On the power law for the kinetic energy spectrum of large scale atmospheric flow

By H. LEONARD STEINBERG, *Department of Environmental Sciences, Tel-Aviv University, Israel*

(Manuscript received February 28; revised version April 19, 1972)

ABSTRACT

The evidence from observational calculations and a numerical simulation experiment point to the dominance of the baroclinic process over the non-linear cascade of kinetic energy in the spectral range $7 \leqslant m \leqslant 15$ in atmospheric flow (m is the wavenumber). It is therefore suggested that the power law governing this portion of the kinetic energy spectrum be derived using C_i, the imaginary part of the phase speed commonly used in studies of baroclinic instability. Evidence is presented to support this view.

1. Introduction

Kraichnan (1967) and Leith (1968) have shown that a two-dimensional fluid containing an inertial subrange between the wavenumbers at which kinetic energy is introduced into the system and the scales at which dissipation is dominant will obey a -3 power law. That is to say that the kinetic energy per unit mass and wavenumber in this subrange may be written as

$$K(k) = Ak^{-3} \quad k_i < k < k_d \tag{1}$$

where $K(k)$ is the kinetic energy per unit mass and wavenumber, $k(=2\pi/\lambda)$ is the wavenumber, λ is the wavelength, A is not a function of wavenumber, k_i is the effective wavenumber at which energy is injected into the system, and k_d is the wavenumber at which dissipation becomes dominant.

Observational studies of the large scale atmospheric flow (Ogura, 1958; Horn & Bryson, 1963; Wiin-Nielsen, 1967; Julian et al., 1970) have shown that the kinetic energy per unit mass and wavenumber may be written in the form

$$K(m) = Bm^{-b} \quad 7 \leqslant m \leqslant 15 \tag{2}$$

where $m = kL/2\pi$, L is the circumference of a latitude circle, and B is not a function of m.

The exponent b has been found to vary between 2.7 and 3.2 depending on the pressure. At a given latitude m/k is constant and the following relationship holds

$$K(m)\,dm = K(k)\,dk \tag{3}$$

Numerical simulations of the general circulation (Manabe et al., 1970) and of two dimensional turbulence (Lilly, 1969) have also yielded power laws for $K(m)$ with exponents close to -3 for the spectral range $8 \leqslant m \leqslant 16$. Steinberg (1971), in a numerical simulation of turbulence using a simplified quasigeostrophic model similar to Phillips (1956) found that the value of b varied between 1.7 and 3.4 depending on the amplitude of the heating function and the value of the coefficient of eddy viscosity.

In regard to the assumptions of isotropy and the presence of an inertial subrange of Kraichnan's theory observational studies of the large-scale atmospheric flow (Ogura, 1958; Saltzman & Teweles, 1964; Yang, 1967; Steinberg et al., 1971) indicate the following:

(a) The assumption of isotropy in the two-dimensional turbulence theory is not observed in atmospheric large scale flow in mid latitudes.

(b) The existence of an inertial subrange in the spectral range under consideration is not substantiated. Indeed, at each wavenumber in the range $7 \leq m \leq 15$ the baroclinic process of $C(A_e, K_i)$, the conversion of eddy available potential energy A_e, to eddy kinetic energy, K_e, is dominant compared to the non-linear transfers. The conversion of A_e to K_e, however, is excluded from a two-dimensional theory. The large scale atmospheric flow may be considered as two-dimensional with respect to velocity and acceleration but not energy transfers.

Therefore, we have a theory whose prediction seems to correspond to observed behavior but whose assumptions to a large degree do not.

2. An alternate theory

We must therefore seek an alternate theory. The direction in which one must look (if indeed a theory exists) is indicated by Leith (1968). He observes that the -3 power law may be derived on purely dimensional grounds simply by assuming the dominant spectral parameter to have dimensions of time^{-1}(t^{-1}). One need not postulate an inertial subrange.

From the above evidence it may be seen that the baroclinic energy transfer, $C(A_e, K_e)$, is the dominant factor in the growth of disturbances with $8 \leq m \leq 15$. A candidate for the important parameter having dimensions of t^{-1} would then be the inverse of the e-folding time, i.e., kC_i where C_i is the imaginary part of the wave speed commonly contained in baroclinic instability theories.

3. Theoretical indications

Both Yang (1967) and Pedlosky (1970), using a model similar to Phillips (1956), obtain the following equation for the equilibrium amplitude of the stream function

$$|A(\infty)|^2 = \frac{C_i}{kN} \tag{4}$$

where k is the wavenumber in the latitudinal direction, $A(\infty)$ is the equilibrium amplitude of the stream function and N is a function of the parameters of the model such as the coefficient of eddy viscosity and the vertical wind shear.

The equilibrium kinetic energy per unit wavenumber and mass on dimensional grounds may be written as

$$K(k) = |A(\infty)|^2 k = \frac{C_i}{N} \tag{5}$$

This is written in terms of the nondimensional wavenumber m as

$$K(m) = K(k)\frac{dk}{dm} = \frac{C_i}{N}\frac{2\pi}{L} = \frac{C_i}{N} a \cos\phi$$

where ϕ is latitude and a is the earth's radius.

Taking $\phi = \phi_0 = 45°$ and letting $D = (a \cos \phi_0)^{-1}$ we write

$$K(m) = \frac{DC_i}{N} \tag{6}$$

Thus, there is theoretical evidence linking the spectrum of $K(m)$ and C_i. If C_i and N can be evaluated for a range of wavenumbers, e.g. $7 \leq m \leq 14$ then (6) and (2) can be compared and a definitive statement made. However, for reasonable values of the static stability, σ, and the vertical wind shear, S, the studies of Yang and Pedlosky have a "short wave cutoff" at $m = 9$, 10. Their values are therefore not useful in verifying the hypothesis.

Suppose that C_i and N may be written in the form

$$C_i = E_1 m^{-c}$$
$$N = E_2 m^{-d} \tag{7}$$

where E_1 and E_2 are not functions of m. A posteriori, N should be a weak function of m so that $-0.5 \leq d \leq 0.5$, depending on the case considered.

Inserting (7) into (6) we have

$$K(m) = Em^{-c-d} = Em^{-b} \tag{8}$$

where $E = (DE_1/E_2)$ and the exponent in the formulation for N is like b $-c$.

Numerical values for C_i are easily obtained from a recent study by Wiin-Nielsen (1971). With the use of a quasi-geostrophic inviscid

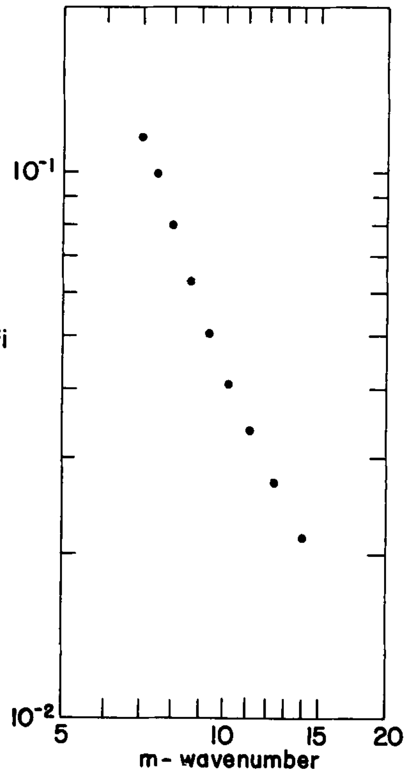

Fig. 1. C_i versus m, $S = 2$ m sec^{-1} km^{-1}, $c = 2.5$.

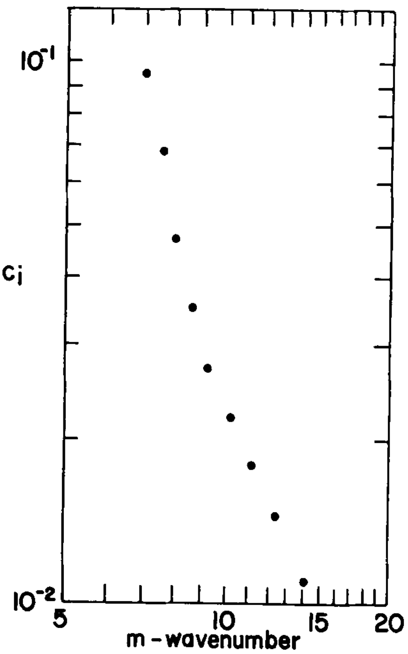

Fig. 2. C_i versus m, $S = 4$ m sec^{-1} km^{-1}, $c = 3.0$.

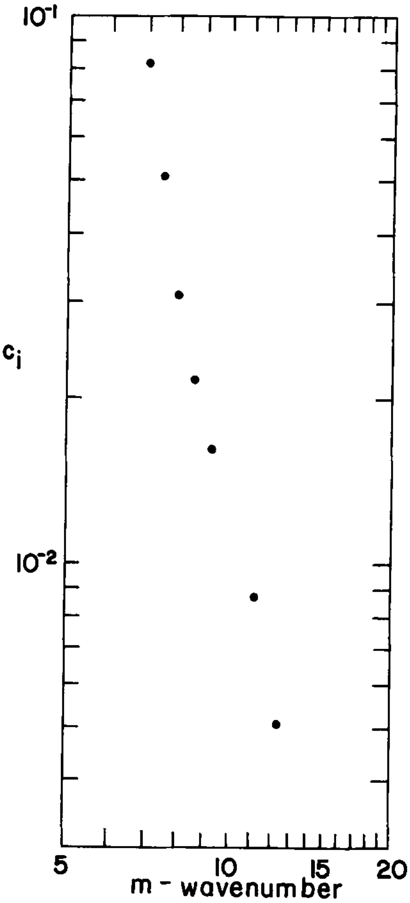

Fig. 3. C_i versus m, $S = 6$ m sec^{-1} km^{-1}, $c = 3.8$.

adiabatic multilayered model C_i was calculated as a function of m. There is no shortwave cutoff in this model.

The values of C_i for $7 \leqslant m \leqslant 14$ are reproduced in Figs. 1, 2 and 3 for $S = 2$, 4 and 6 m sec^{-1} km^{-1}, respectively. These are for the case of the derivative lower boundary condition.

While the model of Wiin-Nielsen is inviscid the models used by Yang & Pedlosky are not. They include frictional dissipation through horizontal eddy diffusivities both in the momentum and thermal equations. The form of $|A(\infty)|^2$ given in (4) will depend on the inclusion of these terms as shown by Pedlosky.

These effects were included in Wiin-Nielsen's model with values of both K_v, the eddy diffusivity for momentum and K_t the thermal eddy diffusivity equal to 10^5 m^2 sec^{-1}. The diabatic heating was omitted.

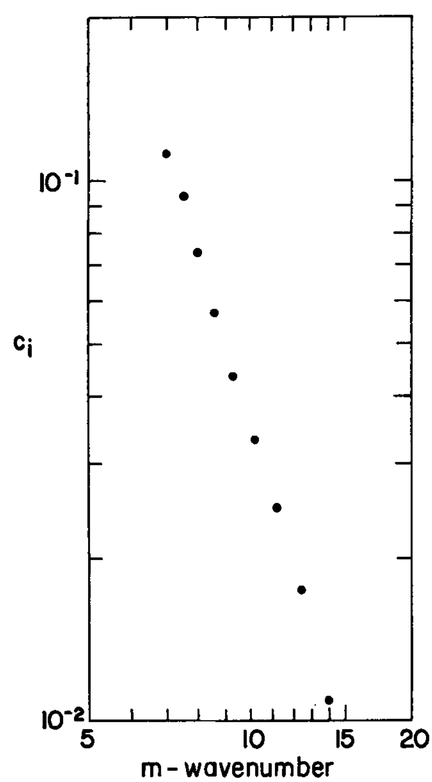

Fig. 4. C_i versus m, $S = 2$ m sec^{-1} km^{-1}, $c = 3.3$.

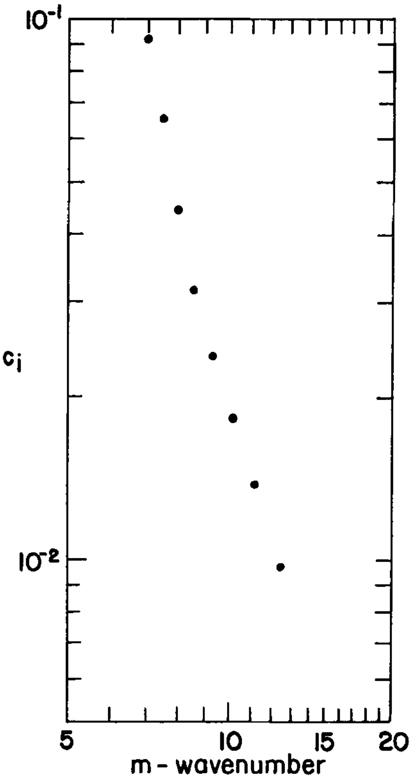

Fig. 5. C_i versus m, $S = 4$ m sec^{-1} km^{-1}, $c = 3.8$.

The values of C_i were then calculated and the results are shown in Figs. 4, 5 and 6 for wind shear values of $S = 2$, 4 and 6 m sec^{-1} km^{-1}, respectively.

It is interesting to note that for $7 \leq m \leq 14$ the values of C_i are well fitted by a power law representation. In each case the exponent was evaluated by the least squares method and is presented in Table 1. The values are surprisingly close to exponents obtained from the power laws governing the kinetic energy spectrum except for the case of $S = 6$ m sec^{-1} km^{-1} including dissipation.

Table 1

Values of c in $C_1 = E_1 m^{-c}$

S (m sec^{-1} km^{-1})	$K_v = K_t = 0$	$K_v = K_t = 10^5$ m^2 sec^{-1}
2	2.5	3.3
4	3.0	3.8
6	3.8	5.4

It should be pointed out that N has not yet been accounted for and it remains to be seen if the assumption made in (7) is correct.

4. Conclusions

Evidence has been presented pointing to the use of C_i as a basis for explaining the shape of the kinetic energy spectrum of large scale atmospheric flow ($7 \leq m \leq 15$).

It remains to investigate the shape of the function N. For the formulations of Yang & Pedlosky N is a slowly varing function of m in the range between $m = 7$ and the cutoff point which unfortunately is around $m = 9$. Formulating an equation analogous to (4) for a multilayered model would permit a better test of the theory presented here.

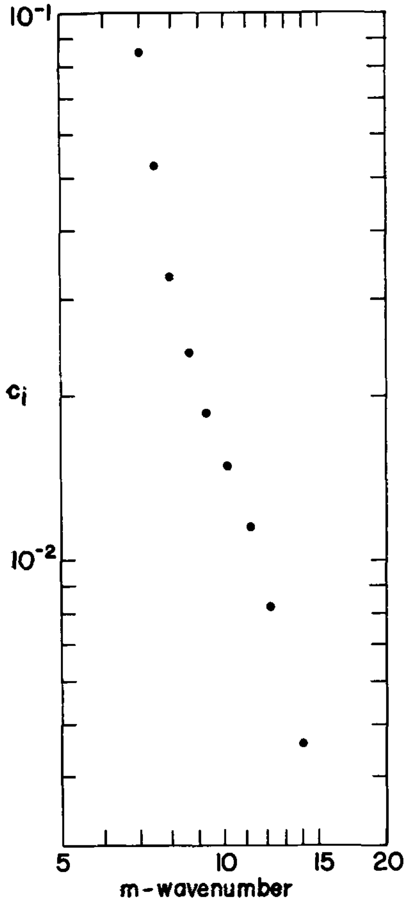

Fig. 6. C_i versus m, $S = 6$ m sec^{-1} km^{-1}, $c = 5.4$.

REFERENCES

Horn, L. H. & Bryson, R. A. 1963. An analysis of the geostrophic kinetic energy spectrum of large-scale atmospheric turbulence. *J. Gecphys. Res. 68*, 1059–1064.

Julian, P., Washington, V. W., Hembree, L. & Ridley, C. 1970. On the spectral distribution of large-scale atmospheric kinetic energy. *J. Atmos. Sci. 27*, 376–387.

Kraichnan, R. H. 1967. Inertial ranges in two-dimen sional turbulence. *Phys. Fluids 10*, 1417–1423.

Leith, C. E. 1968. Diffusion approximation for two-dimensional turbulence. *Phys. Fluids 11*, 671–673.

Lilly, D. K. 1969. Numerical simulation of two-dimensional turbulence. *Phys. Fluids*, Suppl. II, II, *12*, II-240–II-249.

Manabe, S., Smagorinsky, J., Holloway, J. L., Jr & Stone, H. M. 1970. Simulated Climatology of a General Circulation Model with a Hydrological Cycle. *Mon. Wea. Rev. 98*, No. 3, 175–212.

Ogura, Y. 1958. On the isotropy of large-scale disturbances in the upper troposphere. *J. Meteor. 15*, 375–382.

Pedlosky, J. 1970. Finite-amplitude baroclinic waves. *J. Atmos. Sci. 27*, 15–30.

Phillips, N. A. 1956. The general circulation of the atmosphere; a numerical experiment. *Quart. J. Roy. Meteor. Soc. 82*, 123–164.

Saltzman, B. & Teweles, S. 1964. Further statistics on the exchange of kinetic energy between harmonic components of the atmospheric flow. *Tellus 16*, 432–435.

Steinberg, H. L. 1971. On power laws and non-linear cascades in large scale atmospheric flow. Technical Report 002630-4-T, The University of Michigan.

Steinberg, H. L., Wiin-Nielsen, A. and Yang, C. H. 1971. On nonlinear cascades in large-scale atmospheric flow. *J. Geophys. Res. 76*, 36, 8629–8640.

Wiin-Nielsen, A. 1967. On the annual variation and spectral distribution of atmospheric energy. *Tellus 19*, 540–559.

Wiin-Nielsen, A. 1971. A study of the baroclinic instability problem and the lower boundary condition. *J. Geophys. Res. 76*, 6497–6505.

Yang, C. H. 1967. Nonlinear aspects of the large-scale motion in the atmosphere. Technical Report 08759-1-T, The University of Michigan.

О СТЕПЕННОМ ЗАКОНЕ ДЛЯ СПЕКТРА КИНЕТИЧЕСКОЙ ЭНЕРГИИ КРУПНОМАСШТАБНЫХ АТМОСФЕРНЫХ ПОТОКОВ

Расчеты по данным наблюдений и численные эксперименты по моделированию общей циркуляции указывают на преобладание бароклинных процессов в нелинейном каскаде кинетической энергии в спектральном интервале $7 \leqslant n \leqslant 15$ для атмосферных течений (m — волновое число). Поэтому предлагается, что степенной закон, управляющий этой частью спектра кинетической энергии, должен выводиться при использовании C_i, мнимой части фазовой скорости, обычно используемой в исследованиях бароклинной неустойчивости. Приводятся факты в поддержку этой точки зрения.