Stability of a two-layer fluid model to nongeostrophic disturbances

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ABSTRACT

Sufficient conditions for stability of a linear inviscid two-layer model to nongeostrophic disturbances are derived. A bound on the growth rate of an unstable disturbance is also found. When the Rossby number $R_0 \ll 1$, these conditions reduce to the results established by Pedlosky (1964) within the confines of quasi-geostrophic theory. The present analysis provides a theoretical framework for further investigations of fluid systems characterized by strong concentrations of horizontal momentum, such that $R_0 \leqslant 0(1)$. Application to atmospheric jet streams and the Gulf Stream, for example, is suggested by observationally established characteristics of these flow regimes.

1. Introduction

The stability properties of oceanic and atmospheric flows to nongeostrophic disturbances has attracted increasing attention of late. Much of this interest is prompted by the fact that significant deviations from geostrophy are an observable feature of atmospheric and oceanic current systems. Moreover, it is becoming increasingly clear that the transports of mass, momentum and energy in the atmosphere and oceans by nongeostrophic flows, particularly gravity waves, should not necessarily be neglected in comparison with geostrophic transports. It has also been noted recently that energy dissipation in the free atmosphere is the same order of magnitude as dissipation near ground level. Current evidence indicates that this dissipation is the end product in the chain of events beginning with the unstable growth of gravity-type disturbances on an initially laminar flow (Kelvin–Helmholtz instability), followed by a secondary instability in which these perturbations do not grow indefinitely but ultimately break down into smaller scales of turbulent motion.

Although there are other sources for nongeostrophic perturbations, such as waves induced by flow over bottom topography, a significant contribution to the spectrum of these motions is thought to result from various inertial and convective instability mechanisms. As pointed out by Bretherton (1971), however, the evidence to associate clear air turbulence, for example, with Kelvin–Helmholtz instability is circumstantial, as yet not conclusively demonstrated by aircraft measurements. The same difficulty arises in the interpretation of oceanic microstructure, because various mechanisms responsible for the generation and maintenance of waves and localized patches of turbulence have not been clearly delineated.

The present investigation considers the role of various physical processes in relation to the stability of hydrostatic perturbations in a rotating fluid medium. Although the characteristic spatial—and time—scales of these motions are larger than those that generally characterize intermittent turbulence, the association of these motions with instability mechanisms suffers from the same conjectural viewpoint noted above. Ageostrophic baroclinic and barotropic stability problems have been studied (e.g., Stone, 1966; Derome & Dolph, 1970; Sela & Jacobs, 1971; Blumen, 1971*a*). However, the problem of the stability of a zonal flow with *both* vertical and horizontal shear to ageostrophic perturbations has apparently not received attention, with the exception of Orlanski's (1968) study of frontal instability. Attempts by

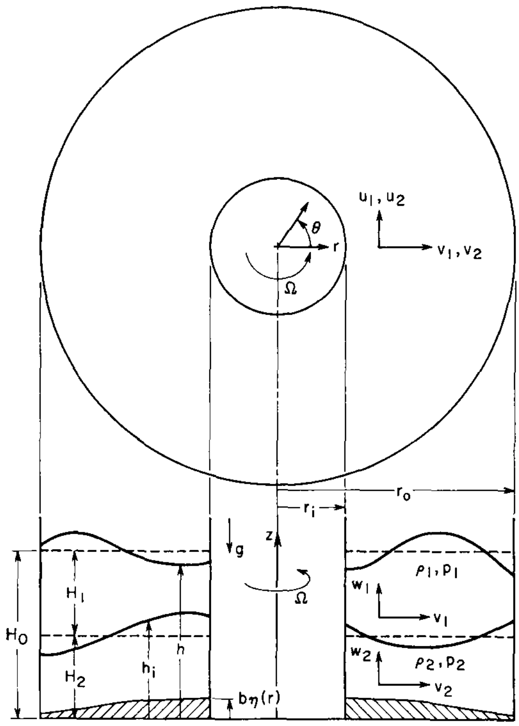

Fig. 1. Schematic representation of the two-layer model.

the author to find sufficient conditions for stability of a continuously varying zonal flow in a stratified fluid have thus far not yielded results as useful as those found by Pedlosky (1964) for a quasi-geostrophic model. Yet Pedlosky's two-layer model can be extended to take nongeostrophic effects into account. Although this model certainly over simplifies the physical structure of atmospheric and oceanic flows, it has proved useful as a starting point in the formulation of many problems of geophysical interst.

In essence, a theoretical framework, for anticipated further investigations of the mixed barotropic–baroclinic stability problem, is provided here by the establishment of sufficient conditions for stability of a two-layer model together with the determination of a bound on the growth rate of an unstable disturbance. The relative importance of higher order terms in a Rossby number expansion are displayed and physical interpretations of the mechanisms responsible for maintaining stability are considered.

Since predictions of linear quasi-geostrophic Ekman-suction theory have recently been verified reasonably well by Hart's (1972) laboratory study, the present study is formulated in cylindrical geometry for possible application to rotating annulus experiments. However, no attempt is made to incorporate thin viscous boundary layers, interfacial tension nor imposed temperature differences on the fluid system.

2. Model

The fluid model under consideration is shown in Fig. 1. Some variables and parameters appearing in the text and in the Appendix are defined in Table 1.

The horizontal scale of allowable disturbance modes is considered to be large compared to the fluid depth. As a consequence, it is assumed that the fluid is in hydrostatic balance and the motion is independent of depth in each layer separated by an interface at height $z = h_i$. The upper surface at $z = h$ is free while the bottom boundary is $z = b\eta(r)$, where b = constant. It is further assumed that the density field satisfies

$$0 \leqslant \frac{\varrho_2 - \varrho_1}{\varrho_2} \ll 1.$$

In layer 1, the equations of motion relative to a coordinate system rotating at constant angular velocity Ω satisfy.

$$\frac{\partial u_1}{\partial t} + \frac{u_1}{r}\frac{\partial u_1}{\partial \theta} + v_1\frac{\partial u_1}{\partial r} + \frac{u_1 v_1}{r} + 2\Omega v_1 = -\frac{1}{r}\frac{\partial P_1}{\partial \theta}, \tag{1}$$

$$\frac{\partial v_1}{\partial t} + \frac{u_1}{r}\frac{\partial v_1}{\partial \theta} + v_1\frac{\partial v_1}{\partial r} - \frac{u_1^2}{r} - 2\Omega u_1 = -\frac{\partial P_1}{\partial r} + \Omega^2 r, \tag{2}$$

$$w_1(h) - w_1(h_i) = -\frac{1}{r}\left(\frac{\partial u_1}{\partial \theta} + \frac{\partial r v_1}{\partial r}\right)(h - h_i). \tag{3}$$

Since the motion is hydrostatic,

$$P_1(r, \theta, z, t) = P_1(r, \theta, h, t) + g(h - z). \tag{4}$$

Table 1. *Symbols not defined in the text*

Coordinates	
t	time
r, θ, z	radial, azimuthal and vertical coordinate axes
Physical variables ($k = 1, 2$)	
u, v, w	velocity components in the direction of (θ, r, z)
p	pressure
ϱ_k	density
$P_k = p/\varrho_k$	
$\omega_1 = \zeta_1 - q_1 \dfrac{(h_1 - h_2)}{H_0}$	perturbation potential vorticity in layer 1
$\omega_2 = \zeta_2 - q_2 \dfrac{h_2}{H_0}$	perturbation potential vorticity in layer 2
$\zeta_k = \dfrac{1}{r}\left(\dfrac{\partial}{\partial r} r u_k - \dfrac{\partial}{\partial \theta} v_k\right)$	perturbation relative vorticity
$q_k = \dfrac{2\Omega + \dfrac{1}{r}\dfrac{d}{dr} r\bar{u}_k}{\Phi_k}$	basic-state potential vorticity, where Φ_k is defined in (21)
Parameters	
g	gravity
$g^* = \dfrac{\varrho_2 - \varrho_1}{\varrho_2} g \ll g$	reduced gravity
$h_e = \dfrac{\Omega^2}{2g}\left(r^2 - \dfrac{r_0^2 + r_i^2}{2}\right)$	
r_i, r_0	radii of the inner and outer cylinders

At the free surface,

$$w_1(h) = \left(\frac{\partial}{\partial t} + \frac{u_1}{r}\frac{\partial}{\partial \theta} + v_1 \frac{\partial}{\partial r}\right) h. \quad (5)$$

Similarly, at the interface

$$w_1(h_i) = \left(\frac{\partial}{\partial t} + \frac{u_1}{r}\frac{\partial}{\partial \theta} + v_1 \frac{\partial}{\partial r}\right) h_i. \quad (6)$$

In view of (4), (5) and (6), the equations for layer 1 become

$$\frac{\partial}{\partial t} u_1 + \frac{u_1}{r}\frac{\partial u_1}{\partial \theta} + v_1 \frac{\partial u_1}{\partial r} + \frac{u_1 v_1}{r} + 2\Omega v_1 = -\frac{1}{r}\frac{\partial}{\partial \theta} gh, \quad (7)$$

$$\frac{\partial v_1}{\partial t} + \frac{u_1}{r}\frac{\partial v_1}{\partial \theta} + v_1 \frac{\partial v_1}{\partial r} - \frac{u_1^2}{r} - 2\Omega u_1 = -\frac{\partial}{\partial r} gh + \Omega^2 r, \quad (8)$$

$$\frac{\partial}{\partial t}(h - h_i) + \frac{1}{r}\left(\frac{\partial}{\partial \theta} u_1 (h - h_i) + \frac{\partial}{\partial r} r v_1 (h - h_i)\right) = 0. \quad (9)$$

Identical arguments applied to layer 2 lead to

$$\frac{\partial u_2}{\partial t} + \frac{u_2}{r}\frac{\partial u_2}{\partial \theta} + v_2 \frac{\partial u_2}{\partial r} + \frac{u_2 v_2}{r} + 2\Omega v_2 = -\frac{1}{r}\frac{\partial}{\partial \theta}\left(g^* h_i + \frac{\varrho_1}{\varrho_2} gh\right), \quad (10)$$

$$\frac{\partial v_2}{\partial t} + \frac{u_2}{r}\frac{\partial v_2}{\partial \theta} + v_2 \frac{\partial v_2}{\partial r} - \frac{u_2^2}{r} - 2\Omega u_2 = -\frac{\partial}{\partial r}\left(g^* h_i + \frac{\varrho_1}{\varrho_2} gh\right) + \Omega^2 r, \quad (11)$$

$$\frac{\partial}{\partial t}(h_i - b\eta) + \frac{1}{r}\left[\frac{\partial}{\partial \theta} u_2 (h_i - b\eta) + \frac{\partial}{\partial r} r v_2 (h_i - b\eta)\right] = 0. \quad (12)$$

The following representation is introduced for the purpose of linearization:

$$h/H_0 = 1 + H_0^{-1}[\bar{h}_1(r) + h_e(r) + h_1'(r, \theta, t)] \quad (13)$$

$$h_i/H_0 = H_0^{-1}[H_2 + \bar{h}_2(r) + h_e(r) + h_2'(r, \theta, t)], \quad (14)$$

where H_0 and H_2 represent the constant undisturbed height of the free surface and interface respectively; $\bar{h}_1(r)$ and $\bar{h}_2(r)$ represent the slopes of these surfaces necessary to sustain a steady balanced relative motion; h_e, defined in Table 1, is the equilibrium slope of the isopycnic surfaces due to rotation of the fluid; and finally $h_1' \ll H_1$, $h_2' \ll H_2$ are deviations due to the nonsteady motion.

In each layer the basic flow $\bar{u}_k$, in gradient balance and independent of height, is given by

$$\bar{u}_k^2/r + 2\Omega \bar{u}_k - d\bar{\varphi}_k/dr = 0, \quad k = 1, 2. \quad (15)$$

Consequently, the vertical shear is concentrated discontinuously at the interface. We seek conditions for stability of this flow to disturbances governed by the linear system ($k = 1,2$):

$$\frac{\partial}{\partial t}u_k + \bar{u}_k \frac{1}{r}\frac{\partial}{\partial \theta}u_k + v_k \frac{1}{r}\frac{d}{dr}\bar{u}_k r + 2\Omega v_k = -\frac{1}{r}\frac{\partial \phi_k}{\partial \theta}, \tag{16}$$

$$\frac{\partial v_k}{\partial t} + \bar{u}_k \frac{1}{r}\frac{\partial}{\partial \theta}v_k - 2\bar{u}_k u_k/r - 2\Omega u_k = -\frac{\partial}{\partial r}\phi_k, \tag{17}$$

$$\frac{\partial}{\partial t}(h_1 - h_2) + \bar{u}_1 \frac{1}{r}\frac{\partial}{\partial \theta}(h_1 - h_2) + \frac{H_0}{r}\left(\frac{\partial}{\partial r} r v_1 \Phi_1 + \frac{\partial}{\partial \theta} u_1 \Phi_1\right) = 0, \tag{18}$$

$$\frac{\partial}{\partial t}h_2 + \bar{u}_2 \frac{1}{r}\frac{\partial}{\partial \theta}h_2 + \frac{H_0}{r}\left(\frac{\partial}{\partial r} r v_2 \Phi_2 + \frac{\partial}{\partial \theta} u_2 \Phi_2\right) = 0, \tag{19}$$

where unprimed variables u_k, v_k and h_k now denote perturbations and

$$\phi_1 = g h_1, \tag{20 a}$$

$$\phi_2 = g^* h_2 + g \frac{\varrho_1}{\varrho_1} h_1, \tag{20 b}$$

$$\Phi_1 = H_0^{-1}[H_1 + (\bar{h}_1 - \bar{h}_2)], \tag{21 a}$$

$$\Phi_2 = H_0^{-1}[H_2 + \bar{h}_2 + h_e - b\eta]. \tag{21 b}$$

The results follow. However, the interested reader will find an outline of the development leading to the proof in the Appendix.

3. Results

The results are displayed most conveniently in nondimensional form. Set

$$\begin{aligned} r &= R r_*, \; t = R t_*/U, \\ u_k, v_k &= U(u_{*k}, v_{*k}), \; k = 1, 2 \\ g(\bar{h}_1 + h_1) &= 2\Omega U R(\bar{h}_{*1} + h_{*1}), \\ g^*(\bar{h}_2 + h_2) &= 2\Omega U R(\bar{h}_{*2} + h_{*2}), \end{aligned} \tag{22}$$

where R denotes the annulus width, U is a characteristic amplitude of the basic flow and the subscripted asterisk, which will be dropped hereafter, indicates a nondimensional quantity. Nondimensional parameters, appearing in the analysis, are:

$$\begin{array}{lll} R_0 = U/2\Omega R & \text{Rossby number} & \\ \Gamma_k = 4\Omega^2 R^2/g^* H_k & \text{Internal rotational Froude number} & \\ \gamma_k = 4\Omega^2 R^2/g H_k & \text{External rotational Froude number} & \\ F_k = R_0^2 \Gamma_k = U^2/g^* H_k & \text{Froude number} & (23) \end{array}$$

As before, k is an index symbol.

Basic state variables satisfy

$$R_0 \bar{u}_k^2/r + \bar{u}_k - d\bar{\phi}_k/dr = 0, \quad k = 1, 2, \tag{24}$$

and the potential vorticity of the basic flow is given by

$$q_k = \frac{1 + R_0 \frac{1}{r}\frac{d}{dr}(r\bar{u}_k)}{\Phi_k}, \quad k = 1, 2 \tag{25}$$

where (21 a, b) may be expressed as

$$\Phi_1 = H_0^{-1} H_1 (1 + R_0(\gamma_1 \bar{h}_1 - \Gamma_1 \bar{h}_2)] = H_0^{-1} H_1 \bar{\Phi}_1, \tag{26}$$

$$\Phi_2 = H_0^{-1} H_2 \left[1 + R_0 \Gamma_2 \bar{h}_2 + \tfrac{1}{8}\gamma_2 \left(r^2 - \frac{r_0^2 + r_i^2}{2}\right) - \frac{b\eta}{H_2}\right] = H_0^{-1} H_2 \bar{\Phi}_2. \tag{27}$$

The development in the Appendix provides the expression

$$\begin{aligned} \frac{\partial}{\partial t}\frac{1}{2}\int_0^{2\pi}\int_{r_i}^{r_0} \Big\{ &\Phi_1[v_1^2 + (1 - 2F_1 \bar{u}_1^2/\bar{\Phi}_1) u_1^2] \\ &+ \Phi_2[v_2^2 + (1 - 2F_2 \bar{u}_2^2/\bar{\Phi}_2) u_2^2] \\ &+ \frac{H_1}{H_0}\gamma_1 (h_1 + R_0 \bar{u}_1 u_1)^2 \\ &+ \frac{1}{2}\frac{H_2}{H_0}\Gamma_2[(h_2 - 2R_0 \bar{u}_1 u_1)^2 \\ &+ (h_2 + 2R_0 \bar{u}_2 u_2)^2] \\ &+ R_0\left[\frac{\bar{u}_1}{dq_1/dr}\omega_1^2 + \frac{\bar{u}_2}{dq_2/dr}\omega_2^2\right]\Big\} \\ &\times r\, dr\, d\theta = 0, \end{aligned} \tag{28}$$

where the nondimensional quantities are defined above.

If all the coefficients in (28) are positive the perturbation energy may increase initially at the expense of the remaining terms but it must remain bounded for all time. As a consequence, stability of the flow (24) is assured if

$$R_0\bar{u}_1/(dq_1/dr) > 0, \tag{29 a}$$

$$R_0\bar{u}_2/(dq_2/dr) > 0, \tag{29 b}$$

$$F_1\bar{u}_1^2/\bar{\Phi}_1 = F_1\bar{u}_1^2/[1 + R_0(\gamma_1\bar{h}_1 - \Gamma_1\bar{h}_2)] \leq \tfrac{1}{2}, \tag{29 c}$$

$$F_2\bar{u}_2^2/\bar{\Phi}_2 = F_2\bar{u}_2^2 \Big/ \left[1 + R_0 + \Gamma_2\bar{h}_2 + \tfrac{1}{8}\gamma_2\left(r^2 - \frac{r_0^2 + r_i^2}{2}\right) - b\eta/H_2\right] \leq \tfrac{1}{2}. \tag{29 d}$$

Note that the derivation leading to (29) requires at least one of $\bar{u}_k \neq 0$.

Before we consider the range of parameters encompassing some realistic geophysical fluid flows, it is instructive to consider some particular cases covered by these criteria (29). In quasi-geostrophic flow $R_0 \ll 1$, $\Gamma_k = 0(1)$, $\gamma_k \ll 1$, $F_k \ll 1$. The stability criteria reduce to Pedloskys (1964) results, without the β term present,

$$\bar{u}_1/(dq_1/dr) > 0, \tag{30 a}$$

$$\bar{u}_2/(dq_2/dr) > 0, \tag{30 b}$$

where

$$dq_1/dr = d\left(\frac{1}{r}\frac{d}{dr}r\bar{u}_1\right) \Big/ dr - \Gamma_1(\bar{u}_1 - \bar{u}_2), \tag{31 a}$$

$$dq_2/dr = d\left(\frac{1}{r}\frac{d}{dr}r\bar{u}_2\right) \Big/ dr - \Gamma_2(\bar{u}_2 - \bar{u}_1) + \frac{b}{R_0 H_2}\frac{d\eta}{dr}. \tag{31 b}$$

The absence of gravity waves as allowable disturbance modes is made evident by the omission of (29c, d) in the present case. (These additional terms will be considered later.)

The reader is referred to Pedlosky's paper for further discussion of the quasi-geostrophic model.

In general, four principal features affect the stability of the basic flow: horizontal and vertical shear of $\bar{u}$, the stretching of vortex tubes due to the non-uniform slope of the interface and free surface and the relative magnitude of the ratio of $|\bar{u}_k|$ to the phase speed of internal gravity waves that may propagate in each fluid layer.

The form of the stability condition expressed by (29a, b) or (30a, b) is a more general representation of the classical Rayleigh-Fjørtoft inflection-point theorem for the stability of homogeneous, rectilinear, two-dimensional shear flow

$$\bar{u}/(d^2\bar{u}/dy^2) > 0 \tag{32}$$

where y is the cross-stream coordinate. The physical interpretation of the criterion in terms of the momentum transfer by the Reynolds stress is discussed, for example, by Taylor (1915) and Drazin & Howard (1966). It is well-known that the same consideration applies when this interpretation is applied to the potential vorticity, which takes into account the first three features of the basic flow noted above. However, two limiting cases are worthy of emphasis here. In homogeneous shear flow barotropically unstable disturbances may grow only at the expense of basic flow kinetic energy, when either (32) or in our cylindrical geometry

$$\bar{u} \Big/ \frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}r\bar{u}\right) > 0 \tag{33}$$

is not satisfied. On the other hand suppose $\bar{u}_1$ and $\bar{u}_2$ are constants, independent of r, γ_1, $\gamma_2 \ll 1$, $b = 0$ and curvature of the flow is ignored. Then (29a, b) reduce to

$$\bar{u}_1/(\bar{u}_2 - \bar{u}_1) > 0, \tag{34 a}$$

$$\bar{u}_2/(\bar{u}_1 - \bar{u}_2) > 0. \tag{34 b}$$

These conditions, expressed by (34), cannot be satisfied simultaneously by any preassigned $\bar{u}_1$ and $\bar{u}_2$. Thus a necessary condition for instability is met which is, in fact, a sufficient one (e.g., Huppert, 1968). This case is usually referred to as baroclinic instability in quasi-geostrophic theory or Kelvin-Helmholtz instability, when gravity-waves are allowable disturbance modes. In either case, the store of available potential energy in the basic flow maintains the unstable disturbances.

The conditions expressed by (29c, d) show that stability is only assured, assuming (29a, b) are satisfied, if $|\bar{u}_k| \leqslant (F_k^{-1}\bar{\Phi}_k/2)^{\frac{1}{2}}$.

However, if either $\bar{u}_2$ or $\bar{u}_1$ is zero, ($\bar{\phi}_2$ or $\bar{\phi}_1 = 0$), then inspection of (A 3) shows that there is stability if

$$R_0\bar{u}_1/(dq_1/dr) > 0, \tag{35 a}$$

$$F_1\bar{u}_1^2/\bar{\Phi}_1 \leqslant 1 \tag{35 b}$$

or

$$R_0\bar{u}_2/(dq_2/dr) > 0, \tag{36 a}$$

$$F_2\bar{u}_2^2/\bar{\Phi}_2 \leqslant 1. \tag{36 b}$$

Moreover, analyses of stably stratified layered models usually establish (35b) or (36b) as an inertial stability criterion (e.g., Stommel, 1965, pp. 129–130) or as a condition that must be met to avoid the possible occurrence of an internal hydraulic jump (e.g., Long, 1954). It is not clear, from physical considerations, why relatively arbitrary depth variations, $\bar{h}_1(r)$ and $\bar{h}_2(r)$, in both layers lead to the more stringent conditions expressed by (29c, d). It is noted however that a reduction in the critical internal Froude number does occur in Long's analysis when depth variations due to bottom topography are permitted.

It seems that little more can be concluded about the stability characteristics of the present model. Computations of the unstable eigenvalues for particular flows must be accomplished before the role of both the nongeostrophic barotropic and baroclinic mechanisms becomes clarified, particularly in relation to the relative magnitude of $F_k\bar{u}_k/\bar{\Phi}_k$.

In spatially continuous flows of a stratified fluid described by, say, a Boussinesq model, the gradient Richardson number

$$J = N^2/(d\bar{u}/dz)^2, \tag{37}$$

where $N^2 = -gd\ln\bar{\varrho}/dz$, rather than $F_k\bar{u}_k/\bar{\Phi}_k$ appears as a relevant parameter in stability analyses. At present, one may only speculate whether or not the criteria for stability may be expressed simply, as in (29), by a restriction on the vorticity distribution together with a critical value of J or if stability conditions depending only on basic-state parameters can, in fact, be established. Computations of the instability characteristics of flows in a nonrotating Boussinesq model, introduced previously (Blumen, 1971*b*) are currently underway with these questions in mind. Extension to the present model and, ultimately, to continuous flows and density stratification is anticipated.

Characteristics of the mid-latitude jet stream and the Gulf Stream, between the Florida straits and Cape Hatteras, have been presented by Newton (1959) and Stommel (1965) respectively. Typical parameter values are similar in both systems, with

$$R_0 \sim 0.4 - 1.0 +$$

$$\Gamma_1, \Gamma_2 = 0(1)$$

$$\gamma_1, \gamma_2 < 1$$

where the higher values are usually more characteristic of the jet stream. Nonetheless, Stommel has indicated that internal Froude numbers close to "critical" values ($F_1, F_2 = 1$) may be attained in the Gulf Stream. It is in flow regimes with these characteristics where the present results should be most applicable.

In laboratory prototypes of atmospheric and oceanic flows as exhibited by Hart (1972), for example, some of the external parameters are constrained to the extent that the free surface should not intersect the interface nor should the interface intersect the bottom boundary. Some limiting values may be estimated from an inspection of (26) and (27).

Finally, a bound on the growth rate follows directly from eq. (A 1). We assume

$$\partial/\partial t \to 2\sigma_i$$

$$\partial/\partial\theta \to im \tag{38}$$

where σ_i denotes the growth rate and m is the azimuthal wave number, a positive integer. We make use of the inequality

$$2|z_1||z_2| \leqslant |z_1|^2 + |z_2|^2 \tag{39}$$

and obtain, in nondimensional terms,

$$\sigma_i \int_0^{2\pi}\int_{r_i}^{r_0} \{\Phi_1[u_1^2+v_1^2) + \Phi_2(u_2^2+v_2^2) + \gamma_0 h_1^2 + \Gamma_0 h_2^2\} \times r\,dr\,d\theta \leqslant \int_0^{2\pi}\int_{r_i}^{r_0} \left\{ \left|\frac{d\bar{u}_1}{dr} - \frac{\bar{u}_1}{r}\right|_{\max} \times \Phi_1\left(\frac{u_1^2+v_1^2}{2}\right) + \left|\frac{d\bar{u}_2}{dr} - \frac{\bar{u}_2}{r}\right|_{\max} \times \Phi_2\left(\frac{u_2^2+v_2^2}{2}\right) + \Gamma_0\left|\frac{\bar{u}_2-\bar{u}_1}{2}\right|_{\max} \times \frac{m}{r}(h_1^2+h_2^2)\right\} r\,dr\,d\theta, \tag{40}$$

where γ_0 and Γ_0 are defined in (23) by $k=0$. Consequently,

$$|\sigma_i| \leqslant \frac{1}{2}\left\{\left|\frac{d\bar{u}_1}{dr}-\frac{\bar{u}_1}{r}\right|_{\max}+\left|\frac{d\bar{u}_2}{dr}-\frac{\bar{u}_2}{r}\right|_{\max} + (\Gamma_0/\gamma_0)\,|\bar{u}_2-\bar{u}_1|_{\max} m/r\right\}. \tag{41}$$

In rectilinear flow m/r is replaced by α, the x-wave number, and the r^{-1} terms do not appear.

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Appendix

The method of proof is a straight-forward extension of the Drazin-Howard (1966) derivation of the Rayleigh-Fjørtoft inflection point theorem for homogeneous shear flow. Following the established procedure with the perturbation equations (16–19) while making use of the assumption $g^* < g$, we obtain the energy equation

$$\frac{\partial}{\partial t}\int_0^{2\pi}\int_{r_i}^{r_0}\left\{\Phi_1\frac{u_1^2+v_1^2}{2}+\Phi_2\frac{u_2^2+v_2^2}{2}+\frac{g}{H_0}\frac{h_1^2}{2}+\frac{g^*}{H_0}\frac{h_2^2}{2}\right\}r\,dr\,d\theta$$

$$= -\int_0^{2\pi}\int_{r_i}^{r_0}\left\{\Phi_1 u_1 v_1\left(\frac{d\bar{u}_1}{dr}-\frac{\bar{u}_1}{r}\right)+\Phi_2 u_2 v_2\left(\frac{d\bar{u}_2}{dr}-\frac{\bar{u}_2}{r}\right)+(\bar{u}_2-\bar{u}_1)\,g\frac{h_1}{H_0}\frac{1}{r}\frac{\partial h_2}{\partial\theta}\right\}r\,dr\,d\theta, \tag{A 1}$$

as well as the desired form of the potential vorticity equation

$$\frac{\partial}{\partial t}\int_0^{2\pi}\int_{r_i}^{r_0}\left(\frac{\bar{u}_1}{dq_1/dr}\frac{\omega_1^2}{2}+\frac{\bar{u}_2}{dq_2/dr}\frac{\omega_2^2}{2}+\bar{u}_1 u_1\frac{(h_1-h_2)}{H_0}+\bar{u}_2 u_2\frac{h_2}{H_0}\right)r\,dr\,d\theta = \int_0^{2\pi}\int_{r_i}^{r_0}\left(\Phi_1 u_1 v_1\times\left(\frac{d\bar{u}_1}{dr}-\frac{\bar{u}_1}{r}\right)+\Phi_2 u_2 v_2\left(\frac{d\bar{u}_2}{dr}-\frac{\bar{u}_2}{r}\right)+(\bar{u}_2-\bar{u}_1)\,g\frac{h_1}{H_0}\frac{1}{r}\frac{\partial h_2}{\partial\theta}\right)r\,dr\,d\theta, \tag{A 2}$$

where variables are defined in Table 1.

Note that

$$\frac{1}{2H_0}\left\{g[h_1^2+2\bar{u}_1u_1h_1]+g^*[h_2^2+2(\bar{u}_2u_2-\bar{u}_1u_1)]\right.$$

$$\approx\frac{1}{2H_0}\left\{g\left(h_1+\frac{\bar{u}_1u_1}{g}\right)^2+\tfrac{1}{2}g^*\left[\left(h_2-2\frac{\bar{u}_1u_1}{g^*}\right)^2+\left(h_2+2\frac{\bar{u}_2u_2}{g^*}\right)^2\right]\right\}-\frac{1}{g^*H_0}(\bar{u}_1^2u_1^2+\bar{u}_2^2u_2^2). \tag{A 3}$$

Then the addition of (A 1) and (A 2), noting (A 3), yields the integral relation (28), which is expressed in nondimensional form.

REFERENCES

Blumen, W. 1971*a*. On the stability of a barotropic shear flow to nongeostrophic disturbances. *Tellus 23*, 295–301.

Blumen, W. 1971*b*. Hydrostatic neutral waves in a parallel shear flow of a stratified fluid. *J. Atmos. Sci. 28*, 340–344.

Bretherton, F. P. 1971. GARP and clear air turbulence. *Bull. Amer. Meteor. Soc. 52*, 17–21.

Derome, J. & Dolph, C. L. 1970. Three dimensional non-geostrophic disturbances in a baroclinic zonal flow. *Geophys. Fluid Dyn. 1*, 91–122.

Drazin, P. G. & Howard, L. N. 1966. Hydrodynamic stability of parallel flow of inviscid fluid. *Advanc. Appl. Mech. 9*, Academic Press, New York.

Hart, J. E. 1972. A laboratory study of baroclinic instability. *Geophys. Fluid. Dyn. 3*, 181–209.

Huppert, H. E. 1968. On Kelvin–Helmholtz instability in a rotating fluid. *J. Fluid Mech. 33*, 353–359.

Long, R. R. 1954. Some aspects of the flow of stratified fluids. II. Experiments with a two-fluid system. *Tellus 6*, 97–115.

Newton, C. W. 1959. Synoptic comparisons of jet stream and Gulf Stream systems. In *The atmosphere and the sea in motion* (ed. B. Bolin), 509. pp. Rockefeller Inst. Press, New York.

Orlanski, I. 1968. Instability of frontal waves. *J. Atmos. Sci. 25*, 178–200.

Pedlosky, J. 1964. The stability of currents in the atmosphere and the ocean. Part I. *J. Atmos. Sci. 21*, 201–219.
Sela, J. & Jacobs, S. J. 1971. Ageostrophic effects on Gulf Stream instability. *J. Atmos. Sci. 28*, 962–967.
Stommel, H. 1965. *The Gulf Stream*. Univ. of California Press, Berkeley and Los Angeles, Cambridge Univ. Press, London. 2nd ed., 248 pp.
Stone, P. H. 1966. On non-geostrophic baroclinic stability. *J. Atmos. Sci. 23*, 390–400.
Taylor, G. I. 1915. Eddy motion in the atmosphere. *Phil. Trans. Roy. Soc. A 215*, 1–26.

УСТОЙЧИВОСТЬ ДВУСЛОЙНОЙ МОДЕЛИ ЖИДКОСТИ ПО ОТНОШЕНИЮ К НЕГЕОСТРОФИЧЕСКИМ ВОЗМУЩЕНИЯМ

Выводятся достаточные условия устойчивости линейной невязкой двуслойной модели относительно негеострофических возмущений. Находится также предельное значение скорости роста неустойчивых возмущений. Когда число Россби $\mathrm{Ro} \ll 1$, эти условия сводятся к результатам, установленным Педлоски (1964) в рамках квазигеострофической теории. Настоящий анализ обеспечивает теоретическую основу для дальнейших исследований потоков, характеризуемых сильной концентрацией горизонтального импульса, когда $\mathrm{Ro} \leqslant 0(1)$. Наблюдаемые характеристики атмосферных струйных течений и Гольфстрима наводят на мысль о применимости к ним излагаемой теории.