12} (\xi^2 - 3\eta^2) \dots \right) \end{pmatrix} \quad f$$

Although no definite statement about the eigenvector error can be made, we observe that the error of the semi-implicit scheme A does decrease for increasing λ , at least for $\lambda|U| < 1/\sqrt{6}$, and that it can never be larger than twice the error of the leap-frog scheme.

It should also be observed that the eigenfrequencies of the short gravity wave modes to the semi-implicit scheme are considerably reduced for increasing λ -values, approaching $\pm(\pi/2\Delta t)$. They do however always have an eigenfrequency of larger magnitude than that of the quasi-geostrophic mode.

4.2.3. Errors in the quasi-geostrophic solution to type B schemes. For the semi-implicit scheme B and the leap-frog scheme B, the quasi-geostrophic solution has the form

$$b_1 f_1 \exp \left\{ i \left(\omega_x x + \omega_y y - \frac{t}{\Delta t} \arcsin(\lambda(U \sin \xi \cos \eta + V \sin \eta \cos \xi)) \right) \right\}$$

where as above $\xi = \omega_x \Delta s$, $\eta = \omega_y \Delta s$.

For the semi-implicit scheme, the eigenvector is

$$f_1 = \begin{pmatrix} -i\omega_y c \frac{\sin \eta}{\eta} (1 - \lambda^2 (U \sin \xi \cos \eta + V \sin \eta \cos \xi)^2)^{\frac{1}{2}} \\ i\omega_x c \frac{\sin \xi}{\xi} (1 - \lambda^2 (U \sin \xi \cos \eta + V \sin \eta \cos \xi)^2)^{\frac{1}{2}} \end{pmatrix} \quad f$$

and for the leap-frog scheme B,

$$f_1 = \begin{pmatrix} -i\omega_y c \frac{\sin \eta}{\eta} \\ i\omega_x c \frac{\sin \xi}{\xi} \\ f \end{pmatrix}$$

The eigenfrequency error formula is for these two schemes

$$R_B = \frac{\arcsin(\lambda(U \sin \xi \cos \eta + V \sin \eta \cos \xi))}{\lambda(U\xi + V\eta)}$$

and for small ξ, η

$$R_B \approx 1 - \frac{1}{2} \frac{U\xi\eta^2 + V\eta\xi^2}{U\xi + V\eta} - \frac{1}{6} \frac{U\xi^3 + V\eta^3}{U\xi + V\eta} + \frac{\lambda^2}{6} (U\xi + V\eta)^2 \dots$$

If $V=0$, $R_B \approx 1 - \frac{1}{6}\eta^2 - \frac{1}{6}\xi^2(1-\lambda^2 U^2)\dots$, and if $U=V$ and $\xi=\eta$, $R_B \approx 1 - \frac{1}{3}\xi^2(1-\lambda^2 U^2)\dots$, which is at least four times as large as the error of the schemes A. As was done for that scheme, we give here the values of R_B for $U=V=0.1c$, $\xi=\eta$. (Then, $c\lambda_{\max}=10$ for semi-implicit B and $=1/1.1$ for leap-frog B.)

	π/ξ					
	2	3	4	6	9	12
$\lambda c = 0.5$	0	0.41	0.64	0.83	0.92	0.96
1	0	0.41	0.64	0.83	0.92	0.96
2.5	0	0.42	0.64	0.83	0.92	0.96
5	0	0.43	0.67	0.86	0.94	0.97
7.5	0	0.45	0.72	0.90	0.96	0.98
10	0	1	1	1	1	1

The eigenfrequency error decreases with increasing λ , favouring the semi-implicit scheme with its much larger $\lambda_{\max}$.

For long waves (small ξ and η), the eigenvector of the semi-implicit scheme B is

$$f_1 = \begin{pmatrix} -i\omega_y c(1 - \frac{1}{6}\eta^2 - \frac{1}{6}\lambda^2(U\xi + V\eta)^2 \dots) \\ i\omega_x c(1 - \frac{1}{6}\xi^2 - \frac{1}{6}\lambda^2(U\xi + V\eta)^2 \dots) \\ f \end{pmatrix}$$

and the eigenvector of the leap-frog scheme B

$$f_1 = \begin{pmatrix} -i\omega_y c(1 - \frac{1}{6}\eta^2 \dots) \\ i\omega_x c(1 - \frac{1}{6}\xi^2 \dots) \\ f \end{pmatrix}$$

so that the eigenvector error of the semi-implicit scheme B increases with λ and is always larger than that of leap-frog scheme B. The optimum choice of λ for semi-implicit scheme B must therefore be based on the relative importance of eigenfrequency errors and eigenvector errors. All these results are for a scheme without splitting. For the "split" scheme, the eigenfrequency and eigenvector are both modified. For long waves, the difference is only of fourth order in Δs and can be neglected. For short waves, the eigenfrequencies are reduced, and in the eigenvector, the velocity components are now also slightly out of phase. The error due to the splitting is roughly proportional to $(f\Delta t)^2$, where f is the Coriolis parameter. For $f\Delta t = 0.18$, corresponding to a time-step of 1/2 hour, we get the following values of R_B . These values

	π/ξ					
	2	3	4	6	9	12
$\lambda c = 1$	0	0.41	0.64	0.83	0.92	0.96
2.5	0	0.42	0.64	0.83	0.92	0.96
5	0	0.39	0.63	0.83	0.92	0.96
7.5	0	0.36	0.59	0.78	0.89	0.94

have been obtained by a computer program, giving the roots to the eigenfrequency equation.

4.2.4 Comparison between the schemes. The possibility of using a larger λ ($=\Delta t/\Delta s$) in the semi-implicit schemes means that both the number of time-steps and the eigenfrequency error ("phase velocity error") may be reduced, compared to the best possible values of the leap-frog schemes. It must however be remembered that the computational work per time-step is approximately twice as much for the semi-implicit schemes, due to the necessity of solving a Helmholtz-type finite-difference equation for φ .

Of the two semi-implicit schemes, scheme A has certain advantages. Both the eigenfrequency error and the eigenvector error normally decrease with increasing values of λ , and the eigen-

frequency error is considerably smaller than that of scheme B, which also has an eigenvector error increasing with λ . To be fair, we must remember that the scheme B uses a grid with only half as many quantities defined and that the stability condition allows us to use a larger λ . We should therefore compare it with a scheme A with $\approx \sqrt{2}$ larger Δs , but still the semi-implicit scheme A seems to be somewhat better. Both of them are however only second-order schemes. For higher accuracy demands or longer integration times, we should modify the schemes by using fourth order approximations to the advection terms in the manner demonstrated by Kreiss & Olinger (1972).

Whether a "split" version should be used or not depends partly on the magnitude of the error terms introduced by the splitting and thus on the value of $f\Delta t$, partly on the difference in computation time. For a fine-mesh model, splitting is certainly advantageous.

5. Limited-area integrations

5.1. Background

In the previous chapters, we discussed the stability and accuracy properties of two semi-implicit schemes (2.4), (2.5) for solving the pure initial-value problem for the shallow-water equations (2.1). If the region of integration is instead limited, we have a mixed initial-boundary value problem for which both the formulation and analysis of difference approximations is more complicated.

The limited-area problems of geophysical interest are of two kinds. In the first case, the boundary of the region is a physical entity. The problem often describes fluid flow in a confined basin, the boundary of which is continuous and piecewise smooth. The relevant boundary condition is then simply $\mathbf{v} \cdot \mathbf{n} = 0$, where $\mathbf{v}$ is the velocity vector and $\mathbf{n}$ is the (outward) normal to the boundary.

We shall not consider this case further but instead study another class of limited-area problems, which has recently gained a considerable interest. Here, the boundary merely encloses a certain subregion of the whole plane, for which we want to make a more exact and detailed forecast than is possible for the large region. Such a "nested" integration technique is e.g. of interest for short-range forecasting of locally small-scale motion systems in an other-

wise large-scale flow. A coarse-grid integration of the large-scale part of the flow is first performed, from which boundary values for a repeated, fine-grid integration over the interesting region are then derived. For the limited-area integration, we may thus assume that (approximative) boundary values of all three quantities, u , v , and ϕ are known. (Some interpolation procedure is necessary to give values at all boundary points of the fine grid.) From these values, we want to find the most suitable boundary conditions for the fine-grid integration, which is a rather delicate problem.

5.2. On the choice of boundary conditions

First, some main properties of the integration region should be stated. We assume for simplicity that it is rectangular. It should be so small that a gravity wave may transverse it within a fraction of the integration time, but also so large that the influence from long-wave, quasi-geostrophic boundary value errors (from the surrounding coarse net) is not felt in the main, interior part of the region before the end of the integration time. The choice of boundary conditions should then be based more on their ability to control the growth of gravity wave disturbances than on their accuracy in describing the quasi-geostrophic part of the motion.

The proposed choices of boundary conditions for the fine-mesh part of a "nested" integration technique can be divided into two separate groups:

(a) *Prescribe all quantities along the whole boundary* (see Wang & Halpern, 1970). This method has the advantage of being simple and stable, but it has some serious shortcomings (Moretti, 1969). Near the outflow boundary, the amplitude of the quasi-geostrophic part of the solution should be determined by the interior solution and not by the prescribed, probably less accurate, values from the coarse grid. A slowly increasing difference between these values will propagate back into the region as a spatially short-wave, quasi-geostrophic error component. (This was noted by Shapiro and O'Brien (1970), although they expected this effect to be due to an instability, which it is not—the amplitude of the error is completely determined by the size of the boundary error.) On the other hand, a high-frequency error at the boundary is transmitted into the region as a long-wave disturbance. These errors may be reduced in magnitude

and speed by a suitable filtering operator, but the effect of such a filter on the real solution may be considerable, and the boundary must anyhow be placed on a fair distance from the region of interest.

(b) *Prescribe only the number of boundary values that the corresponding (linearized) system of differential equations requires* and compute the remaining ones by a suitable extrapolation procedure. If we choose the boundary conditions so that the differential equation problem is well-posed (see Kreiss, 1970), the limited area scheme can then be constructed so that its solution converges to the solution of the differential equation. Boundary conditions of this type were suggested already by Charney-Fjortoft-von Neumann (1950) for limited-area integrations of the barotropic vorticity equation. By a heuristic argument, they concluded that the normal velocity should be prescribed along the whole boundary and the vorticity at inflow parts. At outflow parts, the vorticity was calculated by a linear extrapolation formula. As has later been shown by Sundström (1969), these conditions do actually satisfy the well-posedness criteria (also for the non-linear problem) although they are not the only conditions that do. The extrapolation formula was however chosen in such a way that the scheme became unstable, as Platzman proved (1954). Instead of trying to remove the instability, the forecasters chose the simplest solution and introduced the type *a* condition despite its inefficiency.

For the shallow-water equations, by meteorologists usually called the primitive form of the barotropic forecast equations, Charney (1960) argued that the correct boundary conditions should be similar to those of the balanced forecast equations. The normal velocity should then be given at all boundary points and the "potential vorticity", the quotient between the absolute vorticity and the geopotential φ , at all inflow points. His argument was based on a mixture between a study of characteristics and characteristic quantities for the one-dimensional problem and the observation that the potential vorticity is a materially conserved property of the fluid elements. It was however no proof, and the conditions actually admit a slowly growing eigensolution to the problem, showing up as a gravity wave travelling along the boundary. They are anyhow not the only and prob-

ably not the best choice of boundary conditions.

We have chosen three other possible sets of boundary conditions for further study. In all these cases, the value of the tangential velocity has been given at inflow parts of the boundary, since this is the quantity corresponding to the meteorologically significant characteristic at inflow. (As a comparison, Charney's proposal gives the value of a linear combination between the normal derivative of this quantity and the geopotential, and the actual value of the tangential velocity at inflow is then mainly determined by its initial value. To keep the solution free from errors due to this effect, the initial data must be known very accurately up to the boundary.) A second quantity must be given at all boundary points, determining essentially how a gravity wave type of solution approaching the boundary should interact with the boundary. In the first case (I), we prescribe the normal velocity $v_n = \mathbf{v} \cdot \mathbf{n}$ at the boundary. The wave is then essentially just reflected back into the region. In the second case (II), we give the combination $\varphi - \sqrt{f} \varphi \cdot \mathbf{n}$, corresponding to the ingoing characteristic. A gravity wave approaching the boundary will then essentially disappear out of the region. In the third case (III), we give the geopotential φ , causing a reflection of gravity waves similar to the one in the first case. The stability properties are here however less favourable, and we have actually not been able to construct a stable difference scheme with this boundary condition.

The number of additional conditions required by the difference equations depends on the choice of difference approximation. For an approximation of second order accuracy in space we need the values of the remaining variables at boundary points. If we modify it by using fourth order approximations in the advection terms, we must also decide how to determine the variables at a row of points outside the region. In both cases we may find the extra boundary values by extrapolation rules, which must be designed so that they are both sufficiently accurate and also do not upset the stability of the scheme.

5.3. *Boundary conditions for the semi-implicit schemes*

5.3.1. *Case I-conditions.* We assume now that the normal velocity is given along the whole

boundary, together with the tangential velocity at inflow points.

In the finite-difference formulation, we first give conditions for scheme B. Due to the staggering, the formulae used at odd and even time-steps will be quite different. To simplify the notation, we first rewrite system (2.5) as

$$u^{n+1} = -\Delta t D_{0x}(\varphi^{n+1} - \varphi^{n-1}) + P \quad (5.1)$$

$$v^{n+1} = -\Delta t D_{0y}(\varphi^{n+1} - \varphi^{n-1}) + Q \quad (5.2)$$

$$\varphi^{n+1} - \varphi^{n-1} = -\Delta t \overline{\varphi^{nxy}} (D_{0x} u^{n+1} + D_{0y} v^{n+1}) + R \quad (5.3)$$

where P , Q , R denote the remaining terms (at time t_n and t_{n-1}), and the resulting equation of type (2.3) as

$$(1 - \Delta t^2 \overline{\varphi^{nxy}} (D_{0x}^2 + D_{0y}^2)) (\varphi^{n+1} - \varphi^{n-1}) = S \quad (5.4)$$

with

$$S = R - \Delta t \overline{\varphi^{nxy}} (D_{0x} P + D_{0y} Q) \quad (5.5)$$

We now also assume that the boundary has been placed so that at *odd time-steps*, the only points at the boundary are normal-velocity points. This was a prescribed quantity, and no additional conditions are required. In the derivation of the equation for φ -points next to the boundary, we insert this value and obtain

$$\left(1 - \Delta t^2 \overline{\varphi^{nxy}} \left(D_{0 \text{ tang}}^2 - \frac{1}{2\Delta s} D_{-2n} \right) \right) (\varphi_{b-1}^{n+1} - \varphi_{b-1}^{n-1}) = S \quad (5.6)$$

instead of (5.4), where $D_{0 \text{ tang}}$ is the usual central difference operator in the tangential direction and D_{-2n} is a non-centered $2\Delta s$ difference operator in the direction normal to the boundary. In this and the following formulae, the index b denotes a point at the boundary, $b-1$ the first interior point etc. The term S is given by the previous formula (5.5) if we only define the relevant P or Q at the boundary point to be given by the prescribed normal-velocity value. Combining (5.6) with the usual formula (5.4) for the other φ -points inside the region, we can now solve for φ^{n+1} and then also u^{n+1} , v^{n+1} .

At *even time-steps*, we have to determine the values of the tangential velocity and the geopotential at the boundary. The tangential velocity (v_{tang}) _{b}

is given at t_{n+1} if the normal velocity at t_n has the inward direction, and otherwise we determine it by linear extrapolation from the previous interior values:

$$(v_{\text{tang}})_b^{n+1} = 2(v_{\text{tang}})_{b-1}^n - (v_{\text{tang}})_{b-2}^{n-1} \quad (5.7)$$

Such an inward-backward extrapolation formula was first suggested by B. Gustafsson (see Gustafsson et al., 1972).

The value of the geopotential at the boundary cannot be determined by a similar extrapolation, because it is not a characteristic quantity. Furthermore, we cannot extrapolate the second characteristic quantity, since the staggering means that we do not have φ and v_n defined at the same points. Instead, we proceed as follows. In the solution, the following relation should be approximately true:

$$(v_n)_b^{n+1} = \frac{3}{4}(v_n)_{b-1}^{n+1} - \frac{1}{4}(v_n)_{b-3}^{n+1} \quad (5.8)$$

where $(v_n)_b^{n+1}$ is given. Combining it with (5.1) or (5.2), we get the following "extrapolation formula" for $(\varphi^{n+1} - \varphi^{n-1})_b$:

$$\begin{aligned} & (\varphi^{n+1} - \varphi^{n-1})_{b-4} - 4(\varphi^{n+1} - \varphi^{n-1})_{b-2} + 3(\varphi^{n+1} - \varphi^{n-1})_b \\ &= \frac{2\Delta s}{\Delta t} (-Q_{b-3}^n + 3Q_{b-1}^n - 2(v_n)_b^{n+1}) \end{aligned} \quad (5.9)$$

which may be inserted into (5.4) to give

$$\begin{aligned} & \left(1 - \Delta t^2 \overline{\varphi^{nxy}} \left(D_{0 \text{ tang}}^2 - \frac{1}{3\Delta s} D_{-2n} \right) \right) (\varphi_{b-2}^{n+1} - \varphi_{b-2}^{n-1}) \\ &= S'_{b-2} \end{aligned} \quad (5.10)$$

where S'_{b-2}

$$= S_{b-2} + \left(\frac{\Delta t}{6\Delta s} \right) \overline{\varphi^{nxy}} (-Q_{b-3}^n + 3Q_{b-1}^n - 2(v_n)_b^{n+1})$$

a formula of the same type as (5.6). The values of φ^{n+1} may now be solved at all interior points, and (5.9) then gives also φ_b^{n+1} . At the corner points, we need the value of φ only in calculating the coefficient $\overline{\varphi^{xy}}$ at the next step, and it is thus not critical how it is determined. We have used a simple extrapolation form

$$\varphi_{a,b}^{n+1} = \varphi_{a-2,b}^{n+1} + \varphi_{a,b-2}^{n+1} - \varphi_{a-2,b-2}^{n+1}$$

In the "split" version (5.6) is modified. At *odd time-steps*,

$$(1 - \Delta t^2 \overline{\varphi^{nxy}} D_{0x}^2) \left(1 \mp \frac{\Delta t^2}{2\Delta s} \overline{\varphi^{nxy}} D_{\pm 2y} \right) (\varphi^{n+1} - \varphi^{n-1})_{b-1} = S$$

at boundaries $y = \text{constant}$ and

$$\left(1 \mp \frac{\Delta t^2}{2\Delta s} \overline{\varphi^{nxy}} D_{\pm 2x} \right) (1 - \Delta t^2 \overline{\varphi^{nxy}} D_{0y}^2) (\varphi^{n+1} - \varphi^{n-1})_{b-1} = S$$

at boundaries $x = \text{const.}$, where D_{+2x} , D_{-2y} etc. are non-centered $2\Delta s$ difference operators.

At even time-steps we use instead

$$(1 - \Delta t^2 \overline{\varphi^{nxy}} D_{0x}^2) \left(1 \mp \frac{\Delta t^2}{3\Delta s} \overline{\varphi^{nxy}} D_{\pm 2y} \right) (\varphi^{n+1} - \varphi^{n-1})_{b-2} = S'_{b-2}$$

at boundaries $y = \text{const.}$, and

$$\left(1 \mp \frac{\Delta t^2}{3\Delta s} \overline{\varphi^{nxy}} D_{\pm 2x} \right) (1 - \Delta t^2 \overline{\varphi^{nxy}} D_{0y}^2) (\varphi^{n+1} - \varphi^{n-1})_{b-2} = S'_{b-2}$$

at the boundaries $x = \text{constant}$.

For scheme A, we can use essentially the same boundary conditions, since it may be regarded as two coupled systems of type B. One of these requires "odd" conditions, and the other "even" conditions. The coefficient $\overline{\varphi^{nxy}}$ must however be changed into $\frac{1}{2}(\overline{\varphi^{nx}} + \overline{\varphi^{ny}})$ and the extrapolation formula (5.7) into

$$(v_{\text{tang}})_b^{n+1} = 2(v_{\text{tang}})_{b-1}^n - (v_{\text{tang}})_{b-2}^{n-1}$$

5.3.2. Case II-conditions. Here, we prescribe the value attained by the combination $\varphi - \sqrt{\varphi} v_n = g$, together with v_{tang} at inflow points. The staggering of the grid means however that φ and v_n are not defined at the same points, and we must again use some indirect method.

At odd time-steps, we give a special formula for v_n^{n+1} at the boundary. It differs from the one in (2.5) as follows:

(a) The term $D_{0n}((\varphi^{n+1} + \varphi^{n-1})/2)$ is modified into the non-centered $1\Delta s$ difference operator $D_{-n}((\varphi^{n+1} + \varphi^{n-1})/2)$, in which the undefined value of $\varphi_b^{n+1} + \varphi_b^{n-1}$ is expressed as $\sqrt{\hat{\varphi}_b}(v_n^{n+1} + v_n^{n-1})_b + (g^{n+1} + g^{n-1})_b$, with $\hat{\varphi}_b$ computed as $\overline{\varphi_b^{n\text{tang}}}$.

(b) In the term for advection along the boundary, the difference operator is computed at the first interior point and not as the average across the boundary.

(c) In the term for advection across the boundary, the coefficient v_n is expressed as $\overline{(v_n)_{b-1}^{n\text{tang}}}$ and the difference approximation to $(\partial/\partial n)v_n$ as the non-centered $D_{-n}(v_n)^{n\text{tang}}$, in which the undefined $(v_n)_b^{n\text{tang}}$ is also changed into $\frac{1}{2}(v_n^{n+1} + v_n^{n-1})_b$.

The formula can thus be written

$$\begin{aligned} & \frac{(v_n)_b^{n+1} - (v_n)_b^{n-1}}{2\Delta t} \\ &= -\frac{1}{2\Delta s} (g_b^{n+1} + g_b^{n-1}) \\ &+ \sqrt{\hat{\varphi}_b} ((v_n)_b^{n+1} + (v_n)_b^{n-1}) - \varphi_{b-1}^{n+1} - \varphi_{b-1}^{n-1} \\ &- (v_{\text{tang}})_b^n D_{0\text{tang}}(v_n)_{b-1}^n \\ &- \frac{\overline{(v_n)_{b-1}^{n\text{tang}}} (v_n)_b^{n+1} + (v_n)_b^{n-1} - 2\overline{(v_n)_{b-1}^{n\text{tang}}}}{2\Delta s} \\ &\pm f(v_{\text{tang}})_b^n \end{aligned} \quad (5.11)$$

None of these modifications changes the accuracy of the scheme with more than one order, and the over-all accuracy is then not disturbed.

After having collected all t_{n+1} -terms, (5.11) gives a relation between $(v_n)_b^{n+1}$, φ_{b-1}^{n+1} , and known values. Combining it with the φ -equation for interior points next to the boundary, we get

$$\left(1 + \gamma_{b-1} - \Delta t^2 \overline{\varphi^{nxy}} \left(D_{0\text{tang}}^2 - \frac{1}{2\Delta s} D_{-2n} \right) \right) (\varphi^{n+1} - \varphi^{n-1})_{b-1} = S \quad (5.12)$$

where the term

$$\gamma_{b-1} = \frac{\Delta t^2}{2\Delta s^2} \overline{\varphi^{nxy}} \left/ \left(1 + \frac{\Delta t}{\Delta s} (\sqrt{\hat{\varphi}_b} - \overline{(v_n)_{b-1}^{n\text{tang}}}) \right) \right.$$

is always positive and non-singular. S contains only already known quantities. From (5.12) valid for all points next to the boundary, and (5.4) for other points, we may now compute φ^{n+1} and then obtain the velocities from the original equations and their boundary modifications.

At even time-steps we use the following special formula for φ^{n+1} at the boundary, which is obtained in mainly the same way as (5.11):

$$\begin{aligned}
\frac{\varphi_b^{n+1} - \varphi_b^{n-1}}{2\Delta t} = & -\hat{\varphi}_b \left[D_0 \text{tang} \left(\frac{(v_{\text{tang}})_b^{n+1} + (v_{\text{tang}})_b^{n-1}}{2} \right) \right. \\
& + \frac{1}{2\Delta s} ((\varphi_b^{n+1} + \varphi_b^{n-1} - g_b^{n+1} - g_b^{n-1})/\sqrt{\hat{\varphi}_b} \\
& \left. - ((v_n)_{b-1}^{n+1} + (v_n)_{b-1}^{n-1})) \right] \\
& - (v_{\text{tang}})_{b-1}^n D_0 \text{tang} \varphi_{b-1}^n \\
& - \frac{(v_n)_b^n \text{tang} \varphi_b^{n+1} + \varphi_b^{n-1} - 2\overline{(v_n)_b^n \text{tang}}}}{2\Delta s} \quad (5.13)
\end{aligned}$$

In this case $\hat{\varphi}_b$ is defined to be

$$\hat{\varphi}_b = \frac{3}{2} \overline{(v_n)_b^n \text{tang}} - \frac{1}{2} \varphi_{b-3}^n \text{tang} \quad (5.14)$$

The tangential velocity is given at the inflow part of the boundary (i.e. where $(v_n)_b^n < 0$). At the outflow part we could use the same extrapolation formula (5.7) as before. However, we prefer a non-centered formula, which differs from the one in (2.5) in the same way as (5.11), except that the $D_0 \text{tang}(\varphi^{n+1} + \varphi^{n-1})$ is not affected. The formula will thus be

$$\begin{aligned}
\frac{(v_{\text{tang}})_b^{n+1} - (v_{\text{tang}})_b^{n-1}}{2\Delta t} = & -D_0 \text{tang} \frac{(\varphi_b^{n+1} + \varphi_b^{n-1})}{2} \\
& - \overline{(v_{\text{tang}})_{b-1}^n \text{tang}} D_0 \text{tang} (v_{\text{tang}})_{b-1}^n \\
& - (v_n)_b^n \frac{(v_{\text{tang}})_b^{n+1} + (v_{\text{tang}})_b^{n-1} - 2\overline{(v_{\text{tang}})_{b-1}^n \text{tang}}}}{2\Delta s} \\
& \pm f(v_n)_b^n \quad (5.15)
\end{aligned}$$

Proceeding in the usual way we get a formula of the type

$$\left(1 + \gamma_b + \Delta t^2 \hat{\varphi}_b \frac{D_{-2n}}{2\Delta s}\right) (\varphi_b^{n+1} - \varphi_b^{n-1}) = S_b \quad (5.16)$$

at inflow points, where

$$\gamma_b = \frac{\Delta t}{\Delta s} (\sqrt{\hat{\varphi}_b} - (v_n)_b^n \text{tang}),$$

and

$$\begin{aligned}
\left(1 + \gamma_b - \Delta t^2 \hat{\varphi}_b \left(D_0 \text{tang} \left(1 - \frac{\Delta t}{\Delta s} \overline{(v_n)_b^n \text{tang}}\right)^{-1} D_0 \text{tang} \right. \right. \\
\left. \left. - \frac{D_{-2n}}{2\Delta s}\right)\right) (\varphi_b^{n+1} - \varphi_b^{n-1}) = S_b \quad (5.17)
\end{aligned}$$

at outflow points.

To obtain a useful split version of (5.12), we change (5.3) at the points next to the boundary into

$$\begin{aligned}
& \left(1 - \gamma_{b-1} \Delta t^2 \overline{\varphi^{nxy}} \left(D_0^2 \text{tang} - \frac{1}{2\Delta s} D_{-2n}\right)\right) \\
& (\varphi_{b-1}^{n+1} - \varphi_{b-1}^{n-1}) \\
& = -\Delta t \overline{\varphi^{nxy}} (D_{0x} u^{n+1} + D_{0y} v^{n+1}) + R
\end{aligned}$$

which is one order less accurate than the original formula. We now obtain instead of (5.12) at odd time-steps

$$(1 + \gamma_{b-1}) \left(1 - \Delta t^2 \overline{\varphi^{nxy}} \left(D_0^2 \text{tang} - \frac{1}{2\Delta s} D_{-2n}\right)\right) (\varphi^{n+1} - \varphi^{n-1})_{b-1} = S \quad (5.18)$$

and the addition of the splitting term gives us finally

$$(1 + \gamma_{b-1}) (1 - \Delta t^2 \overline{\varphi^{nxy}} D_{0x}^2) \left(1 \mp \frac{\Delta t^2}{2\Delta s} \overline{\varphi^{nxy}} D_{\pm 2y}\right) (\varphi^{n+1} - \varphi^{n-1})_{b-1} = S$$

at boundaries $y = \text{constant}$, and

$$(1 + \gamma_{b-1}) \left(1 \mp \frac{\Delta t^2}{2\Delta s} \overline{\varphi^{nxy}} D_{\pm 2x}\right) (1 - \Delta t^2 \overline{\varphi^{nxy}} D_{0y}^2) (\varphi^{n+1} - \varphi^{n-1})_{b-1} = S$$

at boundaries $x = \text{constant}$. We can now combine it with the split formulae at other points and obtain φ^{n+1} etc.

We finally get the split version formulae for even time-steps

$$(1 + \gamma_b) (1 - \Delta t^2 \hat{\varphi}_b D_{0x}^2) \left(1 \mp \Delta t^2 \hat{\varphi}_b \frac{D_{\pm 2y}}{2\Delta s}\right) (\varphi_b^{n+1} - \varphi_b^{n-1}) = S_b \quad (5.19)$$

at the boundaries $y = \text{constant}$, and

$$(1 + \gamma_b) \left(1 \mp \Delta t^2 \hat{\varphi}_b \frac{D_{\pm 2x}}{2\Delta s}\right) (1 - \Delta t^2 \hat{\varphi}_b D_{0y}^2) (\varphi_b^{n+1} - \varphi_b^{n-1}) = S_b \quad (5.20)$$

at the boundaries $x = \text{constant}$.

Also in this case, the boundary conditions for scheme A are similar to those presented for scheme B. We only apply both the "even" and

the "odd" conditions at all time steps, change $\overline{\varphi^{nxy}}$ into $\frac{1}{2}(\overline{\varphi^{nx}} + \overline{\varphi^{ny}})$, $\overline{\varphi_b^{n \text{ tang}}}$ into $\overline{\varphi_b^n}$, and modify the advection terms to fit the scheme. We will obtain the same formulae as before, but with

$$\gamma_{b-1} = \frac{\Delta t^2}{2\Delta s^2} \left(\frac{\overline{\varphi_{b-1}^{nx}} + \overline{\varphi_{b-1}^{ny}}}{2} \right) \left(1 + \frac{\Delta t}{\Delta s} (\sqrt{\hat{\varphi}_b} - (v_n)_b^{n \text{ tang}}) \right)$$

and

$$\gamma_b = \frac{\Delta t}{\Delta s} (\sqrt{\hat{\varphi}_b} - (v_n)_b^n)$$

As follows from this presentation, boundary condition II gives a slightly more complicated procedure for determining the φ -equations near the boundary than boundary condition I. This is due to the facts that we use a staggered grid, and that the two variables φ and v_n are mixed in the boundary conditions.

The boundary condition II has however certain properties that should compensate for these complications. It has already been mentioned that its mathematical well-posedness is easy to prove even with a simple energy method and that it essentially lets a gravity wave type of disturbance pass out through the boundary without reflection (or reflection of only a smaller fraction of it). In the Appendix we shall show that the numerical formulation of the boundary condition II given here enables us to prove the stability of the semi-implicit, two-dimensional, limited-area equations (for the constant-coefficient case) by the difference from of the energy method. Actually, the forecast equations at the boundary were formulated so that the contributions to the energy norm from the geopotential force and the advection through the boundary are kept under control. The fact that we use a semi-implicit method instead of a leap-frog scheme is not decisive—as will be shown below, we can also for the leap-frog scheme formulate the numerical boundary conditions so that the same is true.

5.4. Boundary conditions for the leap-frog schemes

Although the explicit schemes A and B are no major subject of this study, we have found stable case I and II boundary conditions also for these schemes. As an example, we show

here a case II condition and assume thus that the combination $\varphi - \sqrt{\varphi} v_n = g$ is given at the whole boundary, together with v_{tang} at inflow parts.

For scheme B, the only variable defined on the boundary at odd time-steps is v_{tang} , which is prescribed at inflow points. At outflow, we determine it by a non-centered formula, constructed similarly to (5.15):

$$\begin{aligned} \frac{(v_{\text{tang}})_b^{n+1} - (v_{\text{tang}})_b^{n-1}}{2\Delta t} = & -D_{0 \text{ tang}} \varphi_b^n \\ & - \overline{(v_{\text{tang}})_{b-1}^{n \text{ tang}}} D_{0 \text{ tang}} (v_{\text{tang}})_{b-1}^n \\ & - (v_n)_b^n \frac{(v_{\text{tang}})_b^{n+1} + (v_{\text{tang}})_b^{n-1} - 2 \overline{(v_{\text{tang}})_{b-1}^{n \text{ tang}}}}{2\Delta s} \\ & \pm f(v_n)_b^n \end{aligned} \quad (5.21)$$

At even time-steps, v_n and φ are defined at successive boundary points.

We determine their values by non-centered formulae, which differ from the ordinary ones in (2.7) through:

(a) the term $D_{0n} \varphi^n$ in the v_n -equation and the term $D_{0n} v_n^n$ in the φ -equation are first modified into the non-centered

$$\frac{\varphi_b^{n+1} + \varphi_b^{n-1} - 2\varphi_{b-1}^n}{2\Delta s}$$

and

$$\frac{(v_n)_b^{n+1} + (v_n)_b^{n-1} - 2(v_n)_{b-1}^n}{2\Delta s},$$

in which the "undefined" quantities $\varphi_b^{n+1} + \varphi_b^{n-1}$ and $(v_n)_b^{n+1} + (v_n)_b^{n-1}$ are changed into $g^{n+1} + g^{n-1} + \sqrt{\hat{\varphi}_b}((v_n)_b^{n+1} + (v_n)_b^{n-1})$ and $(\varphi_b^{n+1} + \varphi_b^{n-1} - g^{n+1} - g^{n-1})/\sqrt{\hat{\varphi}_b}$.

(b) in describing the advection we use the same formulae as in (5.11) and (5.13).

Also here the non-centered difference formulae are constructed so that the energy contributions corresponding to the work done by the geopotential force and the advection of energy into the region are kept under control.

For scheme A, the main difference is that all conditions have to be used at all time-steps.

5.5. Stability proofs for the limited-area problem

For a limited area of integration, there is unfortunately in general no easy, direct way of determining the stability properties of a two-dimensional difference scheme. The energy

method of proving stability works only for boundary condition II, and only if the numerical formulation of that boundary condition is properly designed. Such a formulation was given in section 5.3.2.

In the Appendix, the details of the stability proof are given, and it is shown that the stability conditions are still $\lambda(|u| + |v|) < 1$ for scheme A and $\lambda \max(|u|, |v|) < 1$ for scheme B.

For boundary condition I (and other formulations of boundary condition II), we must instead apply the general method for initial-boundary value stability analysis worked out by H.-O. Kreiss, see e.g. Kreiss (1971). A recent article by Gustafsson et al. (1972) gives a thorough presentation of this theory for the one-dimensional case.

An example of such a one-dimensional stability investigation is given in the Appendix. We can in this way also find some necessary stability conditions for our two-dimensional problem. To proceed further, we may consider the possibility of having unstable solutions which are periodic functions in the tangential coordinate direction. By a Fourier expansion, we obtain a set of one-dimensional problems which should be investigated for all relevant tangential wave-numbers. In certain cases, the formulation of boundary conditions at the corner regions may, however, be critical, and the analysis of stability for a certain set of boundary conditions should always be completed by computer experiments.

5.6. Accuracy considerations for the limited-area problem

For the pure initial value problem, we could discuss the accuracy properties of the finite-difference model in terms of the eigenfrequency and eigenvector of a long-wave solution to the simplified schemes (4.4), (4.5). We found that the main source of error was the truncation errors committed during the integration.

We must now also consider the errors generated at the boundary, due to our inexact knowledge of the solution outside the region and the interaction between the interior solution and the numerical boundary conditions. A general estimate of the rate of error growth, measured by a R.M.S. norm, can be found by methods similar to those used in the stability proofs (see Gustafsson, 1973). However, if we can as before neglect the errors in the initial

data, assume that the boundary errors are slowly varying with time and not too large, and if we use a large enough region of integration, the main source of error is still the truncation error of our scheme. The analysis and comparison of the accuracy in section 4 is then still applicable.

6. Results from numerical experiments

Most of the numerical experiments have been made with the initial values

$$u(x, y, 0) = U_0 - U_1 \sin \frac{2\pi x}{L} \cos \frac{2\pi(y - y_0)}{L}$$

$$v(x, y, 0) = V_0 + U_1 \cos \frac{2\pi x}{L} \sin \frac{2\pi(y - y_0)}{L}$$

$$\begin{aligned} \varphi(x, y, 0) = & \Phi_0 + fV_0 \left(x - \frac{1}{2}L \right) - fU_0 \left(y - \frac{1}{4}L \right) \\ & + \frac{Lf}{2} U_1 \sin \frac{2\pi x}{L} \sin \frac{2\pi(y - y_0)}{L} \\ & + \left(\frac{1}{2} U_1 \right)^2 \left(\cos \frac{4\pi x}{L} + \cos \frac{4\pi(y - y_0)}{L} \right) \end{aligned}$$

describing a single wave on a constant velocity field, with the geopotential chosen so that the non-linear balance equation is initially satisfied.

The area of integration was $0 \leq x \leq L$, $0 \leq y \leq L/2$, $t \geq 0$ and the Coriolis parameter $f = 10^{-4} \text{ s}^{-1}$. The parameter values normally used were

$$\begin{aligned} U_0 &= 5 \text{ m/s} \\ V_0 &= 0 \text{ m/s} \\ U_1 &= 10 \text{ m/s} \\ \Phi_0 &= 10^4 \text{ m}^2/\text{s}^2 \\ L &= 4 \text{ 000 km} \\ y_0 &= 0 \text{ km} \end{aligned}$$

The integration region was usually divided into a 20×10 point grid, with $\Delta s = 100 \text{ km}$ and a time-step Δt of one hour.

All experiments were made with a type B scheme. The reason for this choice was that the stability problems should be similar in the two schemes, and the programming is slightly easier for the type B scheme.

In some integrations, we gave case I or II boundary conditions only at the boundaries $y = 0$ and $y = \frac{1}{2}L$ and used instead periodicity at the boundaries $x = 0$ and $x = L$; in other experi-

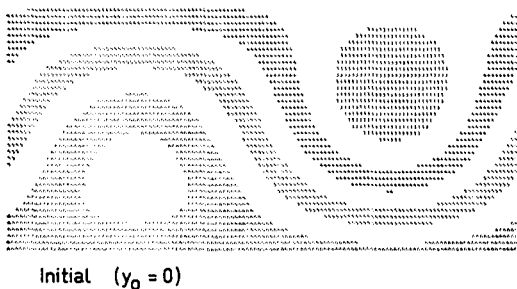


Fig. 1. Initial φ -field for $y_0 = 0$. No flow through the boundaries $y = 0, y = L/2$.

ments, we gave case I or II conditions at all boundaries. The values of the prescribed boundary quantities were chosen as their initial values, advected by ($U_{\text{bnd}}, 0$).

An often used value for U_{bnd} was 4 m/s, being the approximate true speed of the waves.

As a more critical stability test, we have sometimes used initial and boundary values, disturbed in each point by a random variable with average value zero.

The emphasis of the experiments has been to investigate the stability of the different

numerical boundary conditions. Many schemes have been rejected because of instabilities (or a bounded but too fast error growth rate). Some schemes were stable for one-dimensional problems, but the introduction of a second space dimension gave new possibilities for instabilities, usually short-wave in the tangential direction.

The numerical formulations of the case I and II boundary conditions given above did not show any serious instabilities. For the case II conditions, this only confirmed the stability proof above.

Although not being of primary interest for short-range, limited area integrations, most cases could be run up to a considerable amount of time (240 h or more) without showing any instabilities, even without any smoothing at all. This might partially be due to the initial and boundary conditions, which mainly consisted of a long wave. Some experiments were also made with a larger amount of short-wave disturbances superimposed on the main solution. Due to the non-linearity of the equations, non-linear instabilities could then develop. This phenomenon occurred especially if the normal ve-

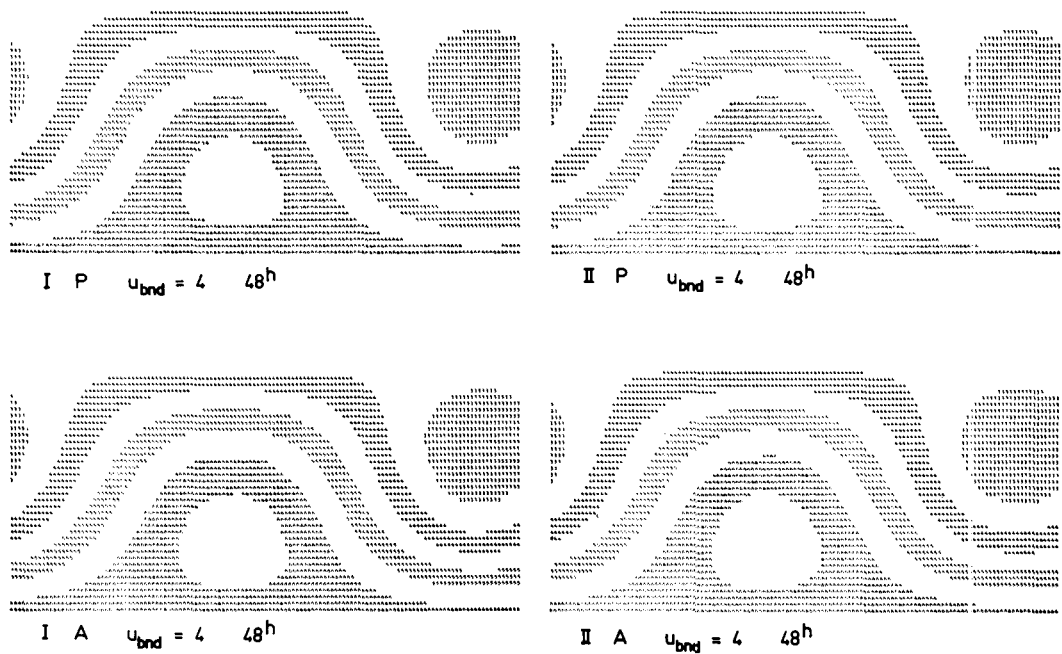


Fig. 2. 48 h forecast of φ with boundary conditions I and II, used either at all boundaries (A) or only at the boundaries $y = 0, y = L/2$ with periodicity conditions (P) at $x = 0, x = L$.

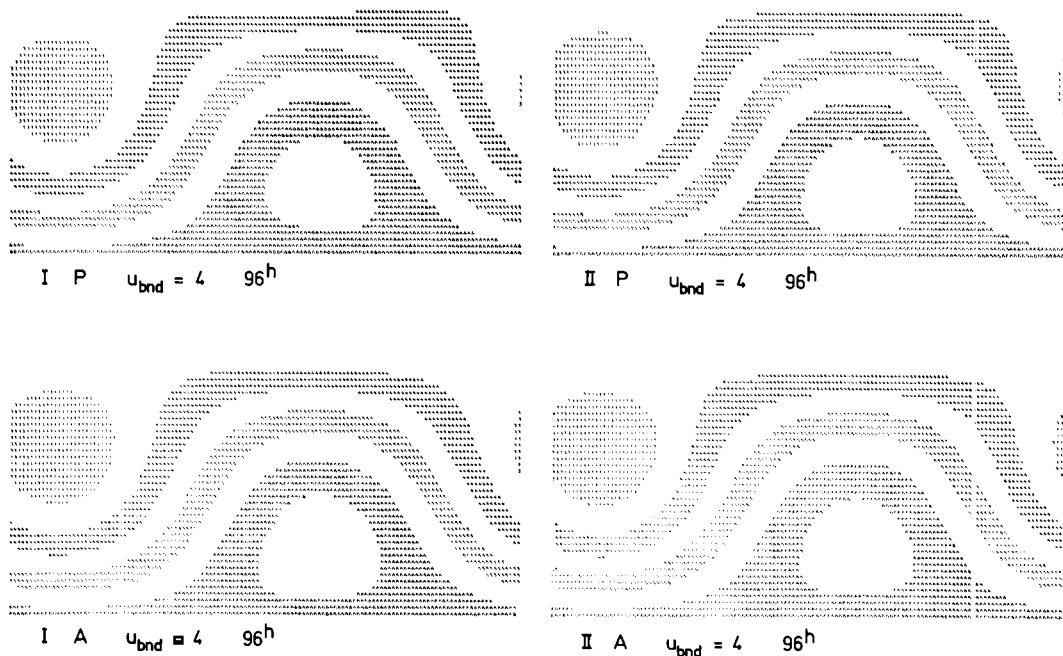


Fig. 3. 96 h forecast from the same initial conditions as in Fig. 2.

locity at the boundary was initially zero. Then the boundary values for the tangential velocity were not prescribed but instead computed. After some time, the truncation errors caused the normal velocity to be slightly positive in some points, negative in others. The tangential velocity should be prescribed at inflow points and computed at outflow points. As a result, the normal velocity became still more unevenly distributed at the next time-step. A type of non-linear instability was formed, similar to the type that can occur in the interior if e.g. the velocity in a coordinate direction is small and the short-wave disturbances are sufficiently large in magnitude. These non-linear disturbances can, however, be taken care of by adding a high-order dissipation term to the equations (Kreiss & Oliger, 1972).

The results of the experiments with $U_{\text{bnd}} = 4$ m/s (Figs. 2, 3, 5) show that boundary conditions I and II give approximately the same results, as long as the quantities specified at the boundaries agree with the interior solution. The experiments with $U_{\text{bnd}} = 0$ (Fig. 6) show larger differences, but no trace of instabilities was observed. The large boundary value errors did however disturb the interior solution considerably.

For a short wave initial solution (Fig. 7), we noticed that the boundary condition I gives a scheme that is more sensitive to disturbances than the scheme with boundary condition II. This may be due to the fact that gravity wave type disturbances, generated inside the region, are reflected at the boundaries when we use boundary condition I but instead transmitted out of the region when we use boundary condition II.

Some experiments have also been run with boundary condition III (the geopotential φ given everywhere, and the tangential velocity at inflow parts of the boundary). However, all



Fig. 4. Initial φ -field with $y_0 = 0.25L$, giving a situation with flow through all boundaries.

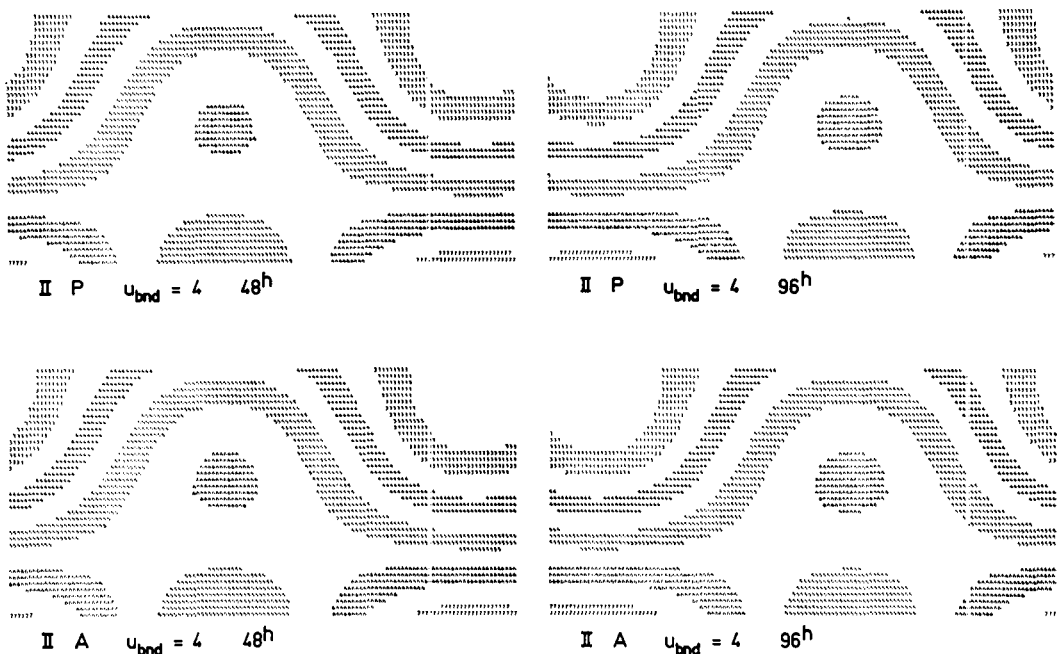


Fig. 5. 48 h and 96 h forecasts of φ with boundary condition II from the initial solution of Fig. 4. The boundary conditions are used either at all boundaries (A) or with periodicity (P) at $x=0$ and $x=L$. No significant differences can be observed.

experiments showed weak instabilities at inflow points, both with the semi-implicit and the leap-frog scheme and even for modifications with a large amount of dissipation added to them. The disturbances appeared also when we had a constant, geostrophically balanced wind field as initial (and stationary) solution. After a while, the disturbances became so large that a non-linear instability was set up (Fig. 8). We are at present of the opinion that these error growths are caused by a weak ill-posedness of the differential equations—prescribing φ is maybe not a permissible boundary condition—which then remains in the difference scheme, and not by any effects of our numerical formulation of the boundary conditions.

7. Conclusions

The comparison between the leap-frog and semi-implicit schemes showed that the semi-implicit scheme A seems to be the most advantageous of these schemes, although the computational work per time-step is roughly twice that of the corresponding leap-frog scheme. From the experiments (with type B schemes) we also found that the reduction of computational work

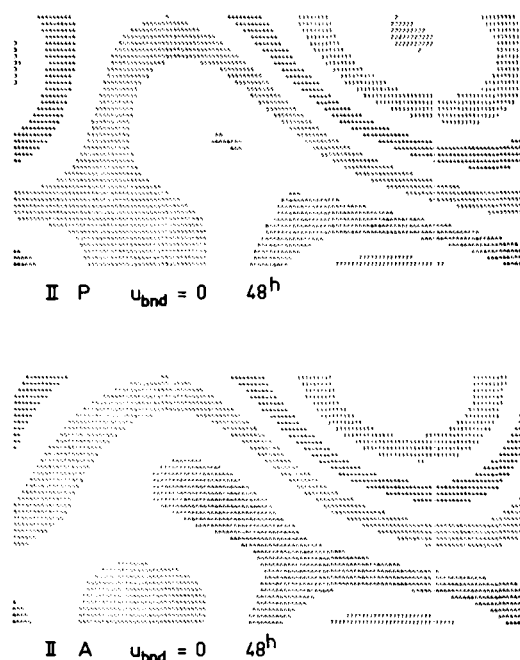


Fig. 6. 48 h forecast from the initial solution of Fig. 4, keeping the prescribed boundary values time-independent. No signs of instabilities, although the large boundary value errors disturb the interior flow considerably.



I A $u_{bnd} = 4 \quad 48^h \quad \Delta t = 10^m \quad 10 \times 20$

II A $u_{bnd} = 4 \quad 48^h \quad \Delta t = 10^m \quad 10 \times 20$



I A $u_{bnd} = 4 \quad 48^h \quad \Delta t = 5^m \quad 20 \times 40$

II A $u_{bnd} = 4 \quad 48^h \quad \Delta t = 5^m \quad 20 \times 40$

Fig. 7. 48 h forecast from a short-wave initial field, using boundary conditions I and II with two different grid lengths. Notice the boundary disturbances for the coarse grid with case I boundary condition.



Explicit III A $22^h 45^m$

Fig. 8. Explosive growth of error with boundary condition III, here for a leap-frog scheme with a constant, geostrophically balanced wind field as initial (and stationary) solution.

was still at least 1:2. Two sets of boundary conditions passed both the theoretical and experimental stability tests, and the best choice between them must be based on further tests with various realistic initial conditions.

Appendix

We shall here first show how the direct "energy method" of proving stability works for a limited-area integration with boundary conditions of type II, in the numerical formulation of section 5.3.2. For simplicity, the proof is only given for the type B staggered grid. For the model equation (3.2) and a region $0 \leq x \leq 2J\Delta s$, $0 \leq y \leq 2K\Delta s$, the equations used at the boundary are at odd time-steps

$$\begin{aligned} \frac{v_{0,k}^{n+1} - v_{0,k}^{n-1}}{2\Delta t} &= -\frac{1}{2\Delta s} (-g^{n+1} + g^{n-1})_{0,k} \\ &+ \sqrt{\Phi}(u^{n+1} + u^{n-1})_{0,k} + (\varphi^{n+1} + \varphi^{n-1})_{1,k} \\ &+ \frac{U}{2\Delta s} ((u^{n+1} + u^{n-1})_{0,k} - (u_{1,k+1}^n + u_{1,k-1}^n)) \\ &- \frac{V}{2\Delta s} (u_{1,k+1}^n - u_{1,k-1}^n) \\ \frac{v_{j,0}^{n+1} - v_{j,0}^{n-1}}{2\Delta t} &= -\frac{1}{2\Delta s} (-g^{n+1} + g^{n-1})_{j,0} \\ &+ \sqrt{\Phi}(v^{n+1} + v^{n-1})_{j,0} + (\varphi^{n+1} + \varphi^{n-1})_{j,1} \\ &- \frac{U}{2\Delta s} (v_{j+1,1}^n - v_{j-1,1}^n) \\ &+ \frac{V}{2\Delta s} ((v^{n+1} + v^{n-1})_{j,0} - (v_{j+1,1}^n + v_{j-1,1}^n)) \end{aligned}$$

etc, and for even time-steps $v_{0,k}^{n+1}$, $v_{2J,k}^{n+1}$, $u_{j,0}^{n+1}$ and $u_{j,2K}^{n+1}$ are given at inflow and computed by

$$\begin{aligned} \frac{u_{j,0}^{n+1} - u_{j,0}^{n-1}}{2\Delta t} &= -\frac{1}{4\Delta s} ((\varphi^{n+1} + \varphi^{n-1})_{j+1,0} \\ &- (\varphi^{n+1} + \varphi^{n-1})_{j-1,0}) - \frac{U}{2\Delta s} (u_{j+1,1}^n - u_{j-1,1}^n) \\ &+ \frac{V}{2\Delta s} ((u^{n+1} + u^{n-1})_{j,0} - (u_{j+1,1}^n + u_{j-1,1}^n)) \end{aligned}$$

$$\begin{aligned} \frac{v_{0,k}^{n+1} - v_{0,k}^{n-1}}{2\Delta t} &= -\frac{1}{4\Delta s} ((\varphi^{n+1} + \varphi^{n-1})_{0,k+1} \\ &- (\varphi^{n+1} + \varphi^{n-1})_{0,k-1}) + \frac{U}{2\Delta s} ((v^{n+1} + v^{n-1})_{0,k} \\ &- (v_{1,k+1}^n + v_{1,k-1}^n)) - \frac{V}{2\Delta s} (v_{1,k+1}^n - v_{1,k-1}^n) \end{aligned}$$

etc. at outflow. The geopotential values $\varphi_{0,k+1}^{n+1}$, $\varphi_{2J,k+1}^{n+1}$, $\varphi_{j+1,0}^{n+1}$ and $\varphi_{j+1,2K}^{n+1}$ are determined by

$$\begin{aligned} \frac{(\varphi^{n+1} - \varphi^{n-1})_{0,k+1}}{2\Delta t} &= -\frac{\Phi}{2\Delta s} (((\varphi^{n+1} + \varphi^{n-1})_{0,k+1} \\ &- (g^{n+1} + g^{n-1})_{0,k+1}) / \sqrt{\Phi} + (u^{n+1} + u^{n-1})_{1,k+1}) \\ &- \frac{\Phi}{4\Delta s} ((v^{n+1} + v^{n-1})_{0,k+2} - (v^{n+1} + v^{n-1})_{0,k}) \\ &+ \frac{U}{2\Delta s} ((\varphi^{n+1} + \varphi^{n-1})_{0,k+1} - (\varphi_{1,k+2}^n + \varphi_{1,k}^n)) \\ &- \frac{V}{2\Delta s} (\varphi_{1,k+2}^n - \varphi_{1,k}^n) \end{aligned}$$

$$\begin{aligned} \frac{(\varphi^{n+1} - \varphi^{n-1})_{j+1,0}}{2\Delta t} &= -\frac{\Phi}{4\Delta s} ((u^{n+1} + u^{n-1})_{j+2,0} \\ &- (u^{n+1} + u^{n-1})_{j,0}) - \frac{\Phi}{2\Delta s} (((\varphi^{n+1} + \varphi^{n-1})_{j+1,0} \\ &- (g^{n+1} + g^{n-1})_{j+1,0}) / \sqrt{\Phi} + (v^{n+1} + v^{n-1})_{j+1,1}) \\ &- \frac{U}{2\Delta s} (\varphi_{j+2,1}^n - \varphi_{j,1}^n) \\ &+ \frac{V}{2\Delta s} ((\varphi^{n+1} + \varphi^{n-1})_{j+1,0} - (\varphi_{j+2,1}^n + \varphi_{j,1}^n)) \end{aligned}$$

etc. Combined with the difference equations (3.2) at interior points, these boundary equations now give a closed problem. Since the model equation is linear, a disturbance on e.g. the initial values satisfies the same equations, with g and the given values of $v_{\text{tang}} = 0$. In the same way as for the Cauchy problem, we can now derive an inequality for the energy norm of the disturbance

$$\sum_{j=0}^{2J} \sum_{k=0}^{2K} W_{jk} (\Phi(u^{n+1} - u^n + v^{n+1} + v^n) + \varphi^{n+1} + \varphi^n)_{jk}$$

where $W_{jk} = 1$ for interior points, $W_{jk} = \frac{1}{2}$ for boundary points and $W_{jk} = \frac{1}{4}$ for the corner

points. After a lengthy but straightforward summation of terms, we obtain

$$\begin{aligned} \sum \sum W_{jk} (\Phi(u^{n+1} + u^n + v^{n+1} + v^n) + \varphi^{n+1} + \varphi^n)_{jk} \\ < \frac{1 + \lambda \max(|U|, |V|)}{1 - \lambda \max(|U|, |V|)} \\ \times \sum \sum W_{jk} (\Phi(u^n + u^{0n} + v^n + v^{0n}) + \varphi^n + \varphi^{0n}) \end{aligned}$$

and the stability condition is as before

$$\lambda \cdot \max(|U|, |V|) < 1.$$

Such "energy method" stability proofs can only be used in special cases. In general, we have to apply the resolvent equation method of H.-O. Kreiss (Kreiss 1971, Gustafsson et al. 1972).

As an example, we give the stability proof for a one-dimensional problem corresponding to (3.1) or (3.2) with boundary condition I. The equations are:

$$\left. \begin{aligned} \frac{u^{n+1} - u^{n-1}}{2\Delta t} &= -D_{0x} \frac{\varphi^{n+1} + \varphi^{n-1}}{2} - U D_{0x} u^n \\ \frac{v^{n+1} - v^{n-1}}{2\Delta t} &= -U D_{0x} v^n \\ \frac{\varphi^{n+1} - \varphi^{n-1}}{2\Delta t} &= -\Phi D_{0x} \frac{u^{n+1} + u^{n-1}}{2} \\ &\quad - U D_{0x} \varphi^n \end{aligned} \right\} \quad (\text{A.1})$$

As boundary conditions, we give at odd time-steps u_0^{n+1} and u_{2J}^{n+1} . At even time-steps, v_0^{n+1} is given and v_{2J}^{n+1} computed by the extrapolation formula $v_{2J}^{n+1} = 2v_{2J-1}^n - v_{2J-2}^n$ if $U > 0$, and if $U < 0$, v_{2J}^{n+1} is instead given and v_0^{n+1} put equal to $2v_1^n - v_2^{n-1}$. The values of φ_0^{n+1} and φ_{2J}^{n+1} are determined indirectly so that the relations $u_0^{n+1} = \frac{3}{2}u_1^{n+1} - \frac{1}{2}u_3^{n+1}$ and $u_{2J}^{n+1} = \frac{3}{2}u_{2J-1}^{n+1} - \frac{1}{2}u_{2J-3}^{n+1}$ are satisfied.

For the scheme (A.1) to be stable with these boundary conditions, it should have no eigen-solutions that grow in amplitude with the number of time-steps. The general eigensolution may however be written down easily: it has the form

$$u_j^n = z^n (A_1 \kappa_1^j + A_2 \kappa_2^j + A_3 \kappa_3^j + A_4 \kappa_4^j)$$

$$v_j^n = z^n (A_5 \kappa_5^j + A_6 \kappa_6^j)$$

$$\varphi_j^n = z^n \sqrt{\Phi} (A_1 \kappa_1^j + A_2 \kappa_2^j - A_3 \kappa_3^j - A_4 \kappa_4^j)$$

where κ_1, κ_2 are the roots of

$$\kappa^2 + \frac{\kappa}{\lambda} (z - z^{-1}) \left/ \left(U + \sqrt{\Phi} \frac{z + z^{-1}}{2} \right) - 1 \right. = 0,$$

κ_3, κ_4 are the roots of

$$\kappa^2 + \frac{\kappa}{\lambda} (z - z^{-1}) \left/ \left(U - \sqrt{\Phi} \frac{z + z^{-1}}{2} \right) - 1 \right. = 0,$$

and κ_5, κ_6 are the roots of

$$\kappa^2 + \frac{\kappa}{\lambda} (z - z^{-1}) / U - 1 = 0.$$

We define $\kappa_1, \kappa_3, \kappa_5$ as the roots of modulus ≤ 1 for $z \geq 1$.

Assuming now that $U > 0$, we get the following conditions for A_1 to A_6 :

$$u_0^{n+1} = 0 \Rightarrow A_1 + A_2 + A_3 + A_4 = 0$$

$$u_{2J}^{n+1} = 0 \Rightarrow A_1 \kappa_1^{2J} + A_2 \kappa_2^{2J} + A_3 \kappa_3^{2J} + A_4 \kappa_4^{2J} = 0$$

$$3u_1^{n+1} - u_3^{n+1} = 0 \Rightarrow A_1 \kappa_1 (3 - \kappa_1^2) + A_2 \kappa_2 (3 - \kappa_2^2) \\ + A_3 \kappa_3 (3 - \kappa_3^2) + A_4 \kappa_4 (3 - \kappa_4^2) = 0$$

$$3u_{2J-1}^{n+1} - u_{2J-3}^{n+1} = 0 \Rightarrow A_1 \kappa_1^{2J-1} (3 - \kappa_1^{-2}) \\ + A_2 \kappa_2^{2J-1} (3 - \kappa_2^{-2}) + A_3 \kappa_3^{2J-1} (3 - \kappa_3^{-2}) \\ + A_4 \kappa_4^{2J-1} (3 - \kappa_4^{-2}) = 0$$

$$v_0^{n+1} = 0 \Rightarrow A_5 + A_6 = 0$$

$$v_{2J}^{n+1} = 2v_{2J-1}^n - v_{2J-2}^n \Rightarrow A_5 \kappa_5^{2J} (1 - 2z^{-1} \kappa_5^{-1} + z^{-2} \kappa_5^{-2}) \\ + A_6 \kappa_6^{2J} (1 - 2z^{-1} \kappa_6^{-1} + z^{-2} \kappa_6^{-2}) = 0$$

These six equations split into two groups. The last two equations can be satisfied for $A_5, A_6 \neq 0$ if and only if

$$(\kappa_5 / \kappa_6)^{2J-2} = (\kappa_6 - z^{-1}) / (\kappa_5 - z^{-1}) \quad (\text{A.2})$$

Since the roots κ_5, κ_6 satisfy $\kappa_5 \kappa_6 = -1$, we must either have $|\kappa_5| < 1$, $|\kappa_6| > 1$ or $|\kappa_5| = |\kappa_6| = 1$. The latter is possible only for $|z| = 1$.

To show that (A.2) has no solution for $|z| > 1$ is rather easy. Writing $\kappa_6 = |\kappa_6| e^{i\alpha}$, $\kappa_5 = -|\kappa_6|^{-1} e^{-i\alpha}$ and noticing that $\text{Re } \kappa_6$ has the opposite sign of $\text{Re } z^{-1}$ for $|z| \geq 1$, the magnitude of the left hand side of (A.2) is < 1 for $|z| > 1$ and all $J > 1$, but that of the right hand side > 1 , leading to a contradiction.

We may also proceed in a different way. Consider instead (A.2) as Δs and $\Delta t \rightarrow 0$, with fixed z , $\lambda = \Delta t / \Delta s$, and $2J \Delta s$. It then goes over into

$$\kappa_6 - z^{-1} = 0 \quad (\text{A.3})$$

which is the eigensolution condition for the right boundary, assuming that the left boundary is removed. If we can show that (A.3) cannot be satisfied for $|z| > 1$ and that it remains unsatisfied as $|z| \rightarrow 1+$, there are no growing eigensolutions to the initial-boundary value problem. It is obvious from the definition of κ_6 that (A.3) is never satisfied for $|z| > 1$. For $|z| = 1$, it would imply $z = 1$ or -1 , but since $\kappa_6 = -1$ for $z = 1$ and $\kappa_6 = 1$ for $z = -1$, (A.3) leads also here to a contradiction. The boundary condition for v is again proved to be stable.

The first four conditions do also separate into two groups. By adding and subtracting the first two and the last two equations and using the relation $\kappa_1 \kappa_2 = \kappa_3 \kappa_4 = -1$, we get

$$\left. \begin{aligned} B_1(1 + \kappa_1^{2J}) + B_2(1 + \kappa_3^{2J}) &= 0 \\ B_1(\kappa_1(3 - \kappa_1^2) - \kappa_1^{2J-1}(3 - \kappa_2^2)) \\ &\quad + B_2(\kappa_3(3 - \kappa_3^2) - \kappa_3^{2J-1}(3 - \kappa_4^2)) = 0 \\ B_3(1 - \kappa_1^{2J}) + B_4(1 - \kappa_3^{2J}) &= 0 \\ B_3(\kappa_1(3 - \kappa_1^2) + \kappa_1^{2J-1}(3 - \kappa_2^2)) \\ &\quad + B_4(\kappa_3(3 - \kappa_3^2) + \kappa_3^{2J-1}(3 - \kappa_4^2)) = 0 \end{aligned} \right\}$$

where $B_1 = A_1 + A_2 \kappa_2^{2J}$, $B_2 = A_3 + A_4 \kappa_4^{2J}$,

$$B_3 = A_1 - A_2 \kappa_2^{2J}, \quad B_4 = A_3 - A_4 \kappa_4^{2J}$$

which has a non-trivial solution if

$$(1 + \kappa_1^{2J})/(1 + \kappa_3^{2J}) = (\kappa_1(3 - \kappa_1^2) - \kappa_1^{2J-1}(3 - \kappa_2^2))/(\kappa_3(3 - \kappa_3^2) - \kappa_3^{2J-1}(3 - \kappa_4^2)) \quad (\text{A.4})$$

or

$$(1 - \kappa_1^{2J})/(1 - \kappa_3^{2J}) = (\kappa_1(3 - \kappa_1^2) + \kappa_1^{2J-1}(3 - \kappa_2^2))/(\kappa_3(3 - \kappa_3^2) + \kappa_3^{2J-1}(3 - \kappa_4^2)) \quad (\text{A.5})$$

The first possibility (A.4) gives

$$\begin{aligned} &(\kappa_1 - \kappa_3)(3 - \kappa_1^2 - \kappa_3^2 - \kappa_1 \kappa_3) \\ &\quad + \kappa_1^{2J}(\kappa_2 - \kappa_3)(3 - \kappa_2^2 - \kappa_3^2 - \kappa_2 \kappa_3) \\ &\quad + \kappa_3^{2J}(\kappa_1 - \kappa_4)(3 - \kappa_1^2 - \kappa_4^2 - \kappa_1 \kappa_4) \\ &\quad + \kappa_1^{2J} \kappa_3^{2J}(\kappa_2 - \kappa_4)(3 - \kappa_2^2 - \kappa_4^2 - \kappa_2 \kappa_4) = 0 \end{aligned}$$

and (A.5)

$$\begin{aligned} &(\kappa_1 - \kappa_3)(3 - \kappa_1^2 - \kappa_3^2 - \kappa_1 \kappa_3) \\ &\quad - \kappa_1^{2J}(\kappa_2 - \kappa_3)(3 - \kappa_2^2 - \kappa_3^2 - \kappa_2 \kappa_3) \\ &\quad - \kappa_3^{2J}(\kappa_1 - \kappa_4)(3 - \kappa_1^2 - \kappa_4^2 - \kappa_1 \kappa_4) \\ &\quad + \kappa_1^{2J} \kappa_3^{2J}(\kappa_2 - \kappa_4)(3 - \kappa_2^2 - \kappa_4^2 - \kappa_2 \kappa_4) = 0 \end{aligned}$$

Both of these are too complicated to analyze directly. Instead we proceed as above, letting Δs and $\Delta t \rightarrow 0$ with z , λ and $2J\Delta s$ fixed. Then, both (A.4) and (A.5) yield the same condition

$$(\kappa_1 - \kappa_3)(3 - \kappa_1^2 - \kappa_3^2 - \kappa_1 \kappa_3) = 0 \quad (\text{A.6})$$

for an eigensolution to exist. Now, $|\kappa_1|$ and $|\kappa_3|$ are both < 1 for $|z| > 1$, and $\text{Re } \kappa_1$ and $\text{Re } \kappa_3$ have the opposite sign for $|z| \geq 1$. The only possibility for a solution to (A.6) is then for $z = \pm i$, giving $\kappa_1 = \kappa_3$.

The eigenvectors belonging to κ_1 and κ_3 are however different, and although there is an eigensolution $u_j^n = v_j^n = 0$, $\varphi_j^n \sim i^{n+J}((1 - \sqrt{1 - \lambda^2 U^2})/(\lambda U))^J$ to (A.1), $z = \pm i$ are no "generalized eigenvalues" that can make the scheme unstable.

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ЭФФЕКТИВНАЯ ВЫЧИСЛИТЕЛЬНАЯ СХЕМА И ГРАНИЧНЫЕ УСЛОВИЯ ДЛЯ БАРОТРОПНОЙ МОДЕЛИ С МАЛЫМ РАЗРЕШЕНИЕМ, ОСНОВАННОЙ НА УРАВНЕНИЯХ МЕЛКОЙ ВОДЫ

Для интегрирования уравнений мелкой воды представлены две эффективные вычислительные схемы — полумплицитная и центральных разностей. Исследуется их точность и устойчивость как в случае задачи с начальными данными, так и для прогноза по ограни-

ченной территории. Для последнего случая также обсуждаются граничные условия. Даны два варианта устойчивых и достаточно точных граничных условий. Практическая полезность этих условий подтверждается вычислительными экспериментами.