On trapped oscillations of a rotating fluid in a thin spherical shell II

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ABSTRACT

The study of the structure of free oscillations of fluid in a spherical shell, trapped near the equatorial plane, carried out by Stewartson (1971) is extended to include all members of the continuous set discovered by Israeli (1971). In the more general trapped mode it is not necessary for the limit cycle bounding the oscillation to touch the inner sphere but the pathological features discussed in I are also present here. The width of the band of continuous periods of oscillation is also found in terms of the number of reflections of the limit cycle at the outer sphere.

1. Introduction

Consider an inviscid fluid confined between two concentric spherical boundaries of nearly equal radii, the whole rotating rigidly about a common diameter. We wish to study the axially symmetric free oscillations of the fluid, about this state of equilibrium, which are relevant to certain problems in geomagnetism and oceanography. In particular we are interested in oscillations confined to the immediate vicinity of the equatorial plane—trapped oscillations—whose existence has been conjectured by Stern (1963) and Bretherton (1964).

As explained there the stream functions ψ of the meridional motion can be written in the form

$$\psi = f((y+\omega)^2 - 2z) + g((y-\omega)^2 - 2z) \quad (1.1)$$

where f, g are arbitrary functions, ω is the reduced frequency of the oscillations and is initially not known, z measures a scaled distance from the inner sphere and y is a scaled co-latitude. Thus ψ is a solution of the wave equation and the characteristics are members of two families (*a*) Γ_+, defined by $(y+\omega)^2 - 2z =$ const., and (*b*) Γ_-, defined by $(y-\omega)^2 - 2z =$ const. These characteristics are parabolas because of the choice of the coordinate system which defines them. In rectangular Cartesian coordinates they would be coaxial and similar cones. The boundary conditions are that the stream function vanishes on both spherical boundaries so that

$$\psi = 0 \quad \text{when} \quad z = 0,1. \quad (1.2)$$

For trapped waves we must choose ω so that

$$\psi = 0 \quad \text{when} \quad |y| > Y_0, \quad (1.3)$$

for some positive Y_0.

Bretherton conjectured that a trapped oscillation could occur if a ray-pattern of the characteristics passed through both of the points $(\omega, 0)$ and $(-\omega, 0)$ for it is then closed. Following up this idea, Stewartson (1971) [we shall subsequently refer to this paper as I] found an infinite number of trapped modes, each bounded externally by the Bretherton closed ray-pattern and the rays through any point converged after appropriate reflections, on to it, so that it may be regarded as a limit cycle. Further the pattern becomes more crowded as the limit cycle is approached and so the stream function becomes pathological in its immediate neighbourhood. The values of ω corresponding to these trapped waves have the asymptotic form

$$\omega^2 \sim \frac{3n}{2} - 0.81$$

where n is a large integer.

Earlier Israeli (1971), in a thesis kindly made available to the author, had demonstrated that the value of ω discovered by Bretherton is not isolated but is the upper limit of a continuous spectrum of ω's each of which permits a trapped oscillation. In this note we extend the study in I by investigating in like manner the properties of these more general oscillations.

2. The lowest continuous band

Suppose that there is a closed ray-pattern passing through the point $(y_1, 0)$ where $0 < y_1 < \omega$. A closed pattern is not possible if $y_1 > \omega$. The ray-pattern consists of Γ_+ and Γ_- characteristics alternately. Following a Γ_+ characteristic from $(y_1, 0)$ this meets $z = 1$ at $(y_2, 1)$ where

$$(y_2 + \omega)^2 - 2 = (y_1 + \omega)^2 \tag{2.1}$$

Now assume that $y_2 > \omega$ and that the Γ_- characteristic from $(y_2, 1)$ does not intersect $z = 0$. These conditions give the lowest modes and if the first is relaxed a higher value of ω is obtained. The second can be realised by choosing y_1 large enough. The Γ_- characteristic through $(y_2, 1)$ then meets $z = 1$ again at $(y_3, 1)$ where

$$y_3 = 2\omega - y_2 \tag{2.2}$$

The Γ_+ characteristic through $(y_3, 1)$ intersects $z = 0$ at $(y_4, 0)$ where

$$(y_4 + \omega)^2 = (y_3 + \omega)^2 - 2 \tag{2.3}$$

A closed ray-pattern is formed if $y_4 = -y_1$

$$\text{i.e. if } (\omega - y_1)^2 = (3\omega - y_2)^2 - 2 \tag{2.4}$$

On comparing (2.1) and (2.4) we find that $y_1 = 2(y_2 - \omega)$, where

$$y_2 = \omega + \tfrac{1}{3}[9\omega^2 - 6]^{\frac{1}{2}}, \quad y_1 = \tfrac{2}{3}[9\omega^2 - 6]^{\frac{1}{2}} \tag{2.5}$$

Thus a closed ray-pattern of this form is possible if

$$\sqrt{\tfrac{2}{3}} < \omega < \tfrac{2}{3}\sqrt{2} \tag{2.6}$$

The upper limit arises from the condition $y_1 < \omega$ and corresponds to the Bretherton limit cycle in which two characteristics touch the inner sphere (at $(\pm\omega, 0)$). The lower limit corresponds to a limit cycle in which a ray from $(\omega, 1)$ travels to the origin, is reflected to $(-\omega, 1)$ and then retraces its path. For other values of ω the closed cycle starts at the point $(y_1, 0)$ where y_1 satisfies (2.5) and $0 < y_1 < \omega$, so that there is a continuous spectrum of values of ω in the range defined by (2.6). A relaxation of the conditions imposed on y_1, y_2 and y_4, which were used to obtain (2.6), does not seem to lead to any lower value of $|\omega|$.

If $\omega = \frac{2}{3}\sqrt{2}$ it has already been shown in I that rays emanating from any point of the line $z = 0$ such that $|y| < y_1 = \omega$, remain within the limit cycle and ultimately converge onto it. Indeed at points on the z axis near the point $(z = 4/9)$ through which the limit cycle passes, the stream function and/or its normal derivative behaves like a periodic function of

$$X = \log\log\left(\frac{1}{4/9 - z}\right) \tag{2.7}$$

and is therefore pathological in the limit $z \to (4/9)-$. A similar argument shows that in the more general case rays emanating from the point $(y, 0)$, where $|y| < y_1$ and y_1 is defined by (2.5), ultimately converge onto the limit cycle and on the z axis the stream function and/or its normal derivative behaves like a periodic function of

$$X_\omega = \log\frac{1}{z_1 - z} \tag{2.8}$$

where $z_1 = \omega y_1 - \frac{1}{2}y_1^2$. Thus again the solution is pathological in the limit $z \to z_1-$, $(0, z_1)$ being on the limit cycle, but it is not so violent as when $\omega = \frac{2}{3}\sqrt{2}$. Further the length of the period in terms of X_ω is

$$\log\frac{4 + 3\omega\sqrt{9\omega^2 - 6}}{4 - 3\omega\sqrt{9\omega^2 - 6}},$$

diminishing from infinity at $\omega = \frac{2}{3}\sqrt{2}-$ to zero at $\omega = \sqrt{\frac{2}{3}}$. In fact at the lower limit the solution formally ceases to be pathological but since it is then confined to two curves it can be regarded at this stage as no more than a bizarre curiosity.

It is noted that one of the two rays emanating from the point $(y, 0)$ where $y > y_1$ also converged on to the limit cycle but the other described a pattern which continually moves away from the cycle and ultimately extends to $|y| = \infty$. Thus disturbances oscillating with a frequency defined by ω and originating from within the limit cycle are trapped there whereas those originating outside spread over the whole flow field. In both cases the disturbances become violent near the limit cycle.

3. The general continuous band

The results of the previous section may be generalised in a straightforward manner. We can expect that there are an infinite number of ranges of values of ω of which (2.6) is the first and the nth can be defined by

$$\Omega_n < \omega < \omega_n \tag{3.1}$$

and has the property that in the limit cycle associated with ω, $2n$ relections occur at the line $z=0$. The upper bound ω_n is given by the nth cycle discussed in I, and the associated limit cycle includes rays which pass through $(\pm\omega_n, 0)$ where one of the characteristics touches the line $z=0$. The lower bound of ω is that value which permits a ray starting at $(\Omega_n, 1)$ to pass through $(-\Omega_n, 1)$ after $(2n-1)$ reflections from the line $z=0$. A few of the smaller values of Ω_n have been kindly computed on behalf of the author by Miss S. M. Burrough and are set out below together with the corresponding values of ω_n reproduced for convenience, after very minor corrections, from I.

When n is large $\omega_n^2 \approx 3/2n - 0.81$ and so the lateral extent of the trapped modes increases without limit as $n \to \infty$. For large enough n the assumptions on which the theory is based need revision in consequence and indeed it seems that trapped modes can extend as far round the spherical shell as we choose. As a first step towards investigating the properties of the spectrum in the more general problem we shall determine the dependence of the width $(\omega_n - \Omega_n)$ of the continuous spectrum with n as $n \to \infty$. The method is to trace the variation of the distance between the rays of the limit cycle for $\omega = \omega_n$ and that for $\omega f = \Omega_n$ as a function of the number of reflections or, equivalently, as a function of y. Since we know the distance between the two rays near $y=\omega$ and $y=-\omega$ we are able to find the value of $\omega_n - \Omega_n$.

We omit the suffix n and write

$$\Omega = \omega + \delta \tag{3.2}$$

and suppose that $\omega \gg 1$, $0 < -\delta < 1$.

Table I

n	Ω_n	ω_n	$-\delta = \omega_n - \Omega_n$	$\left(\frac{-3}{16\delta}\right)^{\frac{1}{4}} - \omega_n$
1	0.8165	0.9428	0.1263	0.198
2	1.4606	1.5004	0.0399	0.176
3	1.9072	1.9275	0.0202	0.172
4	2.2680	2.2807	0.0127	0.170
5	2.5787	2.5876	0.0090	0.170

For ω the limit cycle starts at $(\omega, 0)$ and following it round in a counter-clockwise sense, the successive intersections with the boundaries are at $(y_{-1}, 1)$ $(y_0, 1)$ $(\bar{y}_0, 0)$, $(y_1, 1)$ where

$$y_{-1} = \omega + \frac{1}{2\omega},\ y_0 = \omega - \frac{1}{2\omega},\ \bar{y}_0 = \omega - \frac{1}{\omega},$$

$$y_1 = \omega - \sqrt{2} - \frac{1}{2\omega^2\sqrt{2}} \tag{3.3}$$

higher powers of ω^{-1} being neglected in each case. Subsequent intersections with $z=1$ are defined to occur at $(y_r, 1)$ $(r=2, 3, 4, \ldots)$ and with $z=0$ to occur at $(\bar{y}_r, 0)$ $(r=1, 2, 3, \ldots)$.

For $\omega+\delta$, the limit cycle starts at $(Y_0, 1)$ where $Y_0 = \omega+\delta$ and we define subsequent intersections with $z=1$ to occur at $(Y_r, 1)$ $(r=1, 2, \ldots)$ and with $z=0$ to occur at $(\overline{Y}_r, 0)$ $(r=0, 1, 2, \ldots)$. Thus

$$\overline{Y}_0 = \omega+\delta - \frac{1}{2\omega},\ Y_1 = \omega+\delta - \frac{1}{8\omega^2\sqrt{2}} \tag{3.4}$$

higher powers of ω^{-1} being neglected. Of special interest is

$$\Delta_r = Y_r - y_r \quad \text{and} \quad \overline{\Delta}_r = \overline{Y}_r - \bar{y}_r \tag{3.5}$$

with

$$\Delta_1 = \delta + \frac{3}{8\omega^2\sqrt{2}},\ \Delta_0 = \delta + \frac{1}{2\omega} \tag{3.6}$$

Now consider the change in Δ_r as r increases by unity from a general intersection point of the ray patterns with $z=1$. Provided $|y| < \omega$, it may be shown that

$$\overline{\Delta}_r + \delta \approx (\Delta_r + \delta)\frac{\omega + y_r}{\omega + \bar{y}_r}$$

$$\approx (\Delta_r + \delta)(1 + (\omega + y_r)^{-2}) \tag{3.7}$$

since $y_r - \bar{y}_r = (y_r + \omega)^{-1}$. Similarly

$$\Delta_{r+1} - \delta \approx (\bar{\Delta}_r - \delta)(1 - (\omega - y_r)^{-2}) \qquad (3.8)$$

It follows that

$$\Delta_{r+1} - \Delta_r \approx \Delta_r \frac{-4\omega y_r}{(\omega^2 - y_r^2)^2} + \frac{2\delta(\omega^2 + y_r^2)}{(\omega^2 - y_r^2)^2} \qquad (3.9)$$

while the associated change in y is

$$y_{r+1} - y_r = -\frac{2\omega}{\omega^2 - y_r^2} \qquad (3.10)$$

We may now set up a differential equation for Δ_r as a function of y_r, and we obtain, on dropping the suffices,

$$\frac{d\Delta}{dy} = \frac{2\Delta y}{\omega^2 - y^2} - \frac{\delta(\Delta^2 + y^2)}{\omega(\omega^2 - y^2)} \qquad (3.11)$$

with solution

$$(\omega^2 - y^2)\Delta = -\frac{\delta}{\omega}(\omega^2 y + \tfrac{1}{3}y^3) + A \qquad (3.12)$$

where A is a constant. From the formula for Δ_1 in (3.6) and (3.3) it follows that

$$\frac{3}{4\omega} = A - \frac{4\omega^2}{3}\delta \qquad (3.13)$$

Near $y = -\omega$ the ω-limit cycle passes through $(-\omega + \sqrt{2},\ 1)$ and the Ω-limit cycle passes through $(-\omega - \delta + \sqrt{2} + (8\omega^2\sqrt{2})^{-1},\ 1)$ so that Δ is then equal to $-\delta + (8\omega^2\sqrt{2})^{-1}$. Hence

$$\frac{1}{4\omega} = A + \frac{4\omega^2}{3}\delta \qquad (3.14)$$

i.e.

$$A = \frac{1}{2\omega},\ \delta = -\frac{3}{16\omega^3} \qquad (3.15)$$

Table 1 also includes a set of values of $(-3/16\delta)^{\frac{1}{3}} - \omega_n$ which shows a clear sign of approaching a constant limit as $n \to \infty$.

4. Discussion

As a result of the studies made in I, it was conjectured that all axially symmetric oscillations of the fluid in a thin rotating shell might be trapped in some sense and as an alternative that their pathological structure means that *none* is physically acceptable.

The continuous spectrum whose existence was discovered by Israeli (1971) and whose properties are examined here greatly complicate the physical interpretation of trapped oscillations. First we now know that attached to each of the frequencies discussed earlier there is a continuous spectrum of frequencies for each value of which the associated trapped mode is at least as acceptable, in a physical sense. They are all pathological near their respective limit cycles but their structure implies slightly smaller velocities and at present we are unable to devise an argument for preferring one mode in each range (3.1) to any of the others. Second it would be very surprising if they were all acceptable since it is known (Malkus, 1967) that in a complete sphere there are only a denumerable infinity of modes and the implication of the present study is that in a shell there is a continuum of modes. Israeli has also carried out a numerical integration of the equations governing the motion of a slightly viscous fluid in a thin shell under a particular forced motion and noted a weak resonance near $\omega = \omega_1$. The fluid motion reported showed some signs of trapping but otherwise bore little resemblance to the inviscid motions discussed here. Further studies are clearly needed to resolve the dilemma and difficulties revealed in the theory of trapped oscillations and we report one of them—on the effect of viscosity—elsewhere (Stewartson, 1972).

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СПЕКТР, НЕПРЕРЫВНЫЙ В ДИСКРЕТНЫХ ИНТЕРВАЛАХ, ДЛЯ ЗАХВАЧЕННЫХ КОЛЕБАНИЙ

Стюартсон (1971) детально изучил структуру свободных колебаний в сферической оболочке, захваченных вблизи экваториальной плоскости, в предположении, что колебания ограничены лучами, касающимися внутренней сферы. В настоящем исследовании показано, что каждая мода представляет собой предельный случай непрерывного спектра захваченных мод подобного типа за исключением случая, когда ограничивающие лучи отражаются от внутренней сферы, а не задевают ее. Другим крайним представителем семейства является возмущение, сосредоточенное на лучевой поверхности. Все представители семейства обладают особенностями вблизи ограничивающей поверхности.