On the growth and decay of a deep water wave component

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ABSTRACT

A wave component grows exponentially in time owing to some feedback interaction with the mean wind above the sea. Initially, this growth is opposed primarily by wave breaking. By assuming that the rate of loss of energy at a given wave frequency is proportional to the mean square slope of waves at lower frequencies, an equation is derived for the temporal behaviour of a wave. One prediction of this equation is that a growing wave overshoots its final equilibrium energy level: a phenomenon observed to occur in sea waves.

1. Introduction

Over the last few years the amount of data relevant to the sea wave growth problem has increased to the extent that the present theories of wave generation have been shown to be inadequate. However, measurements do show that the exponential growth of a wave component due to the wind–wave interaction follows a simple law (Snyder & Cox, 1966; Barnett & Wilkerson, 1967; Dobson, 1971). Thus, the rate of input of energy from the wind to a wave component can be modelled empirically at least.

Barnett & Wilkerson report that a wave does not monotonically approach its equilibrium energy level. Indeed, growing exponentially it overshoots its equilibrium value, reaches a maximum and then decays. This implies that the rate of loss of energy is not a local function of wave energy and frequency.

On the other hand, casual observation of a developing wave field suggests that energy is lost by wave breaking at all frequencies. That is, contrary to the case of the spectral density of turbulence, say, energy is not required to be transferred down the spectrum by a cascade process until it reaches scales small enough for viscous dissipation to be effective. Wave breaking permits of the degradation of wave energy at any frequency into turbulent motion.

Assuming that spectral transfer plays a minor role in the growth of deep water waves (at least until the energy of a component reaches its maximum), we find that the input of energy from the mean wind is opposed only by the loss due to wave breaking. A simple model for the rate of loss of energy is presented in § 2. Thus, an equation for the rate of change of energy of a wave component is derived. The predicted behaviour of a wave component is found to be essentially consistent with the data of Barnett & Wilkerson (1967). However, it appears that spectral transfer becomes important for waves approaching final equilibrium.

2. Formulation of growth law

We assume that the temporal behaviour of the wave energy spectral density ϕ at a given frequency σ is described by a first order differential equation. This implies that the time rate of change of energy at time t depends only upon the state of the energy distribution at that time. This seems to be consistent with the common assumption that the amplitude of a wave component is governed by a second order differential system in time.

Hence the rate of increase in the spectral density per radian of a statistically homogeneous wave field is given by the equation

$$\frac{\partial \phi}{\sigma \partial t} = E_P + E_F - E_B + E_T \tag{2.1}$$

where E_P describes the input of energy to a wave component by resonant interaction with turbulent pressure fluctuations in the air (Phillips, 1957); E_F represents the input caused by a feedback interaction with the wave-induced pressure fluctuations in the air (for example, Miles, 1957;) E_B is the rate of loss of energy due to wave breaking; E_T is the net increase in energy by spectral transfer between different wave components (for example, Benjamin & Feir, 1967). The effects of surface tension and viscosity are neglected in (2.1); they are small compared with that of E_B in the range of wave frequencies under consideration.

As we wish to estimate the initial growth rate of the spectral density, we neglect the spectral transfer term E_T in (2.1). The Benjamin & Feir (1967) instability is a probable mechanism for the transfer of energy between components, and it operates in the neighbourhood of a dominant wave component. Its efficacy increases with increasing wave slope. Therefore, such a mechanism affects the energy distribution primarily in the neighbourhood of the peak in the spectral density. It causes the transfer of energy to low frequency (unsaturated) wave components, and it ought to become increasingly important as the phase speed of the waves at the peak of the spectral density approaches the mean wind speed; i.e. as E_F becomes less effective. On the other hand, a transfer mechanism is expected to smooth sharp discontinuities such as the steep forward face observed in the spectral density of a growing wave field. Thus, the appearance of this steep forward face itself implies that the spectral transfer processes are being dominated by processes causing the local input of energy.

We therefore assume that the initial growth of a component to its maximum value at least, is described adequately with E_T set equal to zero. That is, the net growth rate is simply the resultant of the input of energy from the wind and the loss of energy by wave breaking.

Resonant interaction between a wave component and turbulent pressure fluctuations in the air produces a linear increase in wave energy with time. Thus, Phillips (1966) estimates that the frequency σ_0 for which the wave phase speed equals the wind speed,

$$\sigma_0 E_P(\sigma_0) \simeq 240(\varrho_a/\varrho_\omega)^2(u_*^4/g^2) \qquad (2.2)$$

where ϱ_a/ϱ_w is the ratio of the air density to the water density; u_*^2 is the kinematic shear stress in the air flow; g is the gravitational acceleration. The drag coefficient C_D relates u_* to the mean wind speed U such that

$$C_D = (u_*/U)^2.$$

Taking $C_D = 1.2 \times 10^{-3}$ (Miyake et al., 1970) and $\varrho_a/\varrho_w = 1.2 \times 10^{-3}$, we find that

$$\sigma_0 E_P(\sigma_0) \simeq 5 \times 10^{-10}(U^4/g^2).$$

On the other hand, the observations of Barnett & Wilkerson (1967) suggest that

$$\sigma_0 E_P(\sigma_0) \simeq 10^{-7}(U^4/g^2). \qquad (2.3)$$

That is, the constant of proportionality in (2.2) is about 200 times smaller than the observed value.

The magnitude of E_P decreases away from the resonance peak σ_0, and the observations of Barnett & Wilkerson are not inconsistent with the relation

$$\sigma E_P(\sigma)\, g^2/U^4 = 10^{-7} \times \begin{cases} (\sigma/\sigma_0)^3, & \sigma < \sigma_0 \\ (\sigma/\sigma_0)^{-3}, & \sigma > \sigma_0 \end{cases}. \qquad (2.4)$$

We note that, because the resonance term E_P acts only to instigate the growth of a component, its detailed behaviour should not affect greatly the later development of the spectral density.

Observations of wave growth by Snyder & Cox (1966) and Barnett & Wilkerson (1967) and direct measurements of the pressure at the sea surface (Dobson, 1971) suggest that the input of energy to a component by feedback from wave-induced fluctuations in the air is modelled well by

$$E_F(\sigma, t) = (\varrho_a/\varrho_\omega)(U/C-1)H(U/C-1)\phi(\sigma, t) \qquad (2.5)$$

where C is the wave phase speed and $H(x)$ is the Heaviside unit step function. The remarkably simple law (2.5) probably is valid only in the neighbourhood of the peak in the spectral density. In particular, the input rate to high frequency waves should not be as great.

Longuet-Higgins (1969) has argued that the fractional loss of total wave energy per mean wave cycle is essentially constant. Thus, one might attempt to model the term E_B in (2.1) by setting it equal to a constant of proportionality times a measure of the local wave energy, i.e. E_B = constant $\times \sigma\phi(\sigma, t)$. However, Longuet-Higgins' result follows from the assumption that waves of amplitude greater than a_0 break during a wave cycle; here, a_0 is the amplitude at which the slope of waves at the mean frequency becomes so large that these waves must break. This is applied to the probability density function $p(a)$ of the wave amplitude a to obtain the mean loss of energy per cycle. But the tail of $p(a)$ does not consist necessarily of waves of large slope. Thus, rather than $p(a)$, one perhaps should consider the probability density function of the wave slope.

There is some evidence that wave breaking is not a local (in σ-space) phenomenon. Phillips (1966) uses the radiation stress concepts of Longuet-Higgins & Stewart (1960) to show that a long wave causes a short wave to steepen and so break near its crest. There is an interaction between the waves such that the fluctuation in the energy of the short wave is proportional to the slope of the long wave. Although the long wave is the source of energy which produces the wave breaking, the fractional loss of energy is clearly larger for the short than for the long wave.

A Stokes wave has an expansion of the form (Lamb, 1932)

$$\eta = a \cos (kx - \sigma t) + \tfrac{1}{2}ka^2 \cos 2(kx - \sigma t) + \ldots$$

where x is a space co-ordinate. Thus, a primary wave of wavenumber k has energy at wavenumber $2k$ associated with it. The steeper the primary wave, the greater is the fraction of the total energy contained at the higher wavenumber. A free wave of wavenumber $2k$ has a frequency of $\sqrt{2}\sigma$. Hence, the Stokes harmonic is in phase with the free wave of the same wavenumber every $\sqrt{2}/(\sqrt{2}-1)$ wave cycles. This instantaneous apparent increase in steepness could induce breaking of the short free wave when the slope of the primary Stokes wave is large.

Casual observation of a developing sea suggests that wave breaking produces a smoothing of the sea surface. That is, energy is lost primarily from short waves and not from the dominant waves present. If we assume that a section of the sea surface at some instant can be described by a Fourier cosine series, i.e. $\eta = \sum_{n=0}^{\infty} a_n \cos nx$, say, and that wave breaking tends to minimize the mean square slope of the sea surface while the total energy of the surface is decreased by a fixed amount, then it follows that the fundamental component dominates after breaking. That is, short waves rather than the long waves tend to lose their energy when the overall sea slope is reduced.

From the preceeding discussion it seems that the term in (2.1) which represents the rate of loss of energy caused by wave breaking could be modelled by the relation

$$E_B = \phi(\sigma, t) \int_0^\sigma d\sigma' \, k'^2 \phi(\sigma', t) \, \beta(\sigma, \sigma'). \quad (2.6)$$

Eq. (2.6) states that the fractional decrease in energy per radian of the wave at frequency σ is proportional to the mean square slope of waves longer than that wave, weighted by the dimensionless function $\beta(\sigma, \sigma')$. Although $\beta(\sigma, \sigma')$ is perhaps peaked near $\sigma' = \sigma/\sqrt{2}$ because of interaction between Stokes wave components, the most simple assumption is that

$$\beta(\sigma, \sigma') = \text{constant} \quad (2.7)$$

and this form is adopted below.

Neglecting E_T and putting (2.4)–(2.7) into (2.1), we find that the development of the wave spectral density function is given by

$$\frac{\partial \phi}{\sigma \partial t} = 10^{-7} (U^4/g^2) (\sigma/\sigma_0)^{3 sgn(\sigma_0 - \sigma)} \sigma^{-1} + (\varrho_a/\varrho_\omega)(U/C - 1) H(U/C - 1) \phi(\sigma, t) - \beta \phi(\sigma, t) \int_0^\sigma d\sigma' \, k'^2 \phi(\sigma', t) \quad (2.8)$$

where $\sigma_0 = g/U$ is the frequency of waves with phase speed matching the mean wind speed; $C = g/\sigma$ and $k = \sigma^2/g$ for deep water waves. For convenience, (2.8) is normalized such that

$$\frac{\partial F}{\partial \tau} = 10^{-7} \omega^{3 sgn(1-\omega)} + \left\{ (\varrho_a/\varrho_\omega)(\omega - 1) H(\omega - 1) - \beta \int_0^\omega d\omega' \, \omega'^4 F(\omega', \tau) \right\} \omega F(\omega, \tau), \quad (2.9)$$

where $\omega = \sigma/\sigma_0$, $\tau = \sigma_0 t$ and $F = \sigma_0^5 \phi/g^2$. The initial condition on (2.9) is that

$$F(\omega, 0) = 0 \quad (2.10)$$

i.e. the sea surface is undisturbed initially. From the discussion above, it appears that (2.9) adequately describes the behaviour of a wave component until it reaches its maximum value at least.

3. Solution of equation for spectral density function

Although the analytic solution of the system (2.9)–(2.10) is not obvious, the qualitative behaviour of the spectral density F is readily apparent. The wave energy $F(\omega, \tau)$ initially increases linearly with time due to resonant interaction with turbulent pressure fluctuations. After some time the wave-induced fluctuations in the air produce a feedback effect such that the energy of waves travelling slower than the mean wind (i.e. for $\omega > 1$) increases exponentially. The exponent increases with frequency and hence the high frequency waves grow most rapidly.

The effect of wave breaking is not felt at frequency ω until waves of frequencies less than ω have developed significantly. Thus, the wave energy overshoots its equilibrium value and starts to decrease only when the mean square slope of waves longer than that wave is large enough for wave breaking to compensate for the input of energy from the wind. This decrease in F at frequency ω leads to a decrease in the mean square slope of waves with frequencies less than ω_1, say, where $\omega_1 > \omega$. Hence, $F(\omega_1)$ increases because the effect of wave breaking is decreased. Thus, the energy at a given frequency ω is an oscillatory function of time.

To determine the quantitative behaviour of $F(\omega, \tau)$ the system (2.9)–(2.10) may be integrated numerically. The results in § 4 correspond to the time derivative being approximated by a first order forward difference with a time step $\Delta\tau$ of 20 units and to the integral being approximated by the trapezoidal rule with a frequency interval $\Delta\omega$ of 0.1 units. The density ratio ϱ_a/ϱ_ω is set equal to 1.2×10^{-3}.

The value of the wave breaking parameter β is chosen such that the order of magnitude of the spectral density as it approaches equilibrium is comparable with the observed equilibrium density (Phillips, 1966), viz.

$$F_{\text{equ}}(\omega) = 1.17 \times 10^{-2}\omega^{-5}. \qquad (3.1)$$

This criterion places quite a strong restriction on the value of β. As β increases from 0.1 to 0.25, the maximum energy of a component decreases by a factor of at least three. The parameter is taken equal to 0.25.

4. Results and discussion

Fig. 1 shows the behaviour of individual spectral components with time. The energy is seen to oscillate in time about a mean value with decreasing amplitude and frequency. As the wave frequency ω decreases, the amplitude and frequency of the energy fluctuations also decrease.

Barnett & Wilkerson (1967) report that a component rises to a maximum value of between 1.3 and 3 times greater than its equilibrium value. The component then decreases towards its equilibrium level. Thus, the continued oscillation of solutions of (2.9) appears to be unrealistic. However, Barnett & Wilkerson only measure waves for which ω is less than 1.5, and the solutions of (2.9) are much less oscillatory for these waves than for high frequency waves.

Fluctuations of a component have been observed also in wind-water tunnel experiments (Chambers et al., 1970). While the interaction

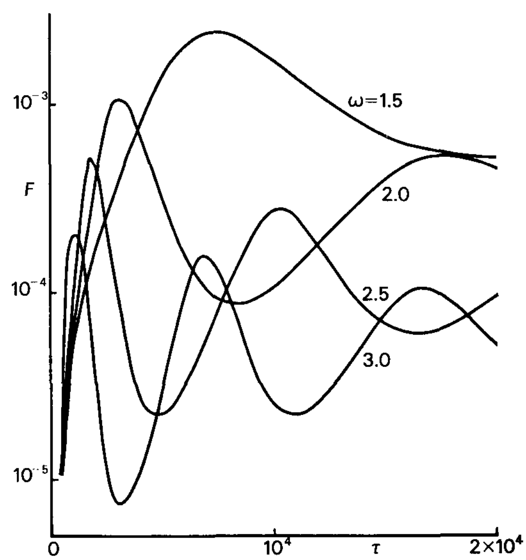

Fig. 1. Development of individual spectral components with time.

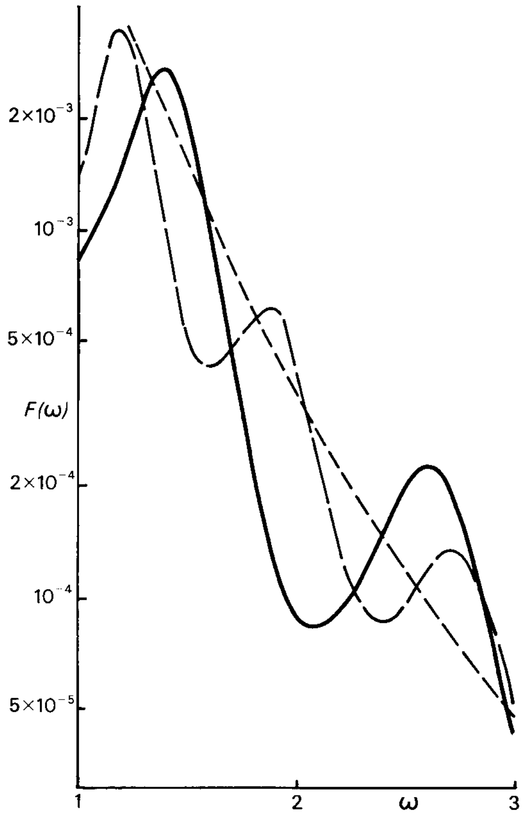

Fig. 2. Development of spectral density function. ——, $\tau = 9\ 000$; ––––, $\tau = 20\ 000$; - - -, F_{equ} from (3.1).

between the wind and waves in such situations is not necessarily equivalent to that in the sea, the qualitative behaviour of a component approaching equilibrium could be reproduced well. The measurements of Chambers et al. indicate that the energy of high frequency components exhibit minima in addition to the initial maxima.

The spectral density $F(\omega)$ obtained from (2.9) at times $\tau = 9\ 000$ and $\tau = 20\ 000$ is shown in Fig. 2. Although the magnitude of the predicted density is comparable with that of the observed equilibrium density (3.1), the predicted density contains maxima and minima which clearly are associated with the fluctuations in the individual spectral components (see Fig. 1). We note that the period of these fluctuations is large (except at very high wave frequencies), and hence a smooth spectral density is not obtained by averaging over several hundred wave cycles.

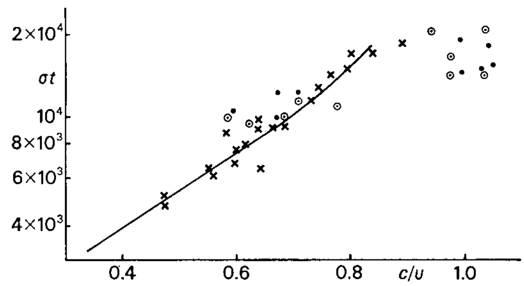

Fig. 3. Time at which wave component occupies peak of spectral density function. Data points: ×, Snyder & Cox (1966); ●, Barnett & Wilkerson (Upwind) (1967); ○, Barnett & Wilkerson (Downwind) (1967).

It is seen that the amplitude of the extrema decreases with time. Nonetheless, if it were a real effect, the predicted appearance of extrema in the spectral density would be measurable. However, such behaviour is not observed in natural sea waves.

The model eq. (2.9) omits the term E_T in (2.1) which describes the transfer of energy throughout the spectrum. It is argued in § 3 that while a component is growing its behaviour is dominated by the input of energy from the wind opposing the loss of energy by wave breaking. Away from the peak in the spectral density, however, the efficacy of the wind-wave interaction decreases and spectral transfer can start to influence the behaviour of a component.

The effect of a spectral transfer process of the form found by Benjamin & Feir (1967) is to remove maxima in the spectral density. Thus, it is expected that spectral transfer tends to reduce, or to eliminate, the extrema in the predicted densities of (2.9). Correspondingly, the amplitude of the oscillations after the initial overshoot in the energy of individual components should be decreased also.

The curve in Fig. 3 represents the time at which a wave component from (2.9) is at the peak of the spectral density. The data points of Snyder & Cox (1966) and Barnett & Wilkerson (1967) in this figure give the observed time at which a component occupies the steep forward face of the spectral density; these times are assumed to be equal.

The predicted time is not very sensitive to the value of β; as the latter varies from 0.1 to 0.25 the former decreases by about 10 %. The curve fits the data well at high frequencies, i.e.

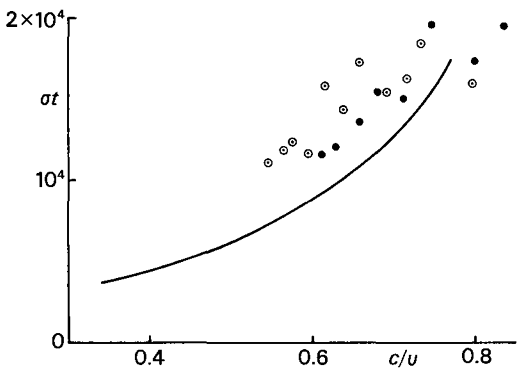

Fig. 4. Time at which wave component reaches its maximum value. Data points from Barnett & Wilkerson (1967): ● Upwind; ⊙ Downwind.

at C/U less than about 0.8. However, as C/U approaches unity the rate of input of energy from the wind, given by (2.5), decreases. Thus, large times are required before such waves occupy the peak of the spectral density. On the other hand, Barnett & Wilkerson observe that waves in the neighbourhood of $C/U = 1$ grow exponentially with time, allowing them to reach the peak of the spectral density more rapidly than predicted by (2.9).

It is difficult to envisage a wind–wave feedback interaction that is efficacious as the wave phase speed approaches the wind speed. However, exponential growth at low frequencies is obtained if energy is input to the sea surface by wind–wave interactions at waves for which C/U is somewhat less than unity, and this energy is subsequently transferred by, say, the Benjamin & Feir (1967) mechanism to waves at lower frequencies. That is, the spectral transfer term E_T in (2.1) is probably important when ω is near unity.

In Fig. 4 the time predicted by (2.9) at which a component reaches its maximum value is compared with the observed times of Barnett & Wilkerson (1967). Barnett & Wilkerson present their results in the form of a graph of frequency f Hz against normalized fetch x/L, where L is the wavelength of a component. The data points of Fig. 4 are obtained by setting $C/U = f_0/f$ and $\sigma t = 4\pi\ x/L$ where f_0 is the frequency corresponding to the Phillips resonance peak.

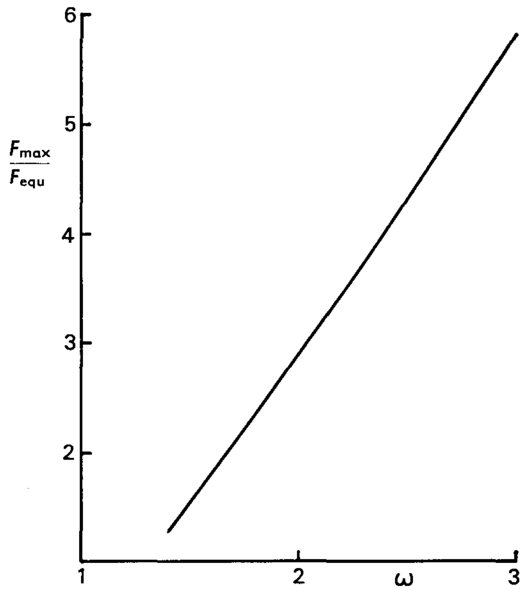

Fig. 5. Variation of ratio $F_{\text{max}}/F_{\text{equ}}$ with frequency ω.

It is seen that the predicted time for a component to reach its maximum is about 25 % less than that observed. However, we note that the results of Barnett & Wilkerson in Fig. 3 give times larger, for $C/U < 0.8$ at least, than those found by Snyder & Cox (1966). Thus, the data points in Fig. 4 might indicate times systematically too large.

Comparison of the curves in Figs. 3 and 4 shows that a wave component reaches its maximum value after it occupies the peak of the spectral density. This time difference increases as the wave frequency decreases.

The predicted ratio $F_{\text{max}}/F_{\text{equ}}$ of the maximum value of a spectral component to its equilibrium value given by (3.1) is shown in Fig. 5. The ratio increases almost linearly with ω. Barnett & Wilkerson state that their data for $F_{\text{max}}/E_{\text{equ}}$ are badly scattered, but have values of order 1.3 to 3. As they consider waves for which ω is less than 1.5 their data are not inconsistent with the predicted results of (2.9).

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О РОСТЕ И ЗАТУХАНИИ ГЛУБОКО-ВОДНОЙ КОМПОНЕНТЫ ВОЛНЫ

Волновая компонента растет экспоненциально со временем благодаря некоторому механизму обратной связи при взаимодействии ее со средним ветром над морем. Первоначально этот рост ограничивается преимущественно опрокидыванием волн. В предположении, что скорость потери энергии для данной частоты пропорциональна среднему квадрату наклона волн меньших частот, выводится уравнение для описания поведения волны со временем. Одним из выводов, следующих из исследования этого уравнения, является то, что растущая волна превосходит свой уровень энергии при конечном равновесии — эффект, который случается наблюдать для морских волн.