On the response of the ocean to impulse

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ABSTRACT

A preliminary investigation on transitory motions of "inertial" period is made with the help of simple models. The amplitudes of the displacements are evaluated when an initial impulse and a sudden cooling are imposed on a non-homogeneous sea. Then the effects of spatial variations or of lateral boundaries are considered. Allowing for a density stratification, the frequencies and phase dispersions corresponding to the diverse inertial modes are given.

1. Introduction

The winter-time surveys "Medoc 69" and "Medoc 70" have shown that the deep Mediterranean water is essentially formed in a region centred at 42° N, 5° E. They have given a detailed and precise description of the different phases of this formation (Medoc Group, 1970; Anati & Stommel, 1970). Nevertheless the dynamics of this process is still not clear. Thus we do not know which perturbation is the real trigger of the deep convection: the surface layer agitation, its instability by cooling, or both of them? In other respects, during the summer regime, the energy imparted to the sea by the wind gusts seems to be concentrated in the surface layer, because of the screen effect of the thermocline. On the contrary, in the winter-time, when the density gradient is very small, this energy is transmitted downwards. The currents are then apparently uncorrelated with the winds; and the mechanism by which they are produced is unknown.

An attempt to give a theoretical answer these questions would necessitate the building of multidimensional and weighty models, which would imply a difficult analytical treatment. Nevertheless it is possible to describe simply, although approximately, the transitory motion of a water mass which is subjected to a sudden impulse or a cooling. This preliminary study brings some insight in the inertial oscillations caused by the disturbing influence of the earth rotation. Indeed, the remarkable character of a rotating fluid is its ability to absorb and transmit energy in the form of inertial oscillations (Chandrasekhar, 1961; Greenspan, 1968). The analysis which is presented below, gives an evaluation of the amplitudes of the vertical and horizontal displacements, and permits the determination of the frequencies and the phase shifts introduced by the presence of spatial variations or lateral boundaries.

2. Impulse in a homogeneous sea

We consider a water mass subjected to a sudden impulse inside an homogeneous sea. In a first approximation we may regard this water mass as a rigid particle movable without friction. Using a system of coordinates $0x$, $0y$, $0z$, pointing respectively to the east, the north, the zenith, and taking into account the rotation of the earth, we write the equations of the motion as follows:

$$\ddot{x} - 2\,\omega \sin\lambda \dot{y} + 2\omega \cos\lambda \dot{z} = 0$$

$$\ddot{y} + 2\,\omega \sin\lambda\, \dot{x} = 0$$

$$\ddot{z} - 2\omega \cos\lambda \dot{x} = 0 \qquad (1)$$

ω denotes the angular velocity of the earth rotation, λ the latitude, and the dots the differentiations with respect to time. If at the instant $t=0$, and impulse defined by the velocity $(\dot{x}_0, \dot{y}_0, \dot{z}_0)$ is communicated to a water mass at the origin, the system (1) takes the form

$$\ddot{x}+4\omega^2 x = 2\omega \sin\lambda\, \dot{y}_0 - 2\,\omega\cos\lambda\, \dot{z}_0$$

$$\dddot{y}+4\omega^2\dot{y} = 4\omega^2\cos^2\lambda\dot{y}_0+4\omega^2\sin\lambda\cos\lambda\dot{z}_0$$

$$\dddot{z}+4\omega^2\dot{z} = 4\omega^2\sin\lambda\cos\lambda\dot{y}_0+4\omega^2\sin^2\lambda\dot{z}_0 \quad (2)$$

These equations are easily integrated, and the solutions are given by

$$x=\dot{x}_0\frac{\sin 2\omega t}{2\omega}+(\dot{y}_0\sin\lambda-\dot{z}_0\cos\lambda)\frac{(1-\cos 2\omega t)}{2\omega}$$

$$y=\dot{y}_0 t-\dot{x}_0\sin\lambda\frac{(1-\cos 2\omega t)}{2\omega}$$

$$+\sin\lambda(\dot{y}_0\sin\lambda-\dot{z}_0\cos\lambda)\left(\frac{\sin 2\omega t}{2\omega}-t\right)$$

$$z=\dot{z}_0 t+\dot{x}_0\cos\lambda\frac{(1-\cos 2\omega t)}{2\omega}$$

$$-\cos\lambda(\dot{y}_0\sin\lambda-\dot{z}_0\cos\lambda)\left(\frac{\sin 2\omega t}{2\omega}-t\right) \quad (3)$$

We see that the water mass makes inertial oscillations with a frequency 2ω and an amplitude which is of the order of $\dot{x}_0/2\omega$, $\dot{y}_0/2\omega$, $\dot{z}_0/2\omega$ along a mean trajectory defined by

$$\bar{x}=\frac{1}{2\omega}(\dot{y}_0\sin\lambda-\dot{z}_0\cos\lambda)$$

$$\bar{y}=-\frac{\dot{x}_0}{2\omega}\sin\lambda+\cos\lambda(\dot{y}_0\cos\lambda+\dot{z}_0\sin\lambda)t$$

$$\bar{z}=\frac{\dot{x}_0}{2\omega}\cos\lambda+\sin\lambda(\dot{y}_0\cos\lambda+\dot{z}_0\sin\lambda)t \quad (4)$$

Concerning the vertical motions, we get from the expressions (3) the following results. In the northern hemisphere, an impulse directed to the east, the north, or the zenith, produces an upward motion in the beginning. Inversely, an impulse directed to the west, the south, or the nadir, produces a downward motion. We also see that an impulse to the east, the north, or the nadir, produces an eastward motion in the beginning (and vice versa), and that an impulse to the north, the west, or the zenith, produces a northward motion (and vice versa). The amplitudes of the vertical displacements corresponding to east–west and north–south impulses are respectively given by $\dot{x}_0\cos\lambda/2\omega$ and $\dot{y}_0\sin\lambda\cos\lambda/2\omega$. At mean latitudes and for an initial velocity of 20 cm/s, the amplitude is about 1 000 m. Besides, it is easy to verify that an impulse parallel or normal to the axis of rotation gives rise respectively, to a displacement parallel to this axis, or to a circular motion in a plane normal to it.

3. Impulse in a non-homogeneous sea

A water mass which is moving about in a stable density field, with a density ϱ_0 equal to that of the surrounding medium at the origin, is subjected to an acceleration $g(\varrho-\varrho_0)$. Considering the simple case of a linear density field, we have

$$\varrho=\varrho_0-\alpha z \quad (5)$$

Then the equations of the motion are the following

$$\ddot{x}-2\omega\sin\lambda\dot{y}+2\omega\cos\lambda\ddot{z}=0$$

$$\ddot{y}+2\omega\sin\lambda\dot{x}=0$$

$$\ddot{z}-2\omega\cos\lambda\dot{x}=-g\alpha z \quad (6)$$

By taking into consideration the initial conditions for the velocity $(\dot{x}_0, \dot{y}_0, \dot{z}_0)$, the system (6) may be written in the form

$$\ddddot{x}+(4\omega^2+g\alpha)\ddot{x}+4\omega^2\sin^2\lambda g\alpha x=2\omega\sin\lambda g\alpha\dot{y}_0$$

$$\ddddot{y}+(4\omega^2+g\alpha)\ddot{y}+4\omega^2\sin^2\lambda g\alpha y=-2\omega\sin\lambda g\alpha\dot{x}_0$$

$$\ddddot{z}+(4\omega^2+g\alpha)\ddot{z}+4\omega^2\sin^2\lambda g\alpha z=0 \quad (7)$$

The integration is made by means of the Laplace transform

$$\bar{f}(s)=\int_0^\infty e^{-st}f(t)\,dt \quad (8)$$

The initial conditions for the second and the third derivatives of x, y, z, come out from the equations (6). Thus we obtain the following solutions:

$$x = \frac{1}{\sigma_*^2 - \sigma_0^2}\left\{\dot{x}_0\left[(g\alpha - \sigma_0^2)\frac{\sin\sigma_0 t}{\sigma_0} - (g\alpha - \sigma_*^2)\frac{\sin\sigma_* t}{\sigma_*}\right]\right.$$

$$+ 2\omega\sin\lambda\dot{y}_0\left[\frac{g\alpha}{\sigma_0^2}(1-\cos\sigma_0 t)\right.$$

$$\left. - \frac{g\alpha}{\sigma_*^2}(1-\cos\sigma_* t) + \cos\sigma_0 t - \cos\sigma_* t\right]$$

$$\left. - 2\omega\cos\lambda\dot{z}_0(\cos\sigma_0 t - \cos\sigma_* t)\right\}$$

$$y = \frac{1}{\sigma_*^2 - \sigma_0^2}\left\{-2\omega\sin\lambda\dot{x}_0\left[\frac{g\alpha}{\sigma_0^2}(1-\cos\sigma_0 t)\right.\right.$$

$$\left. - \frac{g\alpha}{\sigma_*^2}(1-\cos\sigma_* t) + \cos\sigma_0 t - \cos\sigma_* t\right]$$

$$+ \dot{y}_0\left[(g\alpha + 4\omega^2\cos^2\lambda - \sigma_0^2)\frac{\sin\sigma_0 t}{\sigma_0}\right.$$

$$\left. - (g\alpha + 4\omega^2\cos^2\lambda - \sigma_*^2)\frac{\sin\sigma_* t}{\sigma_*}\right]$$

$$\left. + 4\omega^2\sin\lambda\cos\lambda\dot{z}_0\left(\frac{\sin\sigma_0 t}{\sigma_0} - \frac{\sin\sigma_* t}{\sigma_*}\right)\right\}$$

$$z = \frac{1}{\sigma_*^2 - \sigma_0^2}\left\{2\omega\cos\lambda\dot{x}_0(\cos\sigma_0 t - \cos\sigma_* t)\right.$$

$$+ 4\omega^2\sin\lambda\cos\lambda\dot{y}_0\left(\frac{\sin\sigma_0 t}{\sigma_0} - \frac{\sin\sigma_* t}{\sigma_*}\right)$$

$$+ \dot{z}_0\left[(4\omega^2\sin^2\lambda - \sigma_0^2)\frac{\sin\sigma_0 t}{\sigma_0}\right.$$

$$\left.\left. - (4\omega^2\sin^2\lambda - \sigma_*^2)\frac{\sin\sigma_* t}{\sigma_*}\right]\right\} \quad (9)$$

The frequencies σ_* and σ_0 are given by

$$2\sigma^2 = g\alpha + 4\omega^2 \pm [(g\alpha + 4\omega^2)^2 - 16\omega^2\sin^2\lambda g\alpha]^{\frac{1}{2}} \quad (10)$$

with the sign plus and minus respectively. Of course, when the stratification disappears $\alpha \to 0$, $\sigma_0 \to 0$, $\sigma_* \to 2\omega$; and we find again the solution (3). It may be seen on (9) that the water mass oscillates with two distinct frequencies about the mean position:

$$\bar{x} = \frac{\dot{y}_0}{2\omega\sin\lambda}, \quad \bar{y} = -\frac{\dot{x}_0}{2\omega\sin\lambda}, \quad \bar{z} = 0 \quad (11)$$

In the oceans, except in certain regions and at particular times, the density gradient α is greater than the critical value $\alpha_c = 4\omega^2/g$ (1.10^{-11} cm^{-1}). Therefore most often $\alpha \gg \alpha_c$, and the frequencies σ_* and σ_0 which may be calculated from the developments

$$\sigma_*^2 = g\alpha\left(1 + \frac{4\omega^2\cos^2\lambda}{g\alpha} + \dots\right)$$

$$\sigma_0^2 = 4\omega^2\sin^2\lambda/1 - \frac{4\omega^2\sin^2\lambda}{g\alpha} + \dots\Big) \quad (12)$$

are close to the Väisälä and the inertial frequency. The amplitude of the vertical displacements is strongly reduced when the density gradient is greater, though small, than the critical value α_c. This amplitude is approximately equal to $2\omega\cos\lambda\dot{x}_0/g\alpha$ if the impulse is directed to the east or to the west. When the impulse is directed to the north or to the south, the amplitudes corresponding to the inertial and to the Väisälä frequencies are equal to $2\omega\cos\lambda\dot{y}_0/g\alpha$ and $4\omega^2\sin\lambda\cos\lambda_0\dot{y}_0/(g\alpha)^{\frac{3}{2}}$ respectively. With a vertical impulse, the corresponding amplitudes are equal to $8\omega^3\sin\lambda\cos^2\lambda\,\dot{z}_0/(g\alpha)^2$ and $\dot{z}_0/(g\alpha)^{\frac{1}{2}}$. This latter value is the most important one. Hence, with a perturbation $\dot{z}_0$ equal to 20 cm/s, we get an amplitude of 20 m for $\alpha = 1.0^{-7}$ cm^{-1}, and an amplitude of 200 m for $\alpha = 1.10^{-9}$ cm^{-1}.

The critical gradients, which are very small, are found in the regions of deep water formation, for instance in the "chimneys" which have been observed in winter in the Mediterranean Sea (Medoc Group, 1970). In these places the amplitudes of the oscillations are comparable to the ones which are generated in a homogeneous medium, namely $\dot{x}_0/2\omega$, $\dot{y}_0/2\omega$, $\dot{z}_0/2\omega$. Thus an impulse of 10 cm/s gives birth to an amplitude of 1 000 m; and this may be the beginning of a deep convection.

4. Cooling in a non-homogeneous sea

Let us now examine the displacement of a water mass which is suddenly cooled in a stable density field. This water mass, which is still conceived as a particle, is subjected to a sudden increase of density ε. Neglecting the friction, and supposing a linear density field defined by its gradient α, we have equations similar to (6), except for a term coming from ε:

$$\begin{aligned} &\ddot{x} - 2\omega \sin \lambda\, \dot{y} + 2\omega \cos \lambda\, \dot{z} = 0 \\ &\ddot{y} + 2\omega \sin \lambda \dot{x} = 0 \\ &\ddot{z} - 2\omega \cos \lambda \dot{x} = -g(\varepsilon + \alpha z) \end{aligned} \tag{13}$$

The initial velocity being zero, this system is equivalent to

$$\begin{aligned} &\ddddot{x} + (4\omega^2 + g\alpha)\ddot{x} + 4\omega^2 \sin^2 \lambda g\alpha x = 0 \\ &\ddddot{y} + (4\omega^2 + g\alpha)\ddot{y} + 4\omega^2 \sin^2 \lambda g\alpha y = -4\omega^2 \sin \lambda \cos \lambda g\varepsilon \\ &\ddddot{z} + (4\omega^2 + g\alpha)\ddot{z} + 4\omega^2 \sin^2 \lambda g\alpha z = -4\omega^2 \sin^2 \lambda g\varepsilon \end{aligned} \tag{14}$$

The integrations are performed with the help of the Laplace transform, and give the solutions

$$\begin{aligned} x &= \frac{2\omega \cos \lambda g\varepsilon}{\sigma_*^2 - \sigma_0^2}\left(\frac{\sin \sigma_0 t}{\sigma_0} - \frac{\sin \sigma_* t}{\sigma_*}\right) \\ y &= -\frac{4\omega^2 \sin \lambda \cos \lambda g\varepsilon}{\sigma_*^2 - \sigma_0^2}\left\{\frac{(1 - \cos \sigma_0 t)}{\sigma_0^2} - \frac{(1 - \cos \sigma_* t)}{\sigma_*^2}\right\} \\ z &= -\frac{g\varepsilon}{\sigma_*^2 - \sigma_0^2}(\cos \sigma_0 t - \cos \sigma_* t) \\ &\quad - \frac{4\omega^2 \sin^2 \lambda g\varepsilon}{\sigma_*^2 - \sigma_0^2}\left\{\frac{(1 - \cos \sigma_0 t}{\sigma_0^2} - \frac{(1 - \cos \sigma_* t)}{\sigma_*^2}\right\} \end{aligned} \tag{15}$$

where σ_0 and σ_* have the same meaning as previously. Therefore, according to this model a suddenly cooled water mass, oscillates with these two distinct frequencies around a mean position given by

$$\bar{x} = 0, \quad \bar{y} = -\frac{\varepsilon}{\alpha\, tg\lambda}, \quad \bar{z} = -\frac{\varepsilon}{\alpha} \tag{16}$$

$\bar{z}$ is the equilibrium depth and the amplitude of the vertical displacements is approximately ε/α. If the density increase ε is equal to 1.10^{-5}, the amplitude will be 10 m with $\alpha = 1.10^{-8}$, and 1 000 m with $\alpha = 1.10^{-10}$. Moreover, the initial displacement due to the cooling is downwards, eastwards, and southwards, in the northern hemisphere. This results from the gyroscopic forces which oppose to rotations of fluid particles around axes which are not parallel to the axis of rotation of the earth. It is interesting to note that a deep descent of cold water has been recently observed in a canyon of the Gulf of Lions in the Mediterranean Sea (Person, 1972). Unfortunately details are lacking.

5. Perturbation of finite wave-length

We shall now consider the motions produced in a layer of water of finite depth, under the action of a transitory perturbation which presents a spatial variation. If the motions are slow, the viscosity negligible, and the axis of rotation vertical, the equations are simply

$$\begin{aligned} &\varrho_0\left(\frac{\partial u}{\partial t} - 2\omega v\right) = -\frac{\partial p}{\partial x} \\ &\varrho_0\left(\frac{\partial v}{\partial t} + 2\omega u\right) = -\frac{\partial p}{\partial y} \\ &\varrho_0 \frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} - g\varepsilon \\ &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \\ &\frac{\partial \varepsilon}{\partial t} - \alpha \varrho_0 w = 0 \end{aligned} \tag{17}$$

where

$$\varrho = \varrho_0(z) + \varepsilon, \quad \alpha = -\frac{1}{\varrho_0}\frac{d\varrho_0}{dz} \tag{18}$$

Eliminating p, ε, u, v we obtain a wave equation for the vertical component of the velocity in the form

$$\left\{\frac{\Delta \partial^2}{\partial t^2} + 4\omega^2 \frac{\partial^2}{\partial z^2} - \alpha\left(\frac{\partial^2}{\partial t^2} + 4\omega^2\right)\frac{\partial}{\partial z} + g\alpha\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\right\} w = 0 \tag{19}$$

The water is bounded at the bottom by a rigid wall ($z = -h$). And at the surface ($z = 0$) a perturbation of duration τ, which is a sinusoïdal function of x, is applied to w. Therefore the conditions to be fulfilled are the following

$$w\big|_{z=0} = Y(t)\, Y(\tau - t) \sin kx, \quad w\big|_{z=-h} = 0 \qquad (20)$$

where Y denotes the step function. Using the Laplace transform

$$\overline{w}(s) = \int_0^\infty e^{-st} w(t)\, dt \qquad (21)$$

we have from (19) and (20) change the formula (22) into:

$$\frac{\partial^2 \overline{w}}{\partial z^2} - \alpha \frac{\partial \overline{w}}{\partial z} - \frac{s^2 + g\alpha}{s^2 + 4\omega^2} \frac{\partial^2 \overline{w}}{\partial x^2} = 0 \qquad (22)$$

$$\overline{w}\big|_{z=0} = \frac{1}{s}(1 - e^{-s\tau}) \sin kx, \quad \overline{w}\big|_{z=-h} = 0 \quad (23)$$

With an exponential density ϱ_0, α is a constant, and the solution of the system (22), (23), is given by

$$\overline{w} = \frac{1}{s}(1 - e^{-s\tau})\, e^{\alpha z/2} \frac{Sh\mu(h+z)}{Sh\mu h} \sin kx$$

$$\mu = \left\{\frac{\alpha^2}{4} + \frac{k^2(s^2 + g\alpha)}{s^2 + 4\omega^2}\right\}^{\frac{1}{2}} \qquad (24)$$

Then the original function w is obtained by means of a calculus of residues. The poles of the expression (24) are

$$s = \pm 2\omega i \nu_n,$$

$$\nu_n = \left(\frac{4\omega^2 n^2 \pi^2 + \omega^2 \alpha^2 h^2 + g\alpha k^2 h^2}{4\omega^2 n^2 \pi^2 + \omega^2 \alpha^2 h^2 + 4\omega^2 k^2 h^2}\right)^{\frac{1}{2}} \qquad (25)$$

And finally we get

$$w = \left\{\sum_{n=1}^{\infty} (-1)^n \frac{16\omega^2(4\omega^2 - g\alpha)\, n\pi k^2 h^2 \nu_n^2}{(4\omega^2 n^2 \pi^2 + \omega^2 \alpha^2 h^2 + g\alpha k^2 h^2)^2} \times \sin n\pi\left(1 + \frac{z}{h}\right) \times e^{\alpha z/2} \sin \omega\nu_n \tau \sin 2\omega\nu_n(\tau/2 - t) \sin kx\right\} \qquad (26)$$

Clearly the resulting motion consists of a series of inertial oscillations with the periods

$$T_n = \frac{\pi}{\omega \nu_n} \qquad (27)$$

If the depth is small with respect to the wavelength $2\pi/k$, $kh \to 0$, $\nu_n \to 1$ and $T_n \to \pi/\omega$. The periods are close to the limit of the inertial period, and the phases are in coincidence. If $kh \sim 2\omega/N$ with $N = (g\alpha)^{\frac{1}{2}}$ and $\alpha > \alpha_c$, the periods T_n have distinct values which are smaller than the limit period π/ω, and the phases are also separate. And if $kh \sim 1$, $\alpha > \alpha_c$, the periods ar close to the Väisälä period $2\pi/N$. Moreover, as observed (Gonella, Crepon & F. Madelain, 1969), the periods are smaller than the limit of the inertial period. In short, the distribution of the frequencies and phases is strongly connected to the ratio of the depth to the wave-length. One obtains slightly different results in a homogeneous medium (Saint-Guily, 1971).

The amplitude of the mode of index n is maximum when the perturbation has a duration $\tau = (2m + 1)T_n/2$ (m integer), i.e. a duration equal to an odd number of half inertial periods. Inversely the amplitude of the mode of index n is null when the perturbation has a duration $\tau = mT_n$ (m integer), i.e. a duration equal to an even number of half inertial periods. Besides, if the density gradient is equal the critical value α_c, that is if the Väisälä period is equal to the limit of the inertial period, there is no motion.

Therefore the motion generated by a perturbation in a layer of water is different according as the depth is small or not in comparison with the wave-length. In the first case the inertial oscillations have the same phase and the same period, and the energy is concentrated in a peak in the spectrum. In the second case, the phases and the periods corresponding to the different modes are distinct, and the energy is distributed in a rather wide band in the spectrum. Furthermore the perturbation is generally composed of a set of different wave-lengths, and the occurring motions are complex. Similar results may be expected in a bounded sea. The indications given by this model are in a qualitative agreement with the observations made on the Bouée-Laboratoire in the western part of the Mediterranean Sea. These observations show the presence clearly marked of inertial oscillations above the thermocline, when it is formed, and their absence in winter when the thermocline has disappeared.

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О ПЕРЕХОДНОЙ РЕАКЦИИ ОКЕАНА НА ИМПУЛЬС

На простых моделях сделано предварительное исследование о переходных движениях. Вычисляются амплитуды смещений, когда к неоднородному морю прикладываются начальный импульс и внезапное охлаждение. Затем рассматривается влияние пространственных изменений и боковых границ. С учетом стратификации плотности находятся частоты и фазы для различных мод инерционных колебаний.