The effect of finite vertical Ekman number on the coastal boundary layers of a lake

By GERALD S. JANOWITZ, *Division of Fluid, Thermal, and Aerospace Sciences, Case Western Reserve University, Cleveland, Ohio 44106, USA*

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ABSTRACT

The linear hydrostatic coastal boundary layers in a shallow lake were previously investigated by the author under the assumption of infinitesimal vertical Ekman number; overlapping layers of thickness $E_H^{\frac{1}{3}}$ and $E_H^{\frac{1}{2}}/E_v^{\frac{1}{2}}$ were found (Janowitz, 1970). In this work the boundary layers are investigated for $E_v = 0(1)$. The $E_H^{\frac{1}{2}}/E_v^{\frac{1}{2}}$ layer is distinguishable, though thinner. The $E_H^{\frac{1}{3}}$ layer is also present in a modified form; each vertical component present has a different horizontal length scale, though each is $0(E_H^{\frac{1}{3}})$. Numerical calculations of boundary layer thicknesses for $E_v = 0.1$, 0.5, 1.0 were performed.

Introduction

Linear vertical shear layers in almost rigidly rotating homogeneous fluids have been known for some years. Stewartson (1957) considered the case of a fluid in a container of depth to width ratio (H/L) of order one with uniform viscosity, density, and infinitesimal Ekman number (E). He found vertical shear layers of (dimensionless) thicknesses $E^{\frac{1}{3}}$ and $E^{\frac{1}{4}}$. Pedlosky (1968) in considering oceanic flows considered a container with $H/L < E^{\frac{1}{3}}$ and with horizontal and vertical Ekman numbers of the same order of magnitude. He found, among other results, that the $E^{\frac{1}{3}}$ layer is replaced by layers of thicknesses $E^{\frac{1}{2}}$ (hydrostatic) and H/L (non-hydrostatic).

Janowitz (1970), in investigating lake flows, considered the parameter range $H/L < E_H^{\frac{1}{3}} < E_v^{\frac{1}{2}} < 1$. Hydrostatic layers of thickness $E_H^{\frac{1}{3}}$ and $E_H^{\frac{1}{2}}/E_v^{\frac{1}{2}}$ were found. This work expands the previous work to the case $E_v = 0$ (1). There are two reasons for considering the finite vertical Ekman number case. First, it completes the theoretical description of linear vertical shear layers in homogeneous rotating fluids over the parameter ranges for E_H, H/L, and E_v that are likely to be of geophysical interest. Secondly, the case of finite vertical Ekman number is indeed encountered in some coastal regions. The discussion of whether a linear homogeneous model is relevant for lake coastal flows is postponed until after the calculations have been presented.

The model

We consider the flow, in a basin of uniform depth H and horizontal dimension L, of a homogeneous fluid acted upon by a steady non-uniform wind stress applied at the free surface. The fluid is in a state of turbulent motion characterized by uniform horizontal and vertical eddy viscosities A_H and A_v. The local Coriolis parameter, assumed constant, is f. The motion is viewed from a cartesian reference frame with the origin at some point on the coast at the free surface, the z' axis is positive upwards, the x' axis horizontal, normal to the shore, and pointing into the lake, and the y' axis horizontal, parallel to the coast, and counterclockwise from the x' axis (Fig. 1). The dimensional (mean) velocities in the (x', y', z') directions are (u', v', w'). The flow is driven by the horizontal wind stress (τ_x', τ_y') applied at the free surface. We non-dimensionalize the variables as follows.

$$(x, y) = (x', y')/L, z = z'/H$$

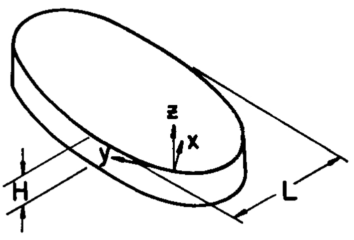

Fig. 1. Geometry of the basin.

$$\begin{aligned} &(u, v) = (u', v')/U,\ w = w'/(UH/L) \\ &p = (p' + \varrho g z')/\varrho U f L \\ &\tau_M = MAX[\tau_x'(x', y'), \tau_y'(x', y')] \\ &U = \tau_M/\varrho \sqrt{A_v f},\ (\tau_x, \tau_y) = (\tau_x', \tau_y')/\tau_M \\ &\varepsilon = U/fL,\ E_H = A_H/fL^2,\ E_v = A_v/fH^2 \end{aligned} \tag{1}$$

Requiring that $\varepsilon \ll 1$, and $H/L < E_H^{\frac{1}{2}} \ll 1$, non-linear and non-hydrostatic terms may be neglected and the governing equations are as follows.

$$-v = -\frac{\partial p}{\partial x} + E_H \nabla_H^2 u + E_v \frac{\partial^2 u}{\partial z^2} \tag{2 a}$$

$$u = -\frac{\partial p}{\partial y} + E_H \nabla_H^2 v + E_v \frac{\partial^2 v}{\partial z^2} \tag{2 b}$$

$$o = -\frac{\partial p}{\partial z} \tag{2 c}$$

$$w = -\int_{-1}^{z} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz' \tag{2 d}$$

where

$$\nabla_H^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

The boundary conditions are as follows.

At

$$z = 0,\ E_v^{\frac{1}{2}} \frac{\partial u}{\partial z} = \tau_x,\ E_v^{\frac{1}{2}} \frac{\partial v}{\partial z} = \tau_y \tag{3 a}$$

and

$$w = 0 \tag{3 b}$$

At

$$z = -1, \quad u = v = 0 \tag{3 c}$$

At the coast,

$$u = v = 0 \tag{3 d}$$

The solution can be broken into the interior solution ($E_H = 0$) and the boundary layer correction to the interior solution.

The interior solution

The interior solution was obtained by Welander (1957) for $E_v = 0(1)$. His model allows for a variable bottom topography, but we specialize it here to the case of uniform depth. We summarize his solution briefly.

With $E_H = 0$, Welander solves for u and v in terms of $\partial p/\partial x$ and $\partial p/\partial y$ from equations (2a, b) and satisfies conditions (3a, c). He finds

$$S = \tau H - i p_n G \tag{4}$$

where

$$S = u + iv$$

$$\tau = \tau_x + i\tau_y$$

$$p_n = \frac{\partial p}{\partial x} + i\frac{\partial p}{\partial y}$$

$$H = H_R + iH_I = \frac{1}{\sqrt{i}} \frac{\sinh \beta(1+z)}{\cosh \beta}$$

$$G = G_R + iG_I = \left(\frac{\cosh \beta z}{\cosh \beta} - 1\right)$$

$$\beta = \sqrt{i/E_v}$$

The volume transport is given by

$$M = M_x + iM_y = \int_{-1}^{0} S dz = -\sqrt{E_v}\, \tau L - i p_n N \tag{5}$$

where

$$L = L_R + iL_I = 1 - (\cosh \beta)^{-1}$$

$$N = N_R + iN_I = \tanh \beta/\beta - 1$$

The last condition to be satisfied is (3b), which can be written as

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} = 0,$$

or, from equation (5).

$$N_I \nabla_H^2 p = -\sqrt{E_v}\, L_R\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right)$$

$$-\sqrt{E_v}\, L_I\left(\frac{\partial \tau_x}{\partial x} + \frac{\partial \tau_y}{\partial y}\right). \qquad (6)$$

We note that as $E_v \to 0$, $N_R \to -1$, $N_I \to -\sqrt{E_v/2}$, $L_R \to 1$, $L_I \to 0$. Equation (6) constitutes a governing equation for $p(x, y)$. As the boundary condition on $p(x, y)$ at the coast, Welander requires that the normal volume flux into the coast be zero. If (n_x, n_y) is the normal to the coast, then from equation (5), at the coast,

$$n_x\left[\sqrt{E_v}(\tau_y L_R + \tau_x L_I) + \left(\frac{\partial p}{\partial y} N_R + \frac{\partial p}{\partial x} N_I\right)\right]$$

$$+ n_y\left[\sqrt{E_v}\,(\tau_y L_I - \tau_x L_R) + \left(\frac{\partial p}{\partial y} N_I - \frac{\partial p}{\partial x} N_R\right)\right] = 0 \qquad (7)$$

Thus, equation (6) may be solved subject to (7) and the interior pressure and velocities are now assumed known. From equation (7), we may then solve for $\partial p/\partial y$ in terms of τ_x, τ_y, and $\partial p/\partial x$ at each point on the coast, and from equation (4), the velocity field at the coast is completely specified in terms of τ_x, τ_y, and $\partial p/\partial x$.

The boundary layer solution

We must now satisfy the condition that the horizontal velocity vanish at each point on the coast for all z. We examine the flow in the neighborhood of (0, 0, z), a typical coastal region. The total velocity and pressure fields in this region (T subscripts) may be written as follows.

$$\begin{aligned} u_T(x, y, z) &= u(0, 0, z) + \bar{u}(x, y, z) + 0(E_H^{\frac{1}{3}}) \\ v_T(x, y, z) &= v(0, 0, z) + \bar{v}(x, y, z) + 0(E_H^{\frac{1}{3}}) \qquad (8) \\ w_T(x, y, z) &= w(0, 0, z) + \bar{w}(x, y, z)/E_H^{\frac{1}{3}} + 0(1) \\ p_T(x, y, z) &= p(0, 0) + E_H^{\frac{1}{3}}\bar{p}(x, z) + 0(E_H). \end{aligned}$$

The solution with the overbar is the boundary layer correction, which is assumed to vanish away from the coast. Letting $\bar{x} = x/E_H^{\frac{1}{3}}$, the governing equations for the boundary layer solution, correct to order $E_H^{\frac{1}{3}}$ are as follows.

$$-\bar{v} = -\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + E_v \frac{\partial^2 \bar{u}}{\partial z^2} \qquad (9\,\text{a})$$

$$\bar{u} = \frac{\partial^2 \bar{v}}{\partial \bar{x}^2} + E_v \frac{\partial^2 \bar{v}}{\partial z^2} \qquad (9\,\text{b})$$

$$0 = -\frac{\partial \bar{p}}{\partial z} \qquad (9\,\text{c})$$

$$\bar{w} = -\frac{\partial}{\partial \bar{x}}\left(\int_{-1}^{z} \bar{u}\, dz'\right) \qquad (9\,\text{d})$$

The boundary conditions on this solution are as follows.

At

$$z = 0, \quad \frac{\partial \bar{u}}{\partial z} = \frac{\partial \bar{v}}{\partial z} = 0 \qquad (10\,\text{a})$$

and

$$w = 0\left(\int_{-1}^{0} \bar{u}\, dz = 0\right) \qquad (10\,\text{b})$$

At

$$z = -1, \quad u = v = 0 \qquad (10\,\text{c})$$

At

$$(0, 0, z), \quad \bar{u} = -u(0, 0, z), \quad \bar{v} = -v(0, 0, z) \qquad (10\,\text{d})$$

To solve these equations for $\bar{u}$, $\bar{v}$, and $\partial \bar{p}/\partial \bar{x}$, we seek solutions of the form:

$$\left(\bar{u}, \bar{v}, \frac{\partial \bar{p}}{\partial \bar{x}}\right) = \sum_{K_n} (U_n(z), V_n(z), P_n) \exp(iK_n \bar{x}) \qquad (11)$$

where $\operatorname{Im} K_n > 0$; the K_n may be regarded as eigenvalues to be determined below. We note that a more general and more cumbersome Fourier transform approach was initially used to solve this problem. However, for conciseness of presentation, at the suggestion of a referee, the above approach was taken; the results are, of course, identical. Substituting (11) into (9) yields the following equations for $V_n(z)$ and $U_n(z)$:

$$E_v^2 \frac{d^4 V_n}{dz^4} - 2E_v K_n^2 \frac{d^2 V_n}{dz^2} + (1+K_n^4)\, V_n = P_n \tag{12a}$$

$$U_n(z) = E_v \frac{d^2 V_n}{dz^2} - K_n^2 V_n \tag{12b}$$

The boundary conditions (10 a, c) become

$$\left.\frac{dV_n}{dz}\right|_{z=0} = \left.\frac{d^3 V_n}{dz^3}\right|_{z=0} = 0 \tag{13a}$$

$$V_n\big|_{z=-1} = \left.\frac{d^2 V_n}{dz^2}\right|_{z=-1} = 0 \tag{13b}$$

Note that the solutions to (12) under (13) are not orthogonal and that (10b) has yet to be satisfied. The solutions to (12) and (13) are

$$V_n = C_n\left(\alpha_+^2\left(\frac{\cosh \alpha_- z}{\cosh \alpha_-} - 1\right) - \alpha_-^2\left(\frac{\cosh \alpha_+ z}{\cosh \alpha_+} - 1\right)\right)\Bigg|_{K=K_n} \tag{14a}$$

$$U_n = -iC_n\left(\alpha_+^2\left(\frac{\cosh \alpha_- z}{\cosh \alpha_-} - 1\right) + \alpha_-^2\left(\frac{\cosh \alpha_+ z}{\cosh \alpha_+} - 1\right)\right)\Bigg|_{K=K_n} \tag{14b}$$

$$p_n = -2iE_v C_n \alpha_+^2 \alpha_-^2 \big|_{K=K_n}$$

$$\alpha_+^2 = \frac{K^2+i}{E_v}, \quad \alpha_-^2 = \frac{K^2-i}{E_v} \tag{14c}$$

The final condition to be satisfied is (10b), i.e.,

$$\int_{-1}^{0} U_n(z) = 0$$

Integrating (14b) and satisfying this condition leads to the eigenvalue equation

$$D(K_n) = 0$$

where

$$D(K) = \alpha_+^2\left(\frac{\tanh \alpha_-}{\alpha_-} - 1\right) + \alpha_-^2\left(\frac{\tanh \alpha_+}{\alpha_+} - 1\right) \tag{15}$$

Hence, to find the boundary layer solutions we must determine the zeros of $D(K)$.

The zeros of $D(K)$

We first note that $K = (\pm 1 + i)/\sqrt{2}$ although zeros of $D(K)$, do not contribute to $\bar{u}$ and $\bar{v}$ since U_n and V_n vanish there.

Secondly we note that if K_M is a zero of $D(K)$ $(\alpha_{+M} \equiv \alpha_+(K_M))$, $(\alpha_{-M} = \alpha_-(K_M))$, then $-K_M^*$ is also a zero and $\alpha_+(-K_M^*) = \alpha_{-M}^*$, $\alpha_-(-K_M^*) = \alpha_{+M}^*$. Hence after some algebra with $K_{RN} \geq 0$, we may write

$$\bar{u}(\bar{x}, z) = \sum_{K_N} \operatorname{Re}\left[D_N \exp(iK\bar{x})\left[\frac{1}{\alpha_-^2}\left(\frac{\cosh \alpha_- z}{\cosh \alpha_-} - 1\right) + \frac{1}{\alpha_+^2}\left(\frac{\cosh \alpha_+ z}{\cosh \alpha_+} - 1\right)\right]\right]_{K_N} \tag{16a}$$

$$\bar{v}(\bar{x}, z) = -\sum_{K_N} \operatorname{Im}\left[D_N \exp(iK\bar{x})\left[\frac{1}{\alpha_-^2}\left(\frac{\cosh \alpha_- z}{\cosh \alpha_-} - 1\right) - \frac{1}{\alpha_+^2}\left(\frac{\cosh \alpha_+ z}{\cosh \alpha_+} - 1\right)\right]\right]_{K_N} \tag{16b}$$

where $D_N = D_{NR} + iD_{NI}$ are arbitrary constants. We note that the terms in brackets also hold if K_M is purely imaginary in which case the coefficients of D_{MI} vanish identically and we take $D_{MI} = 0$.

Now to actually find the zeros of $D(K)$ for finite E_v requires a search of the complex K plane. By examining $D(K)$ for small E_v our search can be shortened.

If K is pure imaginary $\alpha_- = \alpha_+^*$, and we must satisfy

$$\operatorname{Re}\left(\alpha_+^2\left(\frac{\tanh \alpha_-}{\alpha_-} - 1\right)\right) = 0$$

Furthermore, if K is small compared to one $\tanh \alpha_- \approx 1$ and we can readily show that for $E_v \to 0$,

$$K_1 \simeq i(E_v/2)^{\frac{1}{2}}[1 + .75(E_v/2)^{\frac{1}{2}}) \tag{17}$$

We see that this corresponds to the $(E_H)^{\frac{1}{2}}/(E_\nu/2)^{\frac{1}{4}}$ layer found by other means in Janowitz (1970). This result is a good approximation to K_1 for small E_ν. The exact results for $E_\nu = 0.1$, 0.5, 1.0 are respectively $K_1 = 0.550$, 1.12, 1.57, and the approximate results (17) are 0.553, .97, 1.29.

Further, for $E_\nu \to 0$, we also see that if $\alpha_- = 0(1)$, $\alpha_+^2 (= \alpha_-^2 + 2i/E_\nu) = 0(1/E_v)$. Thus, $\alpha_{-N+1} = -i\gamma_N$ is a zero of $D(K)$ to order E_ν, where γ_N is the Nth positive, nonzero root of $\tan \gamma_N = \gamma_N (\gamma_1 < \gamma_2 < \ldots)$. Now $\alpha_{-N+1} = -i\gamma_N$, corresponds to

$$K_{N+1} = \sqrt{i - E_\nu \gamma_N^2}$$

For infinitesimal E_ν, this corresponds to a layer of thickness $E_H^{\frac{1}{2}}$. The effect of increasing E_ν from zero is to split this one layer into infinitely many modes. As N increases the real part of K_N decreases and the imaginary part increases. Physically viscous forces are becoming larger relative to the Coriolis term.

For finite E_ν we can proceed as follows. In the neighborhood of $\alpha_- = -i\gamma_N$, a small change in α_- causes a large change in $\tanh \alpha_-$ but only small changes in α_+ and $\tanh \alpha_+$ (as long as $K = \sqrt{i - E_\nu \gamma_N^2}$ is not near the imaginary K axis). Hence, in $D(K) = 0$, we let $\alpha_- = -i\gamma_N$ and $\alpha_+ = \sqrt{-\gamma_N^2 + 2i/E_\nu}$ in all terms but $\tanh \alpha_-$ and then solve for $\tanh \alpha_-$ and so a new α_-. This will give us a good first guess at α_{-N+1}. This technique is successful for $E_\nu \leqslant 0.5$.

We note that for a fixed E_ν, as N increases, $\operatorname{Im} K_N$ increases and $\operatorname{Re} K_N$ decreases; for $N \geqslant \bar{N}(E_\nu)$ the roots occur as pairs on the imaginary K axis rather than as mirror images across the imaginary K axis. As E_ν increases, $\bar{N}(E_\nu)$ decreases. For $E_\nu = 1.0$ all of the roots corresponding to the splitting of the $E_H^{\frac{1}{2}}$ layer occur as pairs on the imaginary axis.

We find for $E_\nu = 0.1$, $K_1 = 0.550i$, $K_2 = .319 + 1.48i$, $K_3 = .200 + 2.47i$, $K_4 = .142 + 3.47i$, $K_5 = .111 + 4.46i$. For $E_\nu = 0.5$, $K_1 = 1.12i$, $K_2 = .122 + 3.26i$, $K_3 = .0766 + 5.509i$, $K_4 = .0552 + 7.743i$, $K_5 = .0430 + 9.97i$. For $E_\nu = 1.0$, $K_1 = 1.57i$, $K_2 = 4.57i$ (4.64i), $K_3 = 7.78i$ (7.80i), $K_4 = 10.94i$ (10.96i), $K_5 = 14.10i$ (14.11i). These results were arrived at by scanning the complex K plane.

Now that we can consider the roots found for any E_ν we must show how to find the constants D_N.

The values of D_N

Condition (10d) remains to be satisfied. To satisfy this we require

$$\sum_{N=1}^{\infty} [D_{NR} U_{NR}(z) + D_{NI} U_{NI}(z)] = -\operatorname{Re}[\tau H - ip_n G] \tag{18a}$$

$$\sum_{N=1}^{\infty} [D_{NR} V_{NR}(z) + D_{NI} V_{NI}(z)] = -\operatorname{Im}[\tau H - ip_n G] \tag{18b}$$

$$U_{NR} - iU_{NI} = \frac{1}{\alpha_-^2}\left(\frac{\cosh \alpha_- z}{\cosh \alpha_-} - 1\right)\bigg|_{K_N} + \frac{1}{\alpha_+^2}\left(\frac{\cosh \alpha_+ z}{\cosh \alpha_+} - 1\right)\bigg|_{K_N} \tag{18c}$$

$$V_{NI} + iV_{NR} = -\frac{1}{\alpha_-^2}\left(\frac{\cosh \alpha_- z}{\cosh \alpha_-} - 1\right)\bigg|_{K_N} + \frac{1}{\alpha_+^2}\left(\frac{\cosh \alpha_+ z}{\cosh \alpha_+} - 1\right)\bigg|_{K_N} \tag{18d}$$

On the right hand side of (18a, b) $\partial p/\partial y$ is expressed as a linear combination of τ_x, τ_y, $\partial p/\partial x$. To obtain the D_N, we expand U_{NR}, U_{NI}, V_{NR}, V_{NI}, $-u(0, 0, z)$ and $-v(0, 0, z)$ (the right hand side of (18)) in the series of functions $\{\cos (z(2M-1)\pi/2)\}$ and then equate coefficients. We note that

$$\int_{-1}^{0} \frac{\cos Mz}{\alpha^2}\left(\frac{\cosh \alpha z}{\cosh \alpha} - 1\right) dz = -\frac{\sin M}{M(\alpha^2 + M^2)}$$

$$\int_{-1}^{0} \cos Mz \frac{\sinh \beta(1+z)}{\cosh \beta} dz = \frac{1}{\beta} \frac{1}{(\beta^2 + M^2)}$$

if $M = (2m-1)\pi/2$ $(m = 1, 2, 3, \ldots)$. Hence the coefficients are easy to compute. More directly we can multiply (18a, b) by $\cos \pi z/2$ and integrate over z to obtain two equations. We then multiply through by $\cos (3\pi z)/2)$ and integrate over z to obtain two more equations. We truncate this process after $\cos ((2L-1)\pi z/2)$ and so have $2L$ equations to solve. We set $D_{(N+1)R}$, $D_{(N+1)I} = 0$ for $N \geqslant L$, and so have $2L$ unknowns to solve for. Hence we can obtain all the constants. We demonstrate this technique in the next section.

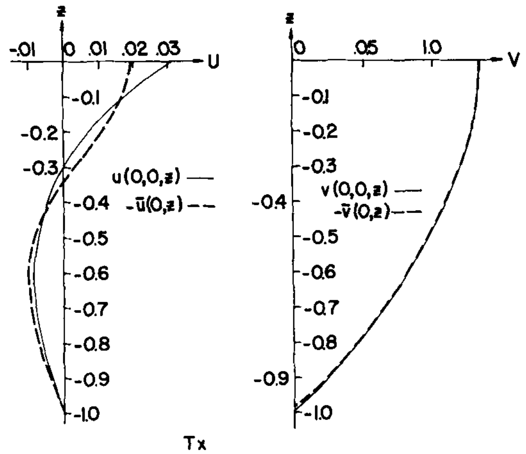

Fig. 2. Comparison of interior velocities at coast and boundary layer corrections for $(\tau_x, \tau_y, \partial p/\partial x) = (1, 0, 0)$.

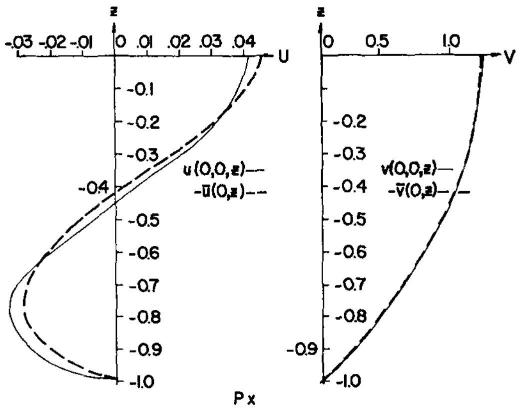

Fig. 3. Comparison of interior velocities at coast and boundary layer corrections for $(\tau_x, \tau_y, \partial p/\partial x) = (0, 0, 1)$.

A numerical example

We consider the case $E_v = 0.5$. For this case

$$\alpha + 1 = .593 + 1.69i, \quad \alpha - 1 = .593 - 1.69i;$$

$$\alpha + 2 = .389 + 4.62i, \quad \alpha - 2 = .0446 - 4.60i;$$

$$\alpha + 3 = .237 + 7.79i, \quad \alpha - 3 = .0200 - 7.79i.$$

Near the origin $n_x = 1$, $n_y = 0$ and hence from (7) we find that

$$\frac{\partial p}{\partial y}(0, 0) = 1.30\,\tau_x(0, 0) + 1.10\,\tau_y(0, 0) - 1.26\frac{\partial p}{\partial x}(0, 0)$$

The series expansions for $-u(0, 0, z)$, $-v(0, 0, z)$ were truncated after $\cos(3\pi z/2)$, leaving four equations in D_{1R}. D_{2R}, D_{2I}, D_{3R} to be solved; the other values were taken to be zero. Three cases of $(\tau_x, \tau_y, \partial p/\partial x) = (1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ were considered. For (1, 0, 0,) we obtained for $(D_{1R}, D_{2R}, D_{2I}, D_{3R})$ the values (1.08, −.408, −.0076, −.084). For (0, 1, 0), we obtained (.0039, .0084, .504, .008). For (0, 0, 1), we obtained (−1.014, .011, −.151, +.020). These values for the D_N's were substituted into equations (16) to obtain $\bar{u}(0, 0, z)$ and $\bar{v}(0, 0, z)$. From equations (4), we obtain $u(0, 0, z)$, $v(0, 0, z)$ and plot the results together with $-\bar{u}(0, 0, z)$, $-\bar{v}(0, 0, z)$ in Figs 2, 3, and 4 for the three cases of $(\tau_x, \tau_y, \partial p/\partial x)$ considered. Considering the fact that the series representations for u and v were truncated after only two terms, agreement is quite good except for u in (1, 0, 0) and v in (0, 1, 0) near $z = 0$. This is not unexpected since the interior flows have nonzero slopes at $z = 0$, and the series representation of u for (1, 0, 0) and v for (0, 1, 0) are not accurate at $z = 0$. To get a better representation near $z = 0$ would require using more terms in the expansion for u and v while changing $D_{1R}, \ldots D_{3R}$ only slightly; however, since the small scale (vertical) features of the boundary layer solution decay most rapidly away from the coast, it is only near (0, 0, 0) that the flow given with the values of D_{1R}, D_{2R}, D_{2I}, D_{3R} obtained here are in error. In this region, a non-hydrostatic boundary layer of thickness H exists which brings the vertical motion to rest and induces horizontal motions of order $(H/L)/E_H^{\frac{1}{2}}$ (<1). Thus, since the flow as predicted will be in error in this region, it makes little sense, physically to use more terms in the expansion.

Final remarks

The question clearly arises as to the applicability of our model to real lakes since we have neglected inertial, and bottom topographical effects as well as stratification.

First to estimate inertial effects we compute

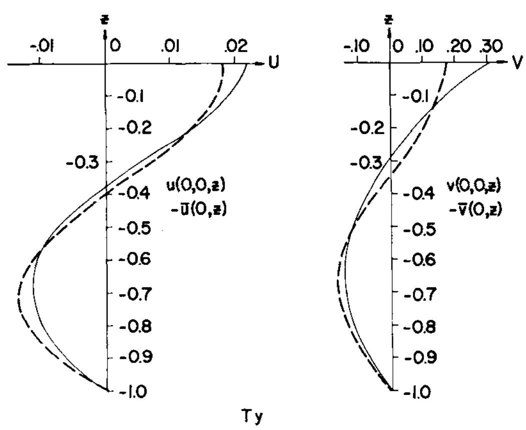

Fig. 4. Comparison of interior velocities at coast and boundary layer corrections for $(\tau_x, \tau_y, \partial p/\partial x) = (0, 1, 0)$.

a Rossby number based on the boundary layer thickness and on $\bar{u}'$ the velocity normal to the coast in the boundary layer. This number is $\bar{u}'/\sqrt{A_H f}$. If $A_H \sim 10^6$ cm²/sec, then we must require that $\bar{u}' < 10$ cm/sec for inertial effects to be small. Now in the interior of the lakes horizontal velocities of the order of 10 cm/sec do occur. Near the shore, however, the presence of the coast generally constrains $\bar{u}'$ to be small compared to typical interior velocities (see Figs. 2, 3, and 4) and hence inertial effects are probably not dominant in the boundary layers considered here.

We have considered the depth to be constant across the boundary layer. If the bottom has a slope of order H/L vertical velocities (dimensionless) of order one are induced. Vertical velocities of order $E_v^{\frac{1}{2}}/E_H^{\frac{1}{4}}(\gg 1)$ are present in these boundary layers and hence bottom topographic effects are also secondary in the coastal boundary layers.

Finally, the effects of stratification are certainly important during much of the year. Lakes in the northern United States are sometimes ice-covered in the winter and may have very pronounced thermoclines in the summer. However, in late spring and fall virtually homogeneous conditions may apply in some lakes (e.g. Lake Erie). Thus the assumptions of our model should be fulfilled for at least a portion of the year and our results applicable then.

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ВЛИЯНИЕ КОНЕЧНОГО ВЕРТИКАЛЬНОГО ЧИСЛА ЭКМАНА НА ПРИБРЕЖНЫЙ ПОГРАНИЧНЫЙ СЛОЙ

Линейные гидростатические прибрежные пограничные слои в мелком озере были ранее изучены автором (Яновиц, 1970) в предположении бесконечной малости вертикального числа Экмана; были найдены непрерывные перекрывающиеся слои с толщинами порядка $E_H^{1/2}$ и $E_H^{1/2}/E_v^{1/4}$. В данной работе погра ничные слои исследуются при $E_v = 0(1)$. Слой порядка $E_H^{1/2}/E_v^{1/4}$ различим, хотя становится тоньше. Слой $E_H^{1/2}$ также присутствует, но в модифицированном виде; каждая вертикальная компонента, имеющаяся здесь, обладает разными горизонтальными масштабами длины, хотя все они порядка $0(E_H^{1/2})$. Проделаны численные расчеты толщин пограничных слоев для $E_v = 0{,}1$, $0{,}5$, $1{,}0$.