

The global thermodynamics of atmospheric motion

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ABSTRACT

Global thermodynamics is the study of the relationships between the total mass, energy, and entropy of the atmosphere. The main result is an explicit, exact relation between the entropy, the kinetic energy, and the static entropic energy, a new form of static energy. The stability of atmospheric motions and the maintenance of the circulation are investigated with the total entropic energy (the sum of the kinetic and static entropic energies). The dynamical significance of Gibbs' concept of thermal equilibrium is that the total entropic energy decreases monotonically to zero in isolated atmospheres as a consequence of the Second Law. Furthermore, the generation of entropic energy is related specifically to the destruction of entropy by heating at high temperature relative to cooling at low temperature. An energy budget for zonal and eddy forms of entropic energy is constructed, and illustrates the catalytic effect for the general circulation of this destruction of entropy. Finally, the argument is reversed and a stability postulate is applied to the total entropic energy to infer the existence of an entropy function. The concept of entropic energy represents a new integral of the equations of atmospheric motion that combines the previous energy and entropy integrals into one statement, giving a mathematical version of the relations between entropy and motion.

1. Introduction

Comprehension of complex thermodynamic processes, such as the general circulation of the atmosphere, depends upon perceiving how they are governed by the constraints and predictions embodied in the concept of entropy. The predominance of entropy over energy in controlling the course of thermodynamic events seems beyond question, but its dictates have not been exploited in fluid dynamics and atmospheric science.

All modern studies of the maintenance of atmospheric motion recognize the thermal forcing as a prime element in atmospheric dynamics, but most emphasize the detailed determinism expressed by the equations of motion, a determinism that is summarized in the conservation of energy. Indeed, no other approach has been possible because the dynamical significance of entropic principles has been apparent only intuitively, not mathematically (except for the classical use of the Second Law to show that the coefficients of viscosity and heat conduction are positive). Thus the current view of atmospheric motion emphasizes its mechanical aspects, it seems to take for granted that the thermal

forcing will produce motion without enquiring into how the motion interacts as part of a thermodynamic process, it attempts to predict how motion fields in being will evolve into new ones. The emphasis is on dynamics and energetics rather than on thermal processes.

This essay presents a theory of atmospheric thermodynamics in which the presence of motion is specifically a thermodynamic phenomenon, as exhibited by the main result which gives explicit relations between both the amounts and rates of change of entropy and kinetic energy. The basic concept of the theory is due to Willard Gibbs (1873, 1875); its development by Coleman & Greenberg (1967) in search of the dynamical implications of thermodynamic stability motivated this application of their ideas to atmospheric motion; the simple properties of an ideal gas permit development of a theory which gives mathematical precision to our intuitive understanding of the role of differential heating and entropy in atmospheric flow.

The equations of momentum, mass, and thermodynamic energy balance yield the conservation of energy principle that has been of great importance in atmospheric science. Modern

large-scale atmospheric energetics is based on the modification of this concept introduced by Lorenz (1955) in his development of a global theory of available potential energy. An attractive aspect of this theory emphasized by Dutton & Johnson (1967) is that the generation of available potential energy occurs through a process similar to the destruction of entropy by differential heating.

Despite the success of the concept of available potential energy in providing a framework for explaining various aspects of the general circulation, it shares the fundamental defect of classical energetics of being an essentially kinematic theory in the sense that conservation of energy makes no prediction about what processes must do, it only says what they may do, and it does not establish a trend in time.

In addition to its participation in the classical energy equation that is an integral of the equations of fluid motion, the First Law of Thermodynamics yields another integral of atmospheric motion—that of entropy. Gibbs used the concept of entropy and the fact that it never decreases in isolated processes to develop a concept of thermodynamic stability that applies when a fluid has an equilibrium state whose entropy, for all states with the same energy and volume, is maximum. An isolated process then, may be expected to evolve so that it approaches this state, and thus the concept of entropy includes a global prediction.

This prediction may be investigated by comparing the entropy, S_0 , of the equilibrium state with the entropy, S , of any natural state of the atmosphere. Coleman & Greenberg (1967) demonstrated that the difference $S_0 - S$ expresses a control exerted by the properties of the equilibrium state on the variations of both thermal and motion patterns that are permitted in natural states. They considered fluid bodies of finite volume, used a very general equation of state, and confined their discussion to bodies in isolation.

Their results are extended in several ways in this article, although the more specific formulation may be possible only for ideal gases. We consider an atmosphere of finite mass, rather than finite volume, and investigate the consequences of thermal forcing of the atmosphere as well as its behaviour in isolation.

The difference $S_0 - S$ is the central quantity in the present theory, and we show that it is equal

to a suitably scaled sum of the kinetic energy and a new static energy form. Thus we have an integral of the equations of motion of an ideal gas that combines the energy and entropy integrals previously available.

For the atmosphere, S_0 represents the entropy of a motionless, hydrostatic, and isothermal state. The dynamical implications of the difference $S_0 - S$ are remarkable because it provides both a stability criterion and an explicit representation of the fact that atmospheric motion is maintained by the entropy destruction resulting from differential heating.

The use of a particular reference state as a standpoint from which to view the state of the atmosphere gives the present theory a similarity in approach to that of available potential energy. But the reference states are quite different, the constraints are different, and the results are different.

The reference state used here is the state of maximum entropy, S_0 , that can be obtained from any natural atmospheric state by readjusting it under the constraints that the total mass and energy must be conserved. In the theory of available potential energy, the reference state is obtained by minimizing the total potential energy subject to a constraint that the distribution of entropy with respect to mass be conserved. The difference between the actual potential energy and the minimum value represents a maximum amount of additional kinetic energy that could be realized for isolated processes, and is called the available potential energy.

The two theories have the fundamental difference that the S_0 reference state is the equilibrium one toward which an atmosphere in isolation will naturally tend; in available energy theory there is no assurance that the reference state has any physical significance other than representing a portion of the total potential energy that could never become kinetic energy in an isolated atmosphere.

This notion is one that has a foundation in entropic principles, for the idea that the kinetic energy is limited for a system in isolation is, as made clear by Coleman & Greenberg (1967), the dynamical consequence of Gibbs' stability concept. It is shown here that the sum of the kinetic energy and the new static energy form, called the static entropic energy, is proportional to, and thus limited by, the difference $S_0 - S$. Therefore the static entropic energy is a form of

energy that might become kinetic energy if the entropy difference remained constant.

The use of entropy in the reference state produces the trend in time, when isolation is imposed, that is lacking in energy theories. Under such conditions, the total energy must remain constant. But in contrast, the sum of the kinetic and the static entropic energies must decrease in an isolated system as it evolves inexorably toward equilibrium. The perhaps surprising implication is that an isolated atmosphere can foresee its future infinitely far ahead.

The consequence of the inexorable trend imposed by the Second Law is that the question of maintenance of circulation becomes equivalent to the question of how the total entropy, S , of the atmosphere is held constant despite the irreversible production of entropy by friction and heat conduction. In the atmosphere, the entropy is maintained because, on the average, radiation is absorbed at high temperature and emitted at low temperature. Here this entropy decreasing process appears explicitly in the budget of kinetic and static entropy as a generation term.

The static entropic energy, like the available potential energy, may be expressed as a sum of a zonal and eddy component, and it is possible to develop an energy budget similar to that given by Lorenz (1955, 1967) for available energy. The crucial difference lies in the generation terms, which in the theory of entropic energy contain explicitly the zonal and eddy components of the rate of change of entropy.

The significance of the present theory is that the energetics which describes the balance of processes involved in atmospheric motion is linked rigorously and directly with the changes in entropy that exert a fundamental control on the intensity of the circulation. With the blending of the basic concepts of atmospheric energetics and the theory of atmospheric entropy, we have a mathematical description of the global thermodynamics of atmospheric motion in which entropy is the leading thermodynamic variable.

2. The state with maximum entropy

The theory to be developed here requires the association with any natural state of the atmosphere of an equilibrium state that possesses a maximum value of the entropy possible under constraints of mass and energy conservation.

Determination of the characteristics of such a state provides the motivation for the definitions from which all else follows.

As a first step then, we seek necessary conditions for an entropy maximum. The basic result was derived by Gibbs (1875); because of the central nature of the maximum entropy state in the present theory we summarize the variational procedure used to discover it.

The total entropy, mass, and energy of an atmospheric state are given by (see Table of Symbols)

$$S = \int \rho (c_v \ln (T/T_c) - R \ln (\rho/\rho_c)) dV + S_c \quad (2.1)$$

$$M = \int \rho dV \quad (2.2)$$

and

$$E = \int \rho \left(c_v T + gz + \frac{v^2}{2} \right) dV \quad (2.3)$$

To determine the distribution of ρ , T , and $\mathbf{v}$ that are necessary for a maximum of S subject to conservation of M and E , Lagrange multipliers λ and μ are used to form the functional

$$\begin{aligned} \mathcal{S} = S - S_c + \lambda \left(M - \int \rho dV \right) \\ + \gamma \left(E - \int \rho \left(c_v T + gz + \frac{v^2}{2} \right) dV \right) \end{aligned} \quad (2.4)$$

The substitutions $\rho = \rho_0 + \varepsilon \delta \rho$, $T = T_0 + \varepsilon \delta T$, $\mathbf{v} = \mathbf{v}_0 + \varepsilon \delta \mathbf{v}$ produce distributions of the variables that depart from the maximizing values denoted by subscript zero, and give the functional in (2.4) a dependence on ε so that $\mathcal{S} = \mathcal{S}(\varepsilon)$. If ρ_0 , T_0 , and $\mathbf{v}_0$ are to maximize $\mathcal{S}$, then

$$\lim_{\varepsilon \rightarrow 0} \frac{d}{d\varepsilon} \mathcal{S}(\varepsilon) = \mathcal{S}'(0) = 0 \quad (2.5)$$

The variations are arbitrary except that they must permit the variables to satisfy the mass and energy constraint. Thus it is easily shown that the integrals resulting from the operation specified in (2.5) can vanish only if the maximizing distributions ρ_0 , T_0 , and $\mathbf{v}_0$ satisfy the Euler equations

$$\delta T: \rho_0 c_v \left(\frac{1}{T_0} - \gamma \right) = 0$$

$$\delta v: \gamma v_0 = 0$$

$$\begin{aligned} \delta q: -R + c_v \ln(T_0/T_c) - R \ln(\varrho_0/\varrho_c) - \lambda \\ - \gamma \left[c_v T_0 + gz + \frac{v_0^2}{2} \right] = 0 \end{aligned} \quad (2.6)$$

The first equation shows that T_0 is constant at value γ^{-1} and the second shows that the maximum entropy state is motionless. These two results allow us to write

$$\begin{aligned} S'(0) = - \int \left\{ \left[R \ln(\varrho_0/\varrho_c) + \frac{gz}{T_0} \right] \right. \\ \left. + [R - c_v \ln(T_0/T_c) + \lambda + c_v] \right\} \delta \varrho dV \end{aligned} \quad (2.7)$$

But the mass constraint

$$M = \int (\varrho_0 + \varepsilon \delta \varrho) dV = \int \varrho_0 dV \quad (2.8)$$

shows that $\delta \varrho$ must integrate to zero. Thus the term in the second set of brackets vanishes since it contains only constants. By the same argument, we may adjust the constants in the first bracket so that we satisfy (2.5) with

$$\varrho_0 = \varrho_0(0) e^{-gz/RT_0} \quad (2.9)$$

Introduction of the thermodynamic pressure with the equation of state

$$p = \varrho RT \quad (2.10)$$

gives

$$p_0 = R\varrho_0(0) T_0 e^{-gz/RT_0} \quad (2.11)$$

which implies that the state of maximum entropy is hydrostatic, and it is clear that the maximizing values of ϱ , p , and T vary only in the vertical. We use an overbar to denote averages per unit area and then find that the constraints require

$$\bar{M} = \int_0^\infty \varrho_0(0) e^{-gz/RT_0} dz = \varrho_0(0) RT_0 / g \quad (2.12)$$

$$p_0(0) = g\bar{M} \quad (2.13)$$

and

$$\begin{aligned} \bar{E}_0 = \int_0^\infty \varrho_0 [c_v T_0 + gz] dz = c_v \bar{M} T_0 + g\varrho_0(0) \\ \times (RT_0/g)^2 = c_p \bar{M} T_0 = \bar{E} \end{aligned} \quad (2.14)$$

so that T_0 , p_0 , and ϱ_0 are determined explicitly by M and E .

The state thus defined, which is still to be rigorously verified as one of maximum entropy, provides the basic reference to which we shall compare natural atmospheric states.

3. Properties of the model atmosphere

A number of assumptions and definitions are required to provide the mathematical model of atmospheric processes to be used in developing a global thermodynamic theory. We shall collect them here, and together they form the definition of what is meant by an atmosphere in the formal statements that follow.

We consider a dry ideal gas confined to a planet by a gravitational field. Use of geopotential height as the vertical coordinate permits the acceleration of gravity to be considered constant. The specific heat, c_v , is assumed constant, and for the dry gas we may choose to measure its specific internal energy as $e = c_v T$.

From the requirements that the kinetic energy,

$$K = \int \varrho \frac{v^2}{2} dV \quad (3.1)$$

internal energy

$$I = \int c_v \varrho T dV = \int \frac{c_v}{R} p dV \quad (3.2)$$

and potential energy

$$P = \int g \varrho z dV \quad (3.3)$$

all be finite, we have the conditions that as $z \rightarrow \infty$

$$\varrho v^2 = o(z^{-1}) \quad p = o(z^{-1}) \quad \varrho = o(z^{-2}) \quad (3.4)$$

Hence the mass is finite as well.

In any natural state of the atmosphere, we may define the *hydrostatic defect*, χ , with the relation

$$\frac{\partial p}{\partial z} = -(g - \chi)\varrho \quad (3.5)$$

Thus (3.2) may be integrated by parts to give

$$I = \frac{c_V}{R} (P - X) \quad (3.6)$$

where

$$X = \int \chi \varrho z dV \quad (3.7)$$

The middle equation of (3.4) shows that the term pz which appears in the integration by parts vanishes as $z \rightarrow \infty$. Thus the total potential energy may always be expressed as

$$\Pi = I + P = \frac{c_p}{c_V} I + X \quad (3.8)$$

Combination of these conditions with those of the previous section allows us to state:

Definition 3.1. *To every natural state of the atmosphere corresponds an associated equilibrium state, a motionless, hydrostatic state with the same mass M and total energy E in which*

$$I_0 = c_V M T_0 = I + \frac{c_V}{c_p} (K + X) \quad (3.9)$$

$$P_0 = \frac{R}{c_V} I_0 = P + \frac{R}{c_p} K - \frac{c_V}{c_p} X \quad (3.10)$$

$$E_0 = \Pi_0 = I + P + K \quad (3.11)$$

$$\varrho_0 = \langle \varrho_0(0) / (RT_0) \rangle e^{-gz/RT_0} \quad (3.12)$$

and

$$p_0(0) = g \bar{M} \quad (3.13)$$

The motivation for the assignment of K and X to I_0 in the proportion shown in (3.9) is the requirement that

$$E_0 = \frac{c_p}{c_V} I_0 = E = \frac{c_p}{c_V} I + K + X \quad (3.14)$$

Now it follows that:

Theorem 3.1. *If two natural states have the same mass M and energies E_1 and E_2 , then the entropies of their associated equilibrium states are related by*

$$S_2 - S_1 = c_p M \ln (E_2/E_1) \quad (3.15)$$

Proof. We have

Tellus XXV (1973), 2

$$\begin{aligned} S_2 - S_1 &= \int \{ c_V [\varrho_2 \ln (T_2/T_c) - \varrho_1 \ln (T_1/T_c)] \\ &\quad - R [\varrho_2 \ln (\varrho_2/\varrho_c) - \varrho_1 \ln (\varrho_1/\varrho_c)] \} dV \\ &= \int \left\{ c_V [\varrho_2 \ln (T_2/T_1) + (\varrho_2 - \varrho_1) \ln (T_1/T_c)] \right. \\ &\quad \left. + gz \left[\frac{\varrho_2}{T_2} - \frac{\varrho_1}{T_1} \right] - R \varrho_2 \ln \left(\frac{\varrho_2(0)}{\varrho_c} \right) \right. \\ &\quad \left. + R \varrho_1 \ln \left(\frac{\varrho_1(0)}{\varrho_c} \right) \right\} dV \end{aligned} \quad (3.16)$$

The second term vanishes immediately because the temperatures are constant and the total masses identical. The third integrates to a difference in surface pressures of the equilibrium states and also vanishes. Hence

$$\begin{aligned} S_2 - S_1 &= c_V M \ln (T_2/T_1) + R M \ln [\varrho_1(0)/\varrho_2(0)] \\ &= c_p M \ln (T_2/T_1) = c_p M \ln (E_2/E_1) \end{aligned} \quad (3.17)$$

which completes the proof.

The model atmosphere is assumed to obey the usual equations of motion, but now we need only specify the rates of change of the total energy and entropy. With the usual boundary conditions that the energy flux $p \mathbf{v} \cdot \mathbf{n}$ vanish at the top of the atmosphere and that $\mathbf{v} \cdot \mathbf{n}$ vanish at the bottom, we have

$$\begin{aligned} \dot{E} &= \int [-\nabla \cdot p \mathbf{v} - \nabla \cdot \mathbf{R} + \nabla \cdot k \nabla T + \frac{\partial}{\partial x_j} (v_i \Delta_{ij})] dV \\ &= \int_B \eta_j \left[-R_j + k \frac{\partial T}{\partial x_j} + v_i \Delta_{ij} \right] dB \end{aligned} \quad (3.18)$$

in which $\mathbf{R}$ is the flux of radiation, k the coefficient of heat conduction, and Δ_{ij} is the viscous tensor defined by

$$\Delta_{ij} = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - \frac{2}{3} (\mu - \mu') \frac{\partial v_k}{\partial x_k} \delta_{ij} \quad (3.19)$$

in which μ is the dynamic shear viscosity coefficient and μ' is the bulk or expansion viscosity coefficient. The heat conduction and dissipation terms are expressed, for convenience, in Cartesian tensor form rather than as covariant derivatives; extension to spherical coordinates is clearly possible but not necessary for the present purposes.

For the rate of change of entropy we have

$$\begin{aligned}\dot{S} &= \int \left\{ -\frac{\nabla \cdot \mathbf{R}}{T} + \frac{1}{T} \nabla \cdot \mathbf{k} \nabla T + \frac{1}{T} \frac{\partial v_i}{\partial x_j} \Delta_{ij} \right\} dV \\ &= \int \left\{ -\frac{\nabla \cdot \mathbf{R}}{T} + \nabla \cdot \left(\frac{\mathbf{k} \nabla T}{T} \right) \right\} dV \\ &\quad + \int \left\{ \frac{k}{T^2} |\nabla T|^2 + \frac{1}{T} \frac{\partial v_i}{\partial x_j} \Delta_{ij} \right\} dV\end{aligned}\quad (3.20)$$

If we define

$$e_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad (3.21)$$

then by virtue of symmetry properties of the viscous tensor we may write

$$\begin{aligned}\varepsilon &= \frac{\partial v_i}{\partial x_j} \Delta_{ij} = e_{ij} [2\mu e_{ij} - \frac{2}{3}(\mu - \mu') e_{kk} \delta_{ij}] \\ &= 2\mu [e_{ij} - \beta e_{kk} \delta_{ij}]^2\end{aligned}\quad (3.22)$$

in which $3\beta = 1 \pm (\mu'/\mu)^{1/2}$. Thus the last integral of (3.20) will be non-negative for $k > 0$, $\mu > 0$, and $\mu' > 0$, which are part of the usual thermodynamic stability criteria for a linear fluid.

The assumptions of this section may now be collected in a set of formal statements:

Definition 3.2. *An atmosphere is composed of a dry ideal gas, confined to a planet by a gravitational field. The coefficient c_V is constant and c_V , k , μ and μ' , are all positive.*

Definition 3.3 *In an admissible or natural process in an atmosphere, the variables $\mathbf{v}(\mathbf{x}, t)$, $T(\mathbf{x}, t)$ and $\varrho(\mathbf{x}, t)$, are required to obey the usual equations of motion and boundary conditions and to yield finite total mass, entropy, and energies. The rates $\dot{E}$ and $\dot{S}$ are specified by (3.18) and (3.20), and assumed finite, and $\dot{M} = 0$ in an admissible process.*

Definition 3.4. *Admissible or natural external forcing fields are those external (incoming) radiation fields and surface temperature fields that lead to admissible processes.*

Definition 3.5. *An admissible process is said to be energetically isolated in an interval of time if $\dot{E} = 0$ in that interval. It is entropically isolated in an interval if*

$$\int \left\{ -\frac{\nabla \cdot \mathbf{R}}{T} + \nabla \cdot \left(\frac{\mathbf{k} \nabla T}{T} \right) \right\} dV = 0 \quad (3.23)$$

in that interval.

It follows from (3.20) that entropically isolated processes are characterized by

$$\dot{S} \geq 0 \quad (3.24)$$

It is this class of processes for which we shall be able to prove global stability theorems.

4. Entropic energy, global stability, and the origin of motion

The Earth's atmosphere, forced into perpetual motion by differential solar heating, will never reach the equilibrium state of Definition 3.1. Nevertheless, this motionless state, forever far in the future of any natural process, provides a reference value of the entropy for comparison with the entropy present at any time in the atmosphere. This comparison, as we shall show, has bountiful rewards, for the remote equilibrium state represents the limit toward which atmospheric flow would tend if natural processes operated in isolation. The existence of this limit and the presence of processes that work toward it exert a control on the evolution of atmospheric flow. This control and its implications are the essence of global thermodynamics.

We denote conditions in the associated equilibrium state with subscript zero, and we shall compute the difference between the entropies of any natural state and its associated equilibrium state. Thus

$$\begin{aligned}S_0(t) - S(t) &= \int [\varrho_0 s_0 - \varrho s] dV \\ &= \int [\varrho(s_0 - s)] dV + \int (\varrho_0 - \varrho) s_0 dV \\ &= \int \varrho [c_V \ln (T_0/T) + R \ln (\alpha_0/\alpha)] dV \\ &\quad + \int (\varrho_0 - \varrho) [c_V \ln (T_0/T_c) \\ &\quad + R \ln (\alpha_0/\alpha_c)] dV\end{aligned}\quad (4.1)$$

In the second integral we have

$$\int (\varrho_0 - \varrho) c_V \ln (T_0/T_c) dV = 0 \quad (4.2)$$

because both T_0 and T_c are constant and $M = M_0$. The other term of that integral is

$$R \int (\varrho_0 - \varrho) \ln (\alpha_0/\alpha_c) dV = R \int (\varrho_0 - \varrho) \times \left[\ln \left(\frac{\alpha_0(0)}{\alpha_c} \right) + \frac{gz}{RT_0} \right] dV = (P_0 - P)/T_0 \quad (4.3)$$

since the first term of the middle expression also vanishes.

An application of Taylor's theorem with Lagrange's form of the remainder gives the exact representation

$$\ln \left(\frac{T_0}{T} \right) = (T - T_0) \left[\frac{\partial \ln (T_0/T)}{\partial T} \right]_{T_0} + \frac{(T - T_0)^2}{2} \times [T_0 + \lambda(T - T_0)]^{-2} \\ = \frac{T_0 - T}{T_0} + \frac{1}{2} \left\{ \frac{(T - T_0)/T_0}{1 + \lambda(T - T_0)/T_0} \right\}^2 \\ (\lambda = \lambda(T, T_0), 0 \leq \lambda \leq 1) \quad (4.4)$$

and therefore

$$\int c_V \varrho \ln (T_0/T) dV = \frac{I_0 - I}{T_0} + \frac{1}{2} \int c_V \varrho \left\{ \frac{(T - T_0)/T_0}{1 + \lambda(T - T_0)/T_0} \right\}^2 dV \quad (4.5)$$

We observe that

$$\int \varrho \frac{(\alpha_0 - \alpha)}{\alpha_0} dV = \int (\varrho - \varrho_0) dV = 0 \quad (4.6)$$

so that use of Taylor's theorem on the specific volume term yields

$$\int R \varrho \ln (\alpha_0/\alpha) dV = \frac{1}{2} \int R \varrho \left\{ \frac{(\alpha - \alpha_0)/\alpha_0}{1 + \kappa(\alpha - \alpha_0)/\alpha_0} \right\}^2 dV \\ (\kappa = \kappa(\alpha, \alpha_0), 0 \leq \kappa \leq 1) \quad (4.7)$$

Collecting results from (4.3), (4.5) and (4.7) gives

$$S_0(t) - S(t) = \frac{I_0 + P_0 - I - P}{T_0} + \Sigma(T, T_0; \alpha, \alpha_0) \quad (4.8)$$

in which we shall define the *static entropic energy* as $T_0 \Sigma$ where

$$\Sigma = \frac{1}{2} \int \varrho \left\{ c_V \left[\frac{(T - T_0)/T_0}{1 + \lambda(T - T_0)/T_0} \right]^2 + R \left[\frac{(\alpha - \alpha_0)/\alpha_0}{1 + \kappa(\alpha - \alpha_0)/\alpha_0} \right]^2 \right\} dV \quad (4.9)$$

But $I_0 + P_0 = E_0$ and $I + P = E - K$, and so the condition that E and E_0 be identical gives

$$N(t) = T_0(t) [S_0(t) - S(t)] = K(t) + T_0(t) \Sigma(t) \quad (4.10)$$

in which T_0 and S_0 also vary in time if $E(t)$ does.

This is the basic statement of the present theory; we shall refer to the quantity in (4.10) as the *entropic energy*. The significance of this result is that we have linked the kinetic energy and a new form of static energy to an entropy difference. It seems justified to view (4.10) as a new integral of the equations of motion, because with the aid of the new energy form the entropy and kinetic energy that previously appeared in separate integrals have been linked together.

Two opportunities are now apparent. The first is that (4.10) offers a characterization of entropy differences by positive definite quantities; the second, and equally important, is that we have discovered an energy form that is characterized by an entropy difference. The first possibility permits verification that the associated state is indeed an equilibrium one of maximum entropy; thereafter it may serve as the fundamental reference to use in exploiting the second alternative.

Definition 4.1. *A static state of an atmosphere is thermodynamically stable in Gibbs' sense if every other state with the same mass and total energy has smaller total entropy.*

Theorem 4.1. *The associated equilibrium state of Definition (3.1) has the maximum entropy of any atmosphere with admissible processes with the same mass and energy, and hence the associated equilibrium state is thermodynamically stable in Gibbs' sense.*

Proof. The right side of (4.10) is obviously positive for any departure from the associated equilibrium state.

4.1. Global stability

Now we may turn to an investigation of the dynamical consequences of the fact that the equilibrium state associated with atmospheric

flow is stable in Gibbs' sense. In other words, we wish to take advantage of the trend in time established by the entropy for isolated systems.

Theorem 4.2. *Let a natural process be energetically and entropically isolated for all time $t \geq \tau$. Then the deviations from the associated equilibrium state of the states produced by the isolated natural processes are for all time $t \geq \tau$ restricted by*

$$\frac{K(t)}{T_0} + \Sigma(t) \leq S_0 - S(\tau) \quad (4.11)$$

Proof. The statement follows immediately from (4.10) and the definitions, since energetic isolation provides an S_0 which is constant for $t \geq \tau$ and entropic isolation ensures that $\dot{S}(t) \geq 0$ so that $S(t) \geq S(\tau)$.

This is a variant of the main Theorem 3 of Coleman & Greenberg (1967), although the measure of departure from equilibrium which appears here is essentially a mean square norm in contrast to the absolute value employed by them. The restriction to an ideal gas makes the proof much easier than that required in their more general case.

Moreover, we have the companion result:

Theorem 4.3. *For all natural processes energetically and entropically isolated for all time $t \geq 0$,*

$$\frac{\dot{K}(t)}{T_0} + \dot{\Sigma}(t) = -\dot{S}(t) \leq 0 \quad (4.12)$$

Proof. Energetic isolation ensures that $\dot{T}_0 = \dot{S}_0 = 0$; entropic isolation implies that $\dot{S} \geq 0$.

In combination, these two theorems provide a specific characterization of the stability and intensity of atmospheric flows whose processes are isolated. Theorem 4.2 has the remarkable consequence that the departures from the equilibrium state yet to be reached are bounded for all time by the entropy difference at the time isolation was imposed or occurred naturally. Thus the equilibrium properties of the atmosphere impose a dynamical control on what may occur in isolated natural processes.

Theorem 4.3 shows that the entropic energy can only decrease in isolated processes, and so the entropic energy is a global variable which, for isolated processes, has a trend in time. The consequence of the basic equation (4.10) is that

in isolation the global stability properties of the atmosphere provide an inequality of dynamical significance in contrast to the kinematic nature of the usual energy integral.

4.2. The origin and maintenance of atmospheric motion

So far, we have shown that the entropic energy of isolated processes is bounded for all time by $T_0[S_0 - S(\tau)]$ and that it can only decrease in isolation. The converse question of whether the destruction of entropy implies the creation of entropic energy is the crucial one in considering the origin and maintenance of atmospheric motion.

Let us begin with an equilibrium state (M, S_e, E_e) at time $t = 0$, and suppose that natural external forcing occurs, generating internal processes that lead to a natural state (M, S, E) at time t . This natural state has an associated equilibrium state (M, S_0, E_0) and (4.10) is valid. But the two entropies S_0 and S_e are, according to Theorem 3.1, related by

$$S_0 - S_e = c_p M \ln (E_0/E_e) = c_p M \ln (E(t)/E_e) \quad (4.13)$$

and so (4.10) implies that

$$\begin{aligned} \frac{K(t)}{T_0(t)} + \Sigma(t) &= S_0(t) - S(t) \\ &= S_e - S(t) + c_p M \ln (E(t)/E_e) \\ &= c_p M \ln \left\{ \frac{E(t)e^{-S(t)/c_p M}}{E_e e^{-S_e/c_p M}} \right\} \end{aligned} \quad (4.14)$$

We have proven:

Theorem 4.4. *Let a state (M, S, E) have evolved by natural processes from an equilibrium state (M, S_e, E_e) which existed at time $t = 0$. Then (4.14) is valid for all $t \geq 0$ and the necessary and sufficient condition for the presence of entropic energy at time t is that*

$$E(t)e^{-S(t)/c_p M} > E_e e^{-S_e/c_p M} \quad (4.15)$$

The implication is that to create entropic energy and thus disturb an equilibrium state we must either increase the energy or decrease the entropy, or do both. The inequality opposite to (4.15) is prohibited by (4.14) in expression of the fact that $S_0 \geq S(t)$ and so with (4.13),

$$Ee^{-S/c_p M} \geq E_0 e^{-S_0/c_p M} = E_e e^{-S_e/c_p M} \quad (4.16)$$

A more specific characterization of the effect of radiation in causing and maintaining atmospheric motion is possible. Terminology is established by:

Definition 4.2. *An atmosphere is in weak motion if the velocity and acceleration do not both vanish identically. It is in motion if $K > 0$. It is evolving in an interval of time if $\dot{K}$ and $\dot{\Sigma}$ do not both vanish identically in that interval. The intensity of the motion is measured by the entropic energy.*

The following result is essentially Jeffreys's (1925) theorem:

Theorem 4.5. *If $\nabla \cdot \mathbf{R}$ varies on any geopotential surface, then the atmosphere is at least in weak motion.*

Proof. Let $\mathbf{v} \equiv \dot{\mathbf{v}} \equiv 0$. Then the equations of motion are

$$\nabla p + g\rho \mathbf{k} = 0$$

$$\partial \varrho / \partial t = 0$$

$$c_V \partial T / \partial t + \nabla \cdot \mathbf{R} = \nabla \cdot k \nabla T$$

$$p = \varrho R T \quad (4.19)$$

The first equation implies that

$$\nabla_z p = 0, \quad \frac{\partial p}{\partial z} = -g\varrho \quad (4.20)$$

and this gives

$$\nabla_z \varrho = \nabla_z T = 0 \quad (4.21)$$

when ∇_z is applied to the second of (4.20) and (4.19)₄ is used.

Furthermore, with (4.19)₂ and (4.20) we have

$$0 = \frac{\partial}{\partial t} \left(\frac{\partial p}{\partial z} \right) = \frac{\partial}{\partial z} \left(\frac{\partial p}{\partial t} \right) = \frac{\partial}{\partial z} \left(\varrho R \frac{\partial T}{\partial t} \right) \quad (4.22)$$

But since $\varrho(\partial T / \partial t) = f(t)$, it must vanish identically because it vanishes as $z \rightarrow \infty$. Hence

$$\nabla \cdot \mathbf{R} = \frac{\partial}{\partial z} \left(k \frac{\partial T}{\partial z} \right) \quad (4.23)$$

and so the radiation divergence is a function of z only.

The contrapositive statement justifies the conclusion, because if $\nabla \cdot \mathbf{R}$ varies on a level surface then both $\mathbf{v}$ and $\dot{\mathbf{v}}$ cannot vanish identically.

Corollary 4.5.1. *If $\nabla \cdot \mathbf{R}$ does not vanish identically in an open interval (t_1, t_2) with $t_2 > t_1$, then the atmosphere is not in its associated equilibrium state in the closed interval $[t_1, t_2]$.*

Proof. Let K and Σ both vanish identically in $[t_1, t_2]$. Then in (t_1, t_2)

$$\dot{K} = \int \varrho \dot{v}^2 dV = 0 \quad (4.24)$$

and so both $\mathbf{v}$ and $\dot{\mathbf{v}}$ vanish in (t_1, t_2) , and (4.23) follows. But $\Sigma = 0$ implies that T is constant, and hence (4.23) shows that $\nabla \cdot \mathbf{R}$ vanishes identically in (t_1, t_2) . The contrapositive statement thus establishes the corollary.

Corollary 4.5.2. *If $\dot{\Sigma}$ is not zero, then there is weak motion.*

Proof. Let $\mathbf{v} = \dot{\mathbf{v}} \equiv 0$. Then (4.19)₂ and (4.22) show that $\dot{\varrho} = \dot{T} = 0$, and hence $\dot{\Sigma} = 0$. Therefore $\dot{\Sigma} \neq 0$ implies the existence of weak motion.

Corollary 4.5.3. *If the atmosphere is evolving through natural processes in an interval $[t_1, t_2]$ with $t_2 > t_1$, then it is in motion in at least a part of $[t_1, t_2]$.*

Proof. Assume the contrary statement that $K \equiv 0$ throughout the interval. Then $\dot{K}$ and $\dot{\Sigma}$ vanish in (t_1, t_2) so the presence of evolution implies that $\dot{\Sigma} \neq 0$ throughout (t_1, t_2) . But then there is weak motion and we have $v \equiv 0$ but $\dot{v} \neq 0$, so

$$\dot{K} = \int \varrho \dot{v}^2 dV > 0 \quad (t_1, t_2) \quad (4.25)$$

which is a contradiction.

We remark that if evolution is defined as occurring when $\dot{K}$ and $\dot{\Pi}$ do not both vanish, then Corollaries 4.5.2 and 4.5.3 are true with $\dot{\Sigma}$ replaced with $\dot{\Pi}$.

The final development of the section is motivated by the observation that the total energy and entropy of the Earth's atmosphere are essentially constant. The following definition allows us to avoid a troublesome problem at infinity.

Definition 4.3. An atmosphere is mechanically isolated if

$$F_B = \int_B \eta_j v_i \Delta_{ij} dB = 0 \quad (4.26)$$

It is mechanically isolated at infinity if the integral over the upper free surface, B_∞ , at infinity vanishes.

Lemma 4.1. In an atmosphere energetically isolated, mechanically isolated at infinity, and bounded below by a non-slip rigid surface, we have

$$\int H dV = \int [-\nabla \cdot \mathbf{R} + \nabla \cdot \mathbf{k} \nabla T] dV \geq 0 \quad (4.27)$$

Proof. The integral of (4.26) need be evaluated only along the lower boundary, B_L , which gives

$$\begin{aligned} \int_{B_L} \eta_j v_i \Delta_{ij} dB &= \int_{B_L} \boldsymbol{\eta} \cdot \left[\frac{\mu}{2} \nabla v^2 + \mu (\mathbf{v} \cdot \nabla) \mathbf{v} \right. \\ &\quad \left. - \frac{2}{3} (\mu - \mu') \mathbf{v} \nabla \cdot \mathbf{v} \right] dB = \int_{B_L} \boldsymbol{\eta} \\ &\quad \times \left[\frac{\mu}{2} \nabla v^2 + \frac{1}{3} \left(\mu' + \frac{\mu}{2} \right) \nabla v_\perp^2 \right] dB \leq 0 \end{aligned} \quad (4.28)$$

where $v_\perp$ is the velocity component normal to the boundary. Since both v^2 and $v_\perp^2$ vanish on B_L , they can only increase away from B_L , giving the result because $\boldsymbol{\eta}$ is an exterior normal vector. Then (3.18) gives (4.27) and establishes the lemma.

For an energetically isolated atmosphere, we have

$$\frac{\dot{K}(t)}{T_0} + \dot{\Sigma}(t) = -\dot{S}(t) \quad (4.29)$$

and if the intensity of the circulation is not decreasing so that the left side is non-negative, then (3.20) and (3.22) show that

$$\int \frac{H}{T} dV \leq - \int \frac{\varepsilon}{T} dV \leq 0 \quad (4.30)$$

With the decomposition of H into the difference $H^+ - H^-$ of two positive functions defined by

$$H^+ = \begin{cases} H & \text{if } H \geq 0 \\ 0 & \text{if } H < 0 \end{cases} \quad H^- = \begin{cases} -H & \text{if } H < 0 \\ 0 & \text{if } H \geq 0 \end{cases} \quad (4.31)$$

and the definitions

$$T^\pm = \frac{\int H^\pm dV}{\int (H^\pm / T) dV} \quad (4.32)$$

we may write

$$\frac{1}{T^+} \int H^+ dV - \frac{1}{T^-} \int H^- dV \leq - \int \frac{\varepsilon}{T} dV \quad (4.33)$$

But then

$$\begin{aligned} T^- &\leq T^+ \left\{ \int H^- dV \left[\int H^+ dV + T^+ \int \frac{\varepsilon}{T} dV \right]^{-1} \right\} \\ &\leq T^+ \left\{ \int H^- dV / \int H^+ dV \right\} \end{aligned} \quad (4.34)$$

By virtue of Lemma 4.1,

$$\int H^+ dV \geq \int H^- dV \quad (4.35)$$

and so (4.34) implies that

$$T^- \leq T^+ \quad (4.36)$$

Thus we have proven:

Theorem 4.6. For the maintenance or intensification of motion, in the sense that (4.29) is non-negative, in an atmosphere which is energetically isolated, mechanically isolated at infinity, bounded below by a non-slip rigid surface, and in which H^- does not vanish identically, it is necessary and sufficient that the average temperature T^- (defined by 4.32) in the region of cooling be related to the temperature T^+ in the region of heating by (4.34). For this to be true, it is necessary that $T^- \leq T^+$, with strict inequality if frictional dissipation is present on B_L and internally.

This theorem provides a rigorous verification of the intuitively attractive hypothesis that the general circulation is maintained by the entropy destruction resulting from radiation convergence at high temperature coupled with radiation divergence at low temperature. It is thus a formal version of the common statement that the atmosphere derives its energy from the sun, and makes it clear that the effect on the entropy of the differential heating induced by the spherical shape of the Earth stands at the beginning of the events involved in global thermodynamics.

5. The budget of entropic energy

The concept of entropic energy provides a new view of the global processes involved in the maintenance of atmospheric motion, as demonstrated in the preceding section. It has essential differences from the classical approach to energetics or from that of the theory of available potential energy, and study of the mechanisms involved in the budget of entropic energy will demonstrate how the processes that maintain the budgets of energy and entropy interact.

5.1. Rederivation of the basic result

The definition of Σ given in (4.9) had the advantage of emphasizing the positive definite nature of the integrand. An alternate form is preferable for calculating temporal rates of change, however. The combination of (4.5) and (4.7) with (4.9) gives

$$\Sigma = \int \rho \left[c_v \left(\frac{T - T_0}{T_0} - \ln \frac{T}{T_0} \right) - R \ln \frac{\alpha}{\alpha_0} \right] dV \quad (5.1)$$

This is an exact expression, and clearly may be computed for any atmospheric state for which the total energy and the temperature and density fields are known.

Differentiation of (5.1) with the aid of the transport theorem (see Table of Symbols) and use of the fact that T , T_0 , α , $\alpha_0(z)$, and $\alpha_0(0)$ all may change with time, while due to (3.13) $p_0(0)$ may not, gives

$$\begin{aligned} \dot{\Sigma} &= \frac{\dot{I}}{T_0} - \frac{(E - K)\dot{T}_0}{T_0^2} - \dot{S} + \frac{c_p M \dot{T}_0}{T_0} + \frac{\dot{P}}{T_0} \\ &= -\frac{\dot{K}}{T_0} + \frac{K\dot{T}_0}{T_0^2} + \frac{c_p M \dot{T}_0}{T_0} - \dot{S} \end{aligned} \quad (5.2)$$

Direct computation or use of (3.17) shows that

$$\dot{S}_0 = c_p M \dot{T}_0 / T_0 \quad (5.3)$$

and so (5.2) is

$$\dot{\Sigma} + \frac{d}{dt} \left(\frac{K}{T_0} \right) = \dot{S}_0 - \dot{S} \quad (5.4)$$

Integration of this equation, and use of the fact that Σ , K , and $S - S_0$ all vanish in the equilibrium state to eliminate the constant, produces (4.10) again.

5.2. Rate of change and maintenance of motion

It is convenient to scale the total entropic energy by T_0 and to define

$$\mathcal{E} = \frac{K}{T_0} + \Sigma = S_0 - S \quad (5.5)$$

and thus (5.4) and (5.3) imply that

$$\dot{\mathcal{E}} = \frac{\dot{E}}{T_0} - \dot{S} \quad (5.6)$$

Combination of this result with (3.18) and (3.20) produces

$$\begin{aligned} \dot{\mathcal{E}} &= \int \left\{ \left(\frac{1}{T} - \frac{1}{T_0} \right) \nabla \cdot \mathbf{R} + \left(\frac{1}{T_0} - \frac{1}{T} \right) \nabla \cdot k \nabla T \right. \\ &\quad \left. + \frac{1}{T_0} \frac{\partial}{\partial x_j} (v_i \Delta_{ij}) - \frac{1}{T} \frac{\partial v_i}{\partial x_j} \Delta_{ij} \right\} dV \\ &\quad - \int \left(\frac{1}{T} - \frac{1}{T_0} \right) \nabla \cdot \mathbf{R} dV \\ &\quad + \int \left\{ k \left(\frac{1}{T_0} - \frac{1}{T} \right) \frac{\partial T}{\partial x_j} + \frac{1}{T_0} v_i \Delta_{ij} \right\} n_j dB \\ &\quad - \int \left\{ \frac{k}{T^2} |\nabla T|^2 + \frac{1}{T} \frac{\partial v_i}{\partial x_j} \Delta_{ij} \right\} dV \end{aligned} \quad (5.7)$$

It is possible to ascertain the sign of each term on the right if the boundary integral vanishes on the upper free surface, B_∞ . The last term is the negative of the expression for irreversible entropy gains. The second term of the boundary integral is shown by (4.28) to be negative. We can expect that on the average in the atmosphere we will have a lower boundary temperature $T_B > T_0$ and that $\partial T / \partial z < 0$ at the boundary so that the first term of the boundary integral is positive. The radiation convergence at low altitudes in the equatorial regions where $T > T_0$ and divergence at higher altitudes and in polar regions where $T < T_0$ makes the first term positive.

With the

Definition 5.1. *An atmosphere is isolated at infinity if the portion of the boundary integral in (5.7) over the upper free surface, B_∞ , vanishes.*

we have the result

Theorem 5.1. *In an atmosphere isolated at infinity and not in equilibrium, the maintenance or intensification of motion in the sense that $\dot{\Sigma} \geq 0$ requires that the sum of the first two terms of (5.7)₂ be positive, and hence that the radiation divergence $\nabla \cdot \mathbf{R}$ be positively correlated with $(T_0 - T)/TT_0$, or that, on the average, conduction at the lower boundary be supplying heat to the atmosphere with $T_B > T_0$.*

Two remarks are appropriate. First this result is very similar to that of Theorem 4.6, which using stronger conditions produced a stronger conclusion. Second, the rate of change of entropic energy depends on the internal arrangements of the motion and thermal fields and not just on their boundary values, in contrast to $\dot{E}$.

5.3. The budgets of Σ and K

The basic statement (5.4) of temporal rates of change may be rewritten as

$$\begin{aligned}\dot{\Sigma} &= -\frac{\dot{K}}{T_0} + \frac{K\dot{T}_0}{T_0^2} + \frac{\dot{E}}{T_0} - \dot{S} \\ &= -\frac{\dot{K}}{T_0} + \left(1 + \frac{K}{E}\right)\frac{\dot{E}}{T_0} - \dot{S}\end{aligned}\quad (5.8)$$

because

$$\dot{E} = \dot{E}_0 = c_p M \dot{T}_0 \quad (5.9)$$

We use $C(\Pi, K)$ to denote the rate of conversion of total potential energy to kinetic energy, we write $\Gamma = 1 + (K/E)$, and we denote the last term of (5.7) giving the negative of the irreversible gains of entropy by $-\dot{S}_i$, so that (5.8) becomes

$$\begin{aligned}\dot{\Sigma} &= -C(\Pi, K)/T_0 + \frac{1}{T_0} \int \epsilon dV + \int \left(\frac{1}{T} - \frac{\Gamma}{T_0} \right) \\ &\quad \times \nabla \cdot \mathbf{R} dV + \int_B \left[k \left(\frac{\Gamma}{T_0} - \frac{1}{T} \right) \frac{\partial T}{\partial x_j} + \frac{\Gamma - 1}{T_0} v_i \Delta_{ij} \right] \\ &\quad \times \eta_j dB - \dot{S}_i\end{aligned}\quad (5.10)$$

In symbolic form,

$$\dot{\Sigma} = -C(\Pi, K)/T_0 + G(\Sigma) - D(\Sigma) + B(\Sigma) \quad (5.11)$$

where

$$G(\Sigma) = \int \left(\frac{1}{T} - \frac{\Gamma}{T_0} \right) \nabla \cdot \mathbf{R} dV + \frac{1}{T_0} \int \epsilon dV \quad (5.12)$$

and

$$D(\Sigma) = \dot{S}_i \geq 0 \quad (5.13)$$

and where $B(\Sigma)$ denotes the boundary integral of (5.10).

Thus Σ is generated by radiative heating at high temperatures coupled with cooling at low temperatures. This process creates the forces that give $C(\Pi, K) > 0$ and results in motion. Friction destroys kinetic energy, but that energy returns to Σ through $G(\Sigma)$ and the boundary integral, while Σ itself is dissipated by the same irreversible processes that create entropy.

In this view of the general circulation, the origin of motion lies in the destruction of entropy by differential heating, a process that results in the creation of static entropic energy that is in part converted to kinetic energy and in part destroyed by the irreversible production of entropy. Thus we have a statement of the relation between the components of entropy change and the intensity or maintenance of atmospheric motion.

5.4. Zonal and eddy forms

The nearness of atmospheric structure to that of a zonally symmetric state motivates the common decomposition of motion and thermal fields into the sum of their zonal average and a deviation known as the eddy component. Considerable study has been devoted to the energetics of these components, particularly to their interaction in the conversion of available potential energy. Here we examine zonal and eddy forms of the entropic energy, and in the discussion utilize some ideas developed in a study of such decompositions that is incorporated in a forthcoming work by Johnson & Dutton (1972).

We let an overbar denote a zonal average (along latitude circles at constant height) and define for a variable, f ,

$$\bar{f} = \overline{ef/\bar{e}} \quad f' = f - \bar{f} \quad (5.14)$$

In particular, we shall write the components of the velocity vector as

$$v_i = \bar{v}_i + u_i \quad (5.15)$$

so that

$$K = \frac{1}{2} \int \rho v_i^2 dV = \frac{1}{2} \int [\bar{\rho} \bar{v}_i^2 + \overline{\rho u_i^2}] dV = \bar{K} + K' \quad (5.16)$$

But with the total potential energy this procedure gives

$$\Pi = \int \varrho [c_V T + gz] dV = \int \bar{\varrho} [c_V \hat{T} + gz] dV = \hat{\Pi} \quad (5.17)$$

so that there is no eddy form of Π . One of the contributions of the theory of available potential energy is the provision of an eddy form of static energy, and an eddy form of static entropic energy will appear here as well.

From (5.1) we may write

$$\Sigma = \frac{\hat{I} - I_0}{T_0} - \int \varrho \left[c_V \ln \frac{T}{\hat{T}} + R \ln \frac{\alpha}{\hat{\alpha}} \right] dV - \int \bar{\varrho} \left[c_V \ln \frac{\hat{T}}{T_0} + R \ln \frac{\hat{\alpha}}{\alpha_0} \right] dV \quad (5.18)$$

and thus may define

$$\hat{\Sigma} = \frac{\hat{I} - I_0}{T_0} - \int \bar{\varrho} \left[c_V \ln \frac{\hat{T}}{T_0} + R \ln \frac{\hat{\alpha}}{\alpha_0} \right] dV \quad (5.19)$$

and

$$\Sigma' = - \int \varrho \left[c_V \ln \frac{T}{\hat{T}} + R \ln \frac{\alpha}{\hat{\alpha}} \right] dV \quad (5.20)$$

Use of the Taylor series expansions of (4.4) and (4.7) along with the observation that $\hat{\alpha} = 1/\bar{\varrho}$ shows that $\hat{\Sigma}$ and Σ' are positive and vanish only if $\hat{T} - T_0$ (or T'), and $\hat{\alpha} - \alpha_0$ (or α') vanish identically.

The same process applied to the entropy gives

$$\begin{aligned} S - S_c &= \int \varrho \left[c_V \ln \frac{T}{T_c} + R \ln \frac{\alpha}{\alpha_c} \right] dV \\ &= \int \varrho \left[c_V \ln \frac{T}{\hat{T}} + R \ln \frac{\alpha}{\hat{\alpha}} \right] dV \\ &\quad + \int \bar{\varrho} \left[c_V \ln \frac{\hat{T}}{T_c} + R \ln \frac{\hat{\alpha}}{\alpha_c} \right] dV \\ &= -\Sigma' + S(\hat{T}, \hat{\alpha}) - S_c \end{aligned} \quad (5.21)$$

where $S(\hat{T}, \hat{\alpha}) - S_c$ is defined by the second term of the middle line.

Combination of these decompositions with (5.5) gives the relations

$$\hat{\Sigma} + \frac{\hat{K} + K'}{T_0} = S_0 - S(\hat{T}, \hat{\alpha}) \quad (5.22)$$

and

$$\Sigma' = S(\hat{T}, \hat{\alpha}) - S \quad (5.23)$$

As a first consequence, let us note that in developing corollaries to Theorem 4.5 we could not conclude that $\Sigma > 0$ implied the existence of weak motion because a state satisfying (4.23) provides a counterexample. But now we have

Corollary 4.5.4. *If $\Sigma' > 0$, then there is weak motion.*

Proof. As indicated following (5.20), if $\Sigma' > 0$, then there must be deviations of either temperature or density from their zonal averages and hence either $\nabla_z T \neq 0$ or $\nabla_z \varrho \neq 0$, which shows that $\mathbf{v}$ and $\bar{\mathbf{v}}$ cannot vanish identically.

(It is worth remarking that the same proof and conclusion apply to the eddy form of available potential energy in hydrostatic atmospheres.)

The procedure for developing budgets for the zonal and eddy forms of K and Σ is relatively straightforward although somewhat tedious. When the equations of motion are subjected to averaging of the form (5.14) the results may be used to show that

$$\frac{\partial \hat{K}}{\partial t} = \int \left\{ \overline{\varrho u_i u_k} \frac{\partial \hat{v}_i}{\partial x_k} + \overline{p} \frac{\partial \hat{v}_k}{\partial x_k} - g \bar{\varrho} \hat{v}_3 + \hat{v}_i \frac{\partial \overline{\Delta_{ik}}}{\partial x_k} \right\} dV \quad (5.24)$$

and

$$\frac{\partial K'}{\partial t} = \int \left\{ -\overline{\varrho u_i u_k} \frac{\partial \hat{v}_i}{\partial x_k} + \overline{p} \frac{\partial u_k}{\partial x_k} + u_i \frac{\partial \overline{\Delta_{ik}}}{\partial x_k} \right\} dV \quad (5.25)$$

Because we are expounding theory and not presenting formulas for numerical computation, we retain Cartesian tensor notation; full generality is obtained by interpreting the partial derivatives as covariant derivatives. Details of the derivation of this result appear in Johnson & Dutton (1972) and in Dutton (1972) where an ensemble average rather than a zonal average is employed. The only difference between the formal appearance of the results in these two cases is that the zonal averages are of course constant along latitude circles.

In these equations we have the familiar con-

version via the Reynolds stresses between $\hat{K}$ and K'

$$C(\hat{K}, K') = - \int \overline{\varrho u_i u_k} \frac{\partial \hat{v}_i}{\partial x_k} dV \quad (5.26)$$

the conversion between Π and K

$$\begin{aligned} C(\Pi, K) &= \int \left[\overline{p \frac{\partial v_k}{\partial x_k}} - \overline{g \varrho v_3} \right] dV \\ &= \int \left[\overline{p \frac{\partial \hat{v}_k}{\partial x_k}} - \overline{g \hat{\varrho} \hat{v}_3} \right] dV + \int \overline{p \frac{\partial u_k}{\partial x_k}} dV \\ &= C(\Pi, \hat{K}) + C(\Pi, K') \end{aligned} \quad (5.27)$$

and the frictional losses $-D(\hat{K})$ and $-D(K')$ of kinetic energy given by the last terms of the integrals in (5.24) and (5.25).

To complete the budget, we must compute the rate of change of $S(\hat{T}, \alpha)$. For this purpose, we observe that averaging the continuity equation and the First Law of Thermodynamics gives

$$\frac{\partial \bar{\varrho}}{\partial t} + \frac{\partial}{\partial x_k} (\bar{\varrho} \hat{v}_k) = 0 \quad (5.28)$$

and

$$\begin{aligned} c_v \left[\frac{\partial \hat{T}}{\partial t} + \hat{v}_k \frac{\partial \hat{T}}{\partial x_k} + \frac{1}{\bar{\varrho}} \frac{\partial}{\partial x_k} (\overline{\varrho u_k T'}) \right] \\ + \frac{\bar{p}}{\bar{\varrho}} \frac{\partial \hat{v}_k}{\partial x_k} + \frac{\bar{p}}{\bar{\varrho}} \frac{\partial u_k}{\partial x_k} = \frac{\bar{Q}}{\bar{\varrho}} \end{aligned} \quad (5.29)$$

in which Q now represents the sum of the radiation, conduction, and dissipation terms.

Therefore the expression

$$S(\hat{T}, \hat{\alpha}) - S_c = \int \bar{\varrho} \left\{ c_v \ln \frac{\hat{T}}{T_c} - R \ln \frac{\bar{\varrho}}{\varrho_c} \right\} dV \quad (5.30)$$

gives the rate

$$\begin{aligned} \dot{S}(\hat{T}, \hat{\alpha}) &= \int - \frac{\partial}{\partial x_k} (\bar{\varrho} \hat{v}_k) \left[c_v \ln \frac{\hat{T}}{T_c} + R \ln \frac{\hat{\alpha}}{\alpha_c} \right] dV \\ &\quad + \int \bar{\varrho} \frac{c_v}{\hat{T}} \frac{\partial \hat{T}}{\partial t} dV \end{aligned} \quad (5.31)$$

in which we have used $\dot{M} = 0$. Use of (5.29), the usual boundary conditions, and the fact that

$$\bar{p} = R(\bar{\varrho} \hat{T} + \overline{\varrho' T'}) \quad (5.32)$$

allows some terms to be combined to give

$$\begin{aligned} \dot{S}(\hat{T}, \alpha) &= \int \left\{ \overline{\frac{Q}{\hat{T}}} - \frac{c_v}{\hat{T}} \frac{\partial}{\partial x_k} \overline{\varrho u_k T'} - \overline{\frac{p}{\hat{T}} \frac{\partial u_k}{\partial x_k}} \right. \\ &\quad \left. - R \overline{\frac{\varrho' T'}{\hat{T}}} \frac{\partial \hat{v}_k}{\partial x_k} \right\} dV \end{aligned} \quad (5.33)$$

From (5.23) then

$$\begin{aligned} \frac{\partial \Sigma'}{\partial t} &= \int \left(\overline{\frac{1}{\hat{T}}} - \frac{1}{T} \right) Q dV - \int \left\{ c_v \overline{\frac{\varrho T'}{\hat{T}}} \mathbf{u} \cdot \nabla \ln \hat{T} \right. \\ &\quad \left. + \overline{\frac{p}{\hat{T}}} \nabla \cdot \mathbf{u} + \frac{R \overline{\varrho' T'}}{\hat{T}} \nabla \cdot \hat{\mathbf{v}} \right\} dV \end{aligned} \quad (5.34)$$

It is clear that this result contains no direct interchange between Σ' and K/T_0 , so that Σ' would interact only with $\hat{\Sigma}$. Although this situation does have some attractive implications, it is probably advisable to circumvent it by making use of the fact that $\overline{p \nabla \cdot \mathbf{u} / \hat{T}}$ is likely to be similar to $\overline{p \nabla \cdot \mathbf{u} / T_0}$.

Since we always may, and usually must, add a zero by adding and subtracting the same term to obtain closure in energy budgets (a situation discussed in detail by Johnson & Dutton (1972)), we choose here to rewrite (5.34) as

$$\frac{\partial \Sigma'}{\partial t} = \int \left(\overline{\frac{1}{\hat{T}}} - \frac{1}{T} \right) Q dV + C(\hat{\Sigma}, \Sigma') - C(\Pi, K')/T_0 \quad (5.35)$$

where now

$$\begin{aligned} C(\hat{\Sigma}, \Sigma') &= - \int \left\{ c_v \overline{\frac{\varrho T'}{\hat{T}}} \mathbf{u} \cdot \nabla \ln \hat{T} \right. \\ &\quad \left. + R \overline{\frac{\varrho' T'}{\hat{T}}} \nabla \cdot \hat{\mathbf{v}} - \left(\frac{1}{T_0} - \frac{1}{\hat{T}} \right) \overline{p \nabla \cdot \mathbf{u}} \right\} dV \end{aligned} \quad (5.36)$$

With (5.22) and (5.23), we use $Q = H + \varepsilon$ and

$$\dot{E} = \int Q dV - D(K) = \int H dV - D_B \quad (5.37)$$

where $-D_B$ is the boundary integral of (4.28) to write

$$\begin{aligned} \frac{\partial \hat{\Sigma}}{\partial t} &= - \frac{1}{T_0} \frac{\partial K}{\partial t} + \frac{\Gamma}{T_0} \frac{\partial E}{\partial t} - \frac{\partial S}{\partial t} - \frac{\partial \Sigma'}{\partial t} \\ &= - \frac{C(\Pi, \hat{K})}{T_0} - C(\hat{\Sigma}, \Sigma') + \int \left(\frac{\Gamma}{T_0} - \frac{1}{\hat{T}} \right) \bar{H} dV \\ &\quad + \frac{1-\Gamma}{T_0} D_B + \int \left(\frac{1}{T_0} - \frac{1}{\hat{T}} \right) \bar{\varepsilon} dV \end{aligned} \quad (5.38)$$

We define the sum of the last three terms as $G(\hat{\Sigma})$ and denote this generation when energetic isolation prevails as

$$G_E(\hat{\Sigma}) = \int \left(\frac{1}{T_0} - \frac{1}{\hat{T}} \right) (\bar{H} + \varepsilon) dV \\ = \int \left(\frac{1}{T_0} - \frac{1}{\hat{T}} \right) \bar{Q} dV \quad (5.39)$$

so that

$$G(\hat{\Sigma}) = G_E(\hat{\Sigma}) + \frac{\Gamma-1}{T_0} \dot{E} = \frac{D(K)}{T_0} + \frac{\Gamma}{T_0} \dot{E} - \int \frac{\bar{Q}}{\hat{T}} dV \quad (5.40)$$

Similarly, we set

$$G(\Sigma') = \int \left(\frac{1}{\hat{T}} - \frac{1}{T} \right) Q dV = \int \frac{T' Q}{\hat{T} T} dV \quad (5.41)$$

Combination of (5.39) and (5.41) shows that $G(\hat{\Sigma})$ and $G(\Sigma')$ sum to $D(K)/T_0$, when $\dot{E} = \dot{S} = 0$, as required.

The relations involved in the budget of entropic energy are depicted in Fig. 5.1; we remark, however, that the closure given here is clearly not unique and that others are possible and may be preferable.

This budget has some interesting features. The conversions and dissipations involving $\hat{K}$ and K' are, naturally, the usual ones except for the factor T_0 . But the generation terms are quite different from those of either classical energetics or the theory of available potential energy. In the atmosphere, we have $\dot{E} = \dot{S} = 0$ on the long-term average, and for simplicity we may consider this case. Two interpretations may be given.

First, (5.39) implies that mean heat $\bar{Q}$ added at high mean temperatures relative to heat loss at low mean temperature generates $\hat{\Sigma}$. Because $\varepsilon \geq 0$, while H changes sign in large-scale covariance with the temperature factor, we may expect that the generation of $\hat{\Sigma}$ is dominated by the radiational and conduction heating. Similarly, as shown by (5.41), Σ' is generated if Q is covariant with $T'/T\hat{T}$. On this interpretation, the heating fields must act in the same sense as is required for the destruction of entropy.

As an alternate interpretation, we may use the expressions

$$G_E(\hat{\Sigma}) = \frac{D(K)}{T_0} - \int \frac{\bar{Q}}{\hat{T}} dV \quad (5.42)$$

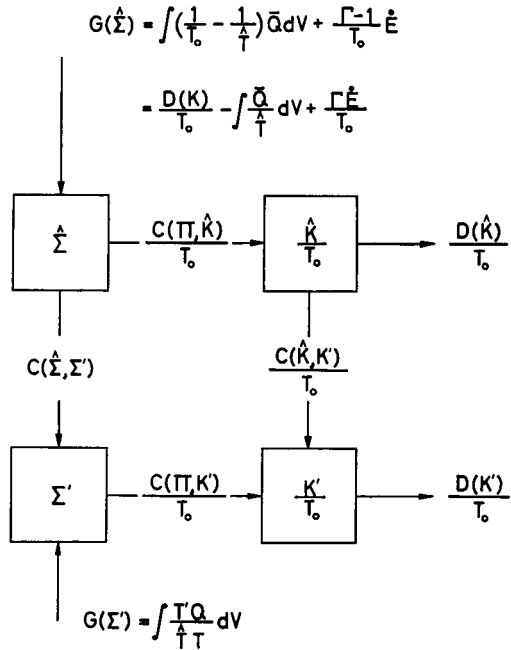


Fig. 5.1. Diagram of the budget of entropic energy. The arrows show the direction of energy flow when the conversion expressions are positive.

and

$$G(\Sigma') = \int \frac{T'}{\hat{T}} \frac{Q}{T} dV = \int \frac{Q T'}{\hat{T} T} dV \quad (5.43)$$

Clearly, the generation closes the kinetic energy cycle by returning the dissipated energy $D(K)$ to the reservoir $T_0\hat{\Sigma}$. But in addition, a heating term appears in each of the two expressions. The condition $\dot{S} = 0$ implies that

$$0 = \int \frac{Q}{T} dV = \int Q \left[\frac{1}{\hat{T}} - \frac{T'}{\hat{T}T} \right] dV = \int \frac{\bar{Q}}{\hat{T}} dV - G(\Sigma') \quad (5.44)$$

which expresses the fact that if the entropy is constant, then the symmetric and eddy components of its rate of change must cancel.

These two heating terms may be interpreted as representing the catalytic effect of external heating which causes the entropy destruction that sets the atmosphere into motion. As a catalyst, this entropy destruction has no further role in the motion it initiates, however, and is lost to the entropic energy budget by appearing in identical form as the loss of Σ' .

As shown by (5.33), the heating term in the

Table 6.1. *The variables of global thermodynamics*

Quantity	Winter (Dec, Jan, Feb)		Summer (Jul, Aug, Sep)		Difference (Winter- Summer)	
E ergs cm^{-2}	2.52	10^{12}	2.58	10^{12}	-6.0	10^{10}
T_0 deg K	243.1		248.9		-5.8	
K ergs cm^{-2}	2.3	10^9	1.0	10^9	1.3	10^9
A ergs cm^{-2}	4.53	10^9	1.84	10^9	2.7	10^9
S ergs cm^{-2} K^{-1}	5.89	10^{10}	5.91	10^{10}	-2.67	10^8
S_0 ergs cm^{-2} K^{-1}	5.98	10^{10}	6.01	10^{10}	-2.43	10^8
$S_0 - S$	9.844	10^8	9.604	10^8	2.04	10^7
K/T_0 ergs $\text{cm}^{-2} \text{K}^{-1}$	9.45	10^6	4.0	10^6	5.45	10^6
Σ ergs cm^{-2} K^{-1}	9.75	10^8	9.56	10^8	1.9	10^7
$T_0 \Sigma$ ergs cm^{-2}	2.37	10^{11}	2.38	10^{11}	1.0	10^9
$(S_0 - S)/S_0$	0.0164		0.0160			

second line of (5.40) gives the effect of the heating $\bar{Q}$ on the entropy $S(\hat{T}, \hat{\alpha})$ and thus decreases in this entropy form due to differential heating are precisely opposite to increases in the amount of $\hat{\Sigma}$. The conversion $C(\hat{\Sigma}, \Sigma')$ represents (apart from the change made in (5.36) in order to obtain closure of the budget) the effect of the eddy component of the flow on $S(\hat{T}, \hat{\alpha})$. Under the constraint that the energy and entropy be constant, these eddy components are arranged so that $C(\hat{\Sigma}, \Sigma') - C(\Pi, K')/T_0$ comes out exactly as expressed by (5.44).

These two interpretations of the results summarized in Fig. 5.1 demonstrate how the entropic energy budget combines the concepts and implications of the theories of energy and entropy. The first interpretation may be referred to as the entropic version, for it emphasizes the covariance between heating and temperature fields. In this version, $\dot{E}$ is presumably unimportant since $\Gamma \sim 1 \sim 10^{-3}$. Hence we have a direct expression in the entropic energy budget of the fact that atmospheric motion is caused and maintained by a global process most easily understood by considering entropy.

The second interpretation is an energetic version, for $\dot{E}/T_0$ appears directly and the input of energy $\dot{D}(K)$ needed for a steady state is explicitly present. In the energetic version, the catalytic effect of entropy processes is still

apparent and shows how destruction of zonal entropy is compensated by production of entropy in the eddy flow.

The results of this section demonstrate the pervasive influence of entropy change on atmospheric processes, even if full comprehension of the internal mechanisms involved is not yet available. Fig. 5.1, as a summary of the budget of entropic energy, manifests the role of entropy as the central variable of global thermodynamics, as the controller of atmospheric events, for there the zonal and eddy components of the global rate of entropy change appear at the beginning of the atmosphere's kinetic energy cycle.

6. Numerical estimates and an approximate form

The primary interest in the quantities under discussion centers on their rates of change and the processes that induce them or balance them so that they are steady. It is possible to obtain presumably accurate estimates of the magnitudes of the global variables involved in the theory of entropic energy from available data. Estimates of the various terms in the entropic energy budget of Fig. 5.1 will also be given, but they are less reliable.

It is also shown in this section that the

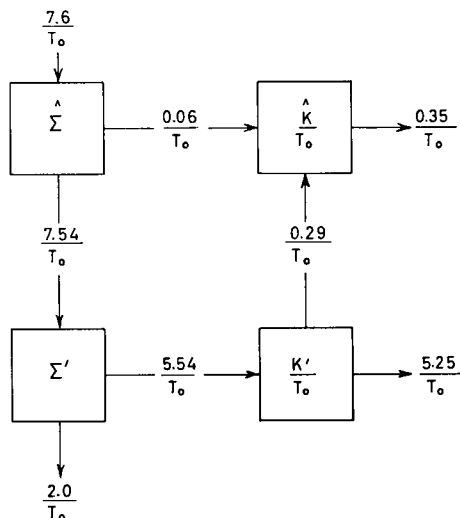


Fig. 6.1. Estimates of the numerical values of the conversions in the budget of entropic energy, obtained as explained in the text.

exact expressions for the amount of static entropic energy may be converted into an approximate form that has both theoretical and practical interest.

6.1. The magnitude of global thermodynamic variables

Estimates of the numerical values of the various quantities discussed in this article are given in Table 6.1. (All variables, both in the table and in the discussion below, are expressed as averages/cm².) With the exception of the kinetic energy values obtained from Kung (1967, Fig. 10), the estimates utilize the mean values computed by Dutton & Johnson (1967) for the Northern Hemisphere along 75 W.

The crucial variable is $S_0 - S$, and the only value of S which seems to be available is that computed for 75 W. The definition used by Dutton & Johnson to compute S makes it clear that in effect they used $\theta_c = 1$ K, so we adopt the same convention in calculating S_0 and we have

$$S_0 = c_p \int_0^\infty \rho_0 \ln \left(\frac{\theta_0}{\theta_c} \right) dz = \frac{p_0(0)}{g} \left[c_p \ln \frac{\theta_0(0)}{\theta_c} + R \right] \quad (6.1)$$

in which we assume that $p_0(0)$ is 1013 mb in both summer and winter, and that $\theta_0(0) = T_0$. Since $K \ll \Pi$, we may use $E = \Pi$, which gives $T_0 = \Pi/c_p M = \Pi g/c_p p_0(0)$.

As is evident from the estimates, the differences between summer and winter in E and S are so slight that correspondingly there is little difference in the seasonal values derived from them. Taking account of the mass flux from the winter to summer hemisphere would presumably not lead to significant variation in the estimates.

The measure $(S_0 - S)/S_0 = 1.6\%$ of departure from equilibrium is, perhaps, surprisingly small, but indicates that the production of kinetic energy and the intensity of motion are very sensitive to small variations in entropic variables.

6.2. Estimates of terms in the budget of entropic energy

There are serious difficulties in attempting to estimate many of the terms in the atmosphere's energy budget. Still, it seems worthwhile to try to develop a preliminary version of Fig. 5.1 with numerical values assigned to the various conversion terms. Specification of only a few of the

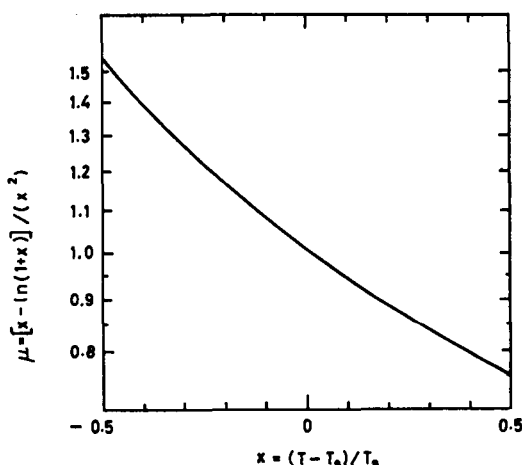


Fig. 6.2. Ratio of the exact form of the integrand to a simple square.

rates in the budget, as demonstrated by Dutton & Johnson (1967), allows estimates of the others to be obtained that are consistent with a steady state. The three basic rates used here are the averages of the global values for $C(\Pi, K)$, and $C(K, K')$ given by Newell et al. (1969, Fig. 19) along with Kung's (1969, Table 3) estimate of 5.6 watts/m² for the annual mean dissipation over North America.

With these values, the budget may be completed except for the second term of (5.42). The only numerical estimate that seems to be suitable for this term is Lettau's (1954) computation of the rate of entropy destruction by heating (including latent heating). This value, expressed in energy units, of 2 watts/m², is probably an underestimate, but allows us to set $G(\Sigma) = 7.6$ watts/m² and complete the budget, as shown in Fig. 6.1.

6.3. Approximate expressions

The two expressions for Σ used in this article are

$$\begin{aligned} \Sigma &= \frac{1}{2} \int \left\{ c_v \left[\frac{(T - T_0)/T_0}{1 + \lambda(T - T_0)/T_0} \right]^2 \right. \\ &\quad \left. + R \left[\frac{(\alpha - \alpha_0)/\alpha_0}{1 + \kappa(\alpha - \alpha_0)/\alpha_0} \right]^2 \right\} dV \\ &= \int \left\{ c_v \left(\frac{T - T_0}{T_0} - \ln \frac{T}{T_0} \right) - R \ln \left(\frac{\alpha}{\alpha_0} \right) \right\} dV \end{aligned} \quad (6.2)$$

The first has the advantage of being obviously positive, but the disadvantage of depending on the values λ and κ which are not readily exhibited. The second is easily computable from observational data, but does not reveal its positive nature as clearly as a sum of squares.

These two expressions derive from the Taylor series for $x = (T - T_0)/T_0$,

$$\ln(1+x) = x - \frac{1}{2} \left[\frac{x}{1+\lambda x} \right]^2 \quad (6.3)$$

which upon comparison with the Maclaurin series

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots \quad |x| < 1 \quad (6.4)$$

suggests that we determine the ratio

$$\mu = [x - \ln(1+x)]/(x^2/2) \quad (6.5)$$

so that we may then write

$$\mu x^2 = \left[\frac{x}{1+\lambda x} \right]^2 \quad (6.6)$$

The ratio (6.5) is shown in Fig. 6.2. If we have

$$\text{Max} \left\{ \left| \frac{T-T_0}{T_0} \right| \text{ and } \left| \frac{\alpha-\alpha_0}{\alpha_0} \right| \right\} < B \quad (6.7)$$

then from the figure we may find values μ_M and μ_m such that

$$\mu_m \Sigma_A \leq \Sigma \leq \mu_M \Sigma_A \quad (6.8)$$

where

$$\Sigma_A = \frac{1}{2} \int \varrho \left\{ c_v \left(\frac{T-T_0}{T_0} \right)^2 + R \left(\frac{\alpha-\alpha_0}{\alpha_0} \right)^2 \right\} dV \quad (6.9)$$

It is clear that Σ and Σ_A vanish together, and at the same rate, as the deviations vanish.

For example, in the atmosphere $\text{Max} |(T - T_0)/T_0| \sim 50/250 = 0.20$ which gives, if $\text{Max} |(T - T_0)/T_0| > \text{Max} |(\alpha - \alpha_0)/\alpha_0|$, the value $\mu_m = 0.86$ and $\mu_M = 1.20$. If the small seasonal variation shown in Table 6.1 is representative, then Σ is nearly constant in the atmosphere and it is likely that $\Sigma \sim \mu \Sigma_A$ will be a suitable approximation for a constant μ , a hypothesis that can be checked by comparing exact and approximate values computed from global observational data.

The fact that (6.8) is true has two significant implications. The first is that the basic stability Theorem 4.2 may be expressed with a L_2 norm. From (4.11) we may write

$$\frac{K(t)}{T_0} + \mu_m(t) \Sigma_A(t) \leq \frac{K(t)}{T_0} + \Sigma(t) \leq \frac{K(\tau)}{T_0} + \Sigma(\tau) \quad (6.10)$$

and hence we have

Corollary 4.2.1. *Under the conditions of the theorem and (6.7), there is a function $\mu_m(t) \leq 1$ such that the deviations from the associated equilibrium state are restricted in mean square by (6.10).*

Perhaps of more interest in atmospheric dynamics is the fact that a form of available potential energy that occurs as part of a quadratic invariant of linearized equations describing perturbation flow resembles Σ_A . We consider only the simplest case of shallow convection in an originally motionless basic state. Then in the notion of Dutton & Fichtl (1969), the perturbation energy is

$$E_p = \frac{1}{2} \int \varrho_0 \left\{ v^2 + \left(\frac{g}{\omega_g} \right)^2 \left(\frac{T-T_0}{T_0} \right)^2 \right\} dV = K + A_p \quad (6.11)$$

and because in this case $(T - T_0)/T_0 = -(\alpha - \alpha_0)/\alpha_0$, from (6.9) we have

$$\Sigma_A = \frac{1}{2} \int \varrho c_p \left(\frac{T-T_0}{T_0} \right)^2 dV \simeq \frac{1}{2} \int \varrho_0 c_p \left(\frac{T-T_0}{T_0} \right)^2 dV \quad (6.12)$$

Although the temperature T_0 is usually allowed to vary as $T_0 = T_0(z)$ in perturbation studies, we use the associated equilibrium state here so that

$$(g/\omega_g)^2 = c_p T_0 \quad (6.13)$$

and therefore

$$A_p \simeq T_0 \Sigma \quad (6.14)$$

The linearized equations describing deep convection and perturbations about steady zonal currents lead to more complex forms of available energy, for which the association with Σ_A is not so apparent. Still the entropic energy or Σ in (6.2) is part of an exact integral of the equations of motion, and it may turn out to be

advantageous to use it or its approximate form to govern the development of approximate equations.

7. Stability and entropy

The combination of entropy and the new energy form in a single integral of the equations of motion is the main result of this article. The entropic energy has a trend in time, when the atmosphere is isolated, that is dictated by the Second Law of Thermodynamics. Since the entropic energy is a positive definite measure of the departure of atmosphere from its associated equilibrium state, this trend implies the global stability of isolated processes. They must tend toward equilibrium.

Thus the maintenance of circulation, in the sense that the entropic energy is constant rather than decreasing, requires that entropy be destroyed by the covariance of temperature and heating fields. The new development is that questions about intensity of the circulation can be phrased as questions about entropic energy—which measures departure from equilibrium—and thus as questions about the entropy difference $S_0 - S$. Entropy is the controlling variable in the present theory of global thermodynamics.

But study of the maintenance of circulation is neither as interesting nor as significant as enquiries about the reasons that particular modes of response are selected as the external forcing is altered. It seems that it might be worthwhile to try to formulate the relevant questions with the concept of entropic energy and to search for the implications of the difference $S_0 - S$ and its rate of change that determine the choice of mode.

In the atmosphere, and in much simpler systems, it is our general experience that motion sets in when the difference $S_0 - S$ becomes large enough, and the intensity of the response to the thermal forcing generally increases as this difference does. It is not unreasonable to suppose that choice of mode is related to the development of entropy production mechanisms that most efficiently hold the difference at a minimum value, consistent with the externally imposed entropy destruction. Such a hypothesis seems consistent with the theorem of minimum entropy production (Prigogine, 1968).

The study of entropy production and its implications for mode choice is more complicated

in the atmosphere than for simpler systems because of the important role of radiation. Although we may assume the solar heating is externally imposed, the loss of long-wave radiation is a functional of the past and present of the motion and thermal fields and depends upon the heat transport mechanisms that create the temperature structure.

The solar heating, by itself, creates entropy, but it also increases S_0 because of the addition of energy. If we let the integral of the solar heating be $\mathcal{H}^\downarrow$, we have

$$(\dot{S}_0 - \dot{S})^\downarrow = \mathcal{H}^\downarrow \left(\frac{1}{T_0} - \frac{1}{T^\downarrow} \right) \quad (7.1)$$

where $T^\downarrow$ is an average similar to that of (4.32). This rate is presumably positive on the grounds that $T^\downarrow > T_0$. The increase in the difference $S_0 - S$ is reflected, in part, by the increasing latitudinal temperature gradient that would result if there were no heat transport. Thus the destruction of these gradients through advection of heat is the primary mode of entropy production in the atmosphere, although it is usually masked by considering the net radiation field as given. But if we consider the integral of the long-wave cooling $-\mathcal{H}^\uparrow$, then

$$(\dot{S}_0 - \dot{S})^\uparrow = -\mathcal{H}^\uparrow \left(\frac{1}{T_0} - \frac{1}{T^\uparrow} \right) \quad (7.2)$$

which is presumably quite small or slightly negative because $\mathcal{H}^\uparrow$ is fairly uniform with latitude. Upon combining (7.1) and (7.2) and using $\mathcal{H} = \mathcal{H}^\downarrow - \mathcal{H}^\uparrow = 0$, we see again that the generation of entropic energy depends on $T^\uparrow < T^\downarrow$.

If we are to study mode selection in the atmosphere from the viewpoint of entropy, we shall have to consider the effect of the various conceivable modes on the outgoing radiation fields. The net radiation field is a prime element in the dynamics of entropic energy, and this field includes both the external forcing and the consequences of the response to that forcing.

The global stability of atmospheric flow under isolation has been demonstrated, but the concept involves the sum of the kinetic and the static entropic energy. In meteorology, we are often interested in local instabilities that involve conversions between these two forms, and the major interest is with the baroclinic instability in which there is an increase of K at the expense of $T_0 \Sigma$.

Such instability may be viewed as a consequence of the competition between entropy destruction and entropy production. The heating of (7.1) is constantly increasing the temperature gradients, while the breakdown of the large-scale patterns into smaller ones provides locally effective heat transport mechanisms that reduce the gradients. Still, this type of oscillating response appears to occur in only a limited range of $S_0 - S$, for as shown by rotating convection experiments, symmetric and essentially steady responses are sufficient when the system is closer to equilibrium.

The global stability that prevails under isolation is a consequence of the properties of entropy. The implication of Theorem 4.2 is that the atmosphere cannot deviate further from equilibrium than it was when isolation is imposed and the consequence of Theorem 4.3 is that it must tend toward equilibrium once it has become isolated. Thus the Gibbs stability of the associated equilibrium state leads to a global stability criterion for the evolution of flow in isolated atmospheres.

The converse question is also of interest and has been studied for simple systems by Coleman & Mizel (1967). Now we ask whether a hypothesis that the atmosphere is globally stable in the sense that the intensity of circulation will decrease in isolation will lead to an entropy concept.

We start with the entropic energy, which can be obtained directly as an integral of the equations of motion, impose a stability postulate, and thus arrive at an entropy function.

Theorem 7.1. *Let an atmosphere have energy E and mass M . Define $T_0 = E/c_p M$ and define $\varrho_0(z)$ and $p_0(0)$ by (3.12) and (3.13). Then the quantity*

$$N = T_0 \mathcal{E} = K + T_0 \Sigma \quad (7.3)$$

where Σ is defined by (5.1) is an integral of the equations of motion that varies in time according to (5.7), and for energetically isolated atmospheres has the rate of change

$$\begin{aligned} \dot{N}/T_0 = \dot{\mathcal{E}} = & \int \left[\frac{\nabla \cdot \mathbf{R}}{T} - \nabla \cdot \left(\frac{k \nabla T}{T} \right) \right] dV \\ & - \int \frac{1}{T} \left[\frac{k |\nabla T|^2}{T} + \varepsilon \right] dV \quad (7.4) \end{aligned}$$

If the energetically isolated atmosphere is stable in the sense that

$$\dot{N} = T_0 \dot{\mathcal{E}} \leq 0 \quad (7.5)$$

when

$$\int \left[\frac{\nabla \cdot \mathbf{R}}{T} - \nabla \cdot \left(\frac{k \nabla T}{T} \right) \right] dV = 0 \quad (7.6)$$

then the coefficients of heat conduction and friction are positive, and there exists a function $S(t)$ which is an absolute maximum when the equilibrium in which N vanishes is reached.

Proof. The statement (7.4) follows from (5.7), which clearly follows from the equations of motion. Hence (7.5) and (7.6) imply that the coefficients are positive. Finally, by defining

$$S(t) = -\mathcal{E} + \text{const} \quad (7.7)$$

we have an entropy function, which upon evaluation when $N = 0$, may be written

$$S_0 - S(t) = \mathcal{E} = N/T_0 \quad (7.8)$$

Stability and entropy, then, are related concepts, the one following from the other in the model atmosphere considered here. The results presented in this article seem to justify an optimism that the study of the internal mechanisms and quasi-stable states of the atmosphere from the viewpoint of global thermodynamics may provide an enhanced perception of the basic criteria by which they are governed.

Acknowledgements

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Table of symbols

All thermodynamic quantities are expressed in mechanical units. Numbers of defining equations are given in parentheses.

Global quantities

S	Entropy (2.1)
M	Mass (2.2)
E	Energy (2.3)
I	Internal energy (3.2)
P	Potential energy (3.3)
Π	Total potential energy $\Pi = I + P$
X	Energy of the hydrostatic defect (3.7)
S_0	Entropy of the associated equilibrium state
$T_0 \Sigma$	Static entropic energy (4.9) or (5.1)
$T_0 \mathcal{E}$	Total entropic energy $T_0 \mathcal{E} = K + T_0 \Sigma$
N	Total entropic energy $N = T_0 \mathcal{E}$
A	Available potential energy

Local variables

p	Thermodynamic pressure
ρ	Density
T	Absolute temperature
α	Specific volume $\alpha = 1/\rho$
s	Specific entropy
θ	Potential temperature $\theta = T(1000/p)^{R/c_p}$
$\mathbf{v}$	Velocity vector
v_s	Vertical component of $\mathbf{v}$
χ	Hydrostatic defect (3.5)
Δ_{ij}	Viscous tensor (3.19)
ε	Dissipation (3.22)
$\mathbf{R}$	Radiant energy flux
H	Heating rate $H = -\nabla \cdot \mathbf{R} + \nabla \cdot k \nabla T$
Q	Total heating rate $Q = H + \varepsilon$
ω_p	Brunt-Väisälä frequency
$\mathbf{u}$	Deviation $\mathbf{u} = \mathbf{v} - \bar{\mathbf{v}}$

Constants and parameters

c_V	Specific heat at constant volume (ergs/gm deg)
c_p	Specific heat at constant pressure
R	Gas constant $R = c_p - c_V$
g	Acceleration of gravity
k	Coefficient of heat conduction
μ	Dynamic shear viscosity coefficient
μ'	Bulk or expansion viscosity

Coordinates

t	Time
τ	A specific time
$\mathbf{x}$	Spatial coordinate
z	Vertical coordinate in geopotential units
$\mathbf{k}$	Vertical unit vector

Subscripts

0	Denotes conditions in the associated equilibrium state
c	Denotes constants used for a reference value of the entropy
B	Denotes evaluation of the boundary

Operators

$\int () dV$	Integral over the entire atmosphere
$\int_B dB$	Integral over the boundary of the atmosphere
$(\dot{})$	Material derivative $(\dot{}) = \left[\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right] ()$
	For the entire atmosphere, the transport theorem implies that if

$$F = \int q f dV$$

then

$$\dot{F} = \int q \dot{f} dV$$

∇	Differentiation operator
∇_z	Differentiation along geopotential surface
$\overline{()}$	In section 2, average per unit area; in Section 5.4, a zonal average
$(\bar{})$	Mass-weighted zonal average (5.14)
$(\bar{})'$	Deviation from a mass weighted zonal average (5.14)

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ГЛОБАЛЬНАЯ ТЕРМОДИНАМИКА АТМОСФЕРНЫХ ДВИЖЕНИЙ

Глобальная термодинамика занимается изучением зависимостей между общей массой, энергией и энтропией атмосферы. Основной результат состоит в явном, точном соотношении между энтропией, кинетической энергией и статической энтропической энергией — новой формой статической энергии. Устойчивость атмосферных движений и поддержание циркуляции исследуются совместно с полной энтропической энергией (суммой кинетической и статической энтропической энергии). Динамический смысл концепции Гиббса о термическом равновесии состоит в том, что полная энтропическая энергия монотонно стремится к нулю в изолированной атмосфере, что является следствием второго начала термодинамики. Более того, генерация энтропической энергии связана спе-

цифическим образом с разрушением энтропии за счет нагрева при высокой температуре относительно охлаждения при низкой температуре. Выводится энергический баланс для зональной и вихревой форм энтропической энергии, который иллюстрирует каталитическое воздействие на общую циркуляцию разрушения энтропии. Наконец, путем обратных рассуждений постулат устойчивости применяется к полной энтропической энергии, чтобы установить существование функции энтропии. Концепция энтропической энергии представляет собою новый интеграл уравнений атмосферных движений, который является комбинацией известных интегралов энергии и энтропии и дает математическую форму соотношения между энтропией и движением.