On the hydrographic response to transient meteorological disturbances

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ABSTRACT

The response to transient forcing in a hydrographic system like the Baltic is considered. We find that under quite general conditions the baroclinic response is confined to a narrow coastal region while the response in the main part of the basin is essentially barotropic. The response is analysed quantitatively under somewhat more stringent conditions using a boundary layer type approach. We find that the barotropic large scale low frequency motions follow contours of constant depth, a feature which might be accessible to empirical verification. The baroclinic coastal response is found to be confined to a region of width 5–10 km (in the Baltic). If the dissipation is sufficiently weak, internal Kelvin waves travelling in a counter clockwise direction may be excited in the coastal region.

1. Introduction

In this paper we consider the hydrographic response in a closed basin to time-dependent forcing caused by synoptic meteorological disturbances. The hydrographic system is assumed to be similar in size to the Baltic. Throughout we have tried to focus the attention on the physically most dominating large scale features, and the results, although rough and sometimes even qualitative in character, are applicable to comparatively realistic types of topography and forcing. To achieve this we make relatively few *a priori* assumptions but try to derive the various permissible simplifications with scale analysis of the general system subject to a general type of forcing. Although the approximations made are sometimes drastic, we have tried to arrive at a mathematically consistent lowest order picture, which might serve as the basis for future refinements.

1.1. *Outline of approach and main results*

In section 2 a linearized set of equations is introduced and cast into a form suitable for the later developments. The conditions for the linearization to be valid are left to section 7, since they depend on the properties of the response.

In section 3 we consider the properties of a solution to the basic equations with prescribed horizontal scale, which is generated directly by the wind and pressure systems acting on the surface of the basin. For horizontal scales of order the width of the basin we find that such a solution must to a first approximation be barotropic and, unless the slope of the bottom is much smaller than the mean slope, the associated motion will be along the contours of constant depth. In subsection 3.1 the problem of predicting this flow is reduced to a single relatively simple differential equation (3.14) in two independent variables (time and one space variable).

The large scale barotropic response discussed in section 3 satisfies the complete inhomogeneous boundary conditions on the surface of the basin. However, as discussed in subsection (4.1), baroclinic forcing will in general appear in the lateral boundary condition, primarily caused by the wind-transport perpendicular to the coast. Thus, the complete solution must have a baroclinic part at least close to the coast.

In section (4.2) we conclude from the nondimensional form of our basic equations that a baroclinic part of the solution necessarily has a much smaller horizontal scale than the basin itself. This suggests a boundary layer approach; i.e. we assume that, besides the interior baro-

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tropic response, we have a baroclinic component which is nonzero only in a relatively narrow region along the coast. From this assumption and the general scaling conditions adopted we derive the equations for the baroclinic "boundary layer" component of the solution.

In section 5 we derive the separate lowest order boundary conditions that should be imposed on the two parts of the solution. The basis for this derivation is that (i) the complete boundary conditions should be satisfied by the two parts of the solution; (ii) each separate set of boundary conditions must be consistent with the corresponding set of lowest order equations.

If the system is internally consistent, (i) and (ii) should give a unique answer to how the boundary conditions should be separated. Being successful with this procedure gives us some confidence in the "boundary layer" approach to the complete problem. We must admit, however that the quantitative treatment of especially the lateral boundary condition (being dependent on mixing in the vicinity of the coast) is far from satisfactory. The width of the coastal region in which baroclinicity is important is found to be of the order of 10 km.

In section 6 some properties of the baroclinic coastal response are discussed. Ignoring dissipation and forcing, we find free internal Kelvin waves travelling in a counterclockwise direction around the basin. If the influence of dissipation is modest the amplitude of waves moving with speed near the Kelvin wave velocity will dominate the response. In general, one might say that there is a one-way transport of information along the coast.

In section 7 finally we discuss the range of validity of the theoretical predictions and associated hydrographical implications.

1.2 *Previous work*

The general quasi-geostrophic equations governing small amplitude time-dependent motion in the ocean were first derived and discussed by Charney (1955). Charney essentially applied results valid for atmospheric motions, developed by himself and others, to the ocean. Some of the results we shall derive here were touched upon by Charney. In particular he discussed the possibility that transient forcing might create coastal baroclinic "jetstreams".

The subject was further developed by Veronis & Stommel (1956) who considered the response of a stratified ocean to a periodic wind field. They discussed in great generality the partition of energy between the various possible types of wave motion but considered only open ocean conditions not influenced by coasts (or topography).

Several authors have discussed time-dependent barotropic motions in the vicinity of a coast, in particular so-called edge waves. For example Reid (1958) discussed such waves on a sloping continental shelf in great detail. The corresponding baroclinic phenomena have been discussed only recently and then notably by Csanady, in an important series of papers (1967, 1968 I and II, 1971), with emphasis on large scale motions in the Great Lakes. These lakes have many features in common with the Baltic and a part of Csanady's work is closely related to the subject of the present paper. Thus, Csanady as well as the present author obtains baroclinic coastal jet streams (or large amplitude Kelvin waves in our formulation) as a dominating feature of the response to a variable wind-stress. The approach of Csanady is, however, strikingly different from the one used here. In the present paper we have tried to deduce results for a general type of system through the use of scale analysis and approximate mathematical methods such as boundary layer analysis. Csanady used the opposite approach and specified a highly idealized system (i.e. a two layer system with constant total depth in a basin of circular shape etc.) and made this the object of an elaborate mathematical analysis. It is believed that both types of approach are much needed to lay the foundation for quantitative predictions of the behaviour of real hydrographic systems.

2. The governing equations

We are interested in small, time-dependent deviations from a state of rigid rotation and stable stratification in a fluid mass of the type represented by the Baltic Sea. As our starting point we take the following linearized set of equations which are supposed to govern the motion in the interior of the fluid region, i.e. everywhere except in the viscous boundary layers close to the top and bottom of the water body.

$$\varrho_m\left(\frac{\partial u}{\partial t}-fv\right)=-\frac{\partial p}{\partial x} \tag{2.1 a}$$

$$\varrho_m\left(\frac{\partial v}{\partial t}+fu\right)=-\frac{\partial p}{\partial y} \tag{2.1 b}$$

$$0=-\frac{\partial p}{\partial z}-qg \tag{2.1 c}$$

$$\frac{\partial q}{\partial t}+w\frac{\partial \varrho_s}{\partial z}=0 \tag{2.1 d}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \tag{2.1 e}$$

$$0=-\frac{\partial p_s}{\partial z}-(\varrho_m+\varrho_s(z))g \tag{2.1 f}$$

where (u, v, w) is the velocity vector in a cartesian coordinate system (x, y, z) oriented in such a way that the vector form of the gravitational acceleration becomes $(0, 0, -g)$, p and q are deviations from the basic pressure and density distributions $p_s(z)$ and $\varrho_m+\varrho_s(z)$, while f is the local coriolis parameter i.e. twice the vertical component of the angular velocity of the earth. The assumptions behind equations (2.1) are:

(i) The scale of the motion is small compared with the radius of the earth, making it possible to neglect the curvature of the geopotential surfaces (i.e. the so-called β-effect).

(ii) The horizontal scale L of the motion is much larger than the depth scale H_0, which gives rise to the hydrostatic approximation (2.1*c*).

(iii) The variation of the density is much smaller than the mean density, which justifies the replacement of the real density $\varrho=\varrho_m+\varrho_s+q$ with ϱ_m in (2.1a and b).

(iv) The time scale is small enough to permit that the dissipative terms due to viscosity and heat conduction in the interior of the system are neglected.

(v) The forcing is sufficiently weak for the linearization to be permissable (see section 7.1).

The derivation of (2.1) follows an expansion procedure quite similar to the one given by Walin (1969) and will not be presented here.

Eliminating u and v from (2a, b and e) and q from (2.1c and d) we obtain

$$\left(\frac{\partial^2}{\partial t^2}+f^2\right)\frac{\partial w}{\partial z}=\frac{1}{\varrho_m}\frac{\partial}{\partial t}\nabla_1^2 p \tag{2.2 a}$$

$$g\frac{\partial \varrho_s}{\partial z}w=\frac{\partial}{\partial t}\frac{\partial p}{\partial z} \tag{2.2 b}$$

where ∇_1^2 denotes the operator

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$$

The properly linearized form of the boundary condition at $z=0$ may be found from the condition at the free surface combined with the hydrostatic equation (2.1c). We obtain the following condition to be applied at the fixed level $z=0$

$$\frac{\partial}{\partial t}(p-p_a)=g\varrho_m(w+w_\tau) \quad \text{at } z=0 \tag{2.3 a}$$

At the bottom we obtain the condition

$$w+w_E+u\frac{\partial H}{\partial x}+v\frac{\partial H}{\partial y}=0 \quad \text{at } z=-H(x,y) \tag{2.3 b}$$

where $H(x, y)$ is the depth of the basin, $p_a(x, y, t)$ is the air pressure above the water surface, while w_τ and w_E are the boundary layer contributions to the vertical velocity at $z=0$ and $z=-H$, created by windstress and bottom friction respectively. Note that we are thinking in terms of an "additive approach". Thus the total velocity field is $\mathbf{v}+\mathbf{v}_\tau+\mathbf{v}_E$ where $\mathbf{v}$ is non-zero (in general) in the whole region $0 \geqslant z \geqslant -H$, while $\mathbf{v}_\tau$ and $\mathbf{v}_E$ are non-zero only close to $z=0$ and $z=-H(x, y)$ respectively. Equations (2.1) describe only the "inviscid" velocity $\mathbf{v}$ with components (u, v, w). The "real" vertical velocity at $z=0$, for example, is thus given by $w(z=0)+w_\tau$ where w_τ is the vertical component of $\mathbf{v}_\tau$ evaluated at $z=0$.

For the application of the bottom boundary condition to equations (2.2) we need expressions for u and v in terms of p. From (2.1) we easily obtain:

$$\left(\frac{\partial^2}{\partial t^2}+f^2\right)u=-\frac{1}{\varrho_m}\left(f\frac{\partial p}{\partial y}+\frac{\partial^2 p}{\partial x\,\partial t}\right) \tag{2.4 a}$$

$$\left(\frac{\partial^2}{\partial t^2}+f^2\right)v=-\frac{1}{\varrho_m}\left(-f\frac{\partial p}{\partial x}+\frac{\partial^2 p}{\partial y\,\partial t}\right) \tag{2.4 b}$$

2.1 *Nondimensional equations*

Let us now introduce nondimensional variables according to the following transformations:

$$(x, y, z, t) = (Lx', Ly', H_0 z', Tt'), \qquad (2.5\,a)$$

$$(p, q, u, v, w) = (P_0 p', Q_0 q', U_0 u', U_0 v', W_0 w'), \qquad (2.5\,b)$$

$$(\varrho_s, H, p_a, w_\tau, w_E) = (Q_s \varrho_s', H_0 H', P_0 p_a', W_0 w_\tau', W_0 w_E'). \qquad (2.5\,c)$$

Insertion in (2.2 a) yields

$$\left(\delta \frac{\partial^2}{\partial t^2} + 1\right)\frac{\partial w}{\partial z} = \frac{P_0}{W_0}\,\delta\,\frac{H_0}{f\varrho_m L^2}\,\frac{\partial}{\partial t}\nabla_1^2 p \qquad (2.6)^*$$

where we have dropped the primes. Instead, we indicate the use of nondimensional variables with an asterisk on the equation number.

The parameter δ, measuring the relative importance of local and coriolis acceleration, is defined by

$$\delta = (Tf)^{-1} \qquad (2.7)$$

We are only interested in time scales larger than the inertial period, f^{-1}, i.e. we require at least

$$\delta < 1 \qquad (2.8)$$

This means that in general the coefficient on the right hand side of (2.6)* has to be of order unity for (2.6)* to have nontrivial time-dependent solutions.

Thus, a consistent choice of W_0/P_0 is given by

$$W_0/P_0 = \delta H_0 / f\varrho_m L^2 \qquad (2.9\,a)$$

Similarly we obtain from (2.4a or b)

$$U_0/P_0 = (f\varrho_m L)^{-1} \qquad (2.9\,b)$$

With the aid of (2.9) we obtain the following nondimensional form of equations (2.2), (2.3) and (2.4).

$$\left(\delta^2 \frac{\partial^2}{\partial t^2} + 1\right)\frac{\partial w}{\partial z} = \frac{\partial}{\partial t}\nabla_1^2 p \qquad (2.10\,a)^*$$

$$\left(\frac{C_B}{fL}\right)^2 \frac{\partial \varrho_s}{\partial z} w = \frac{\partial}{\partial t}\frac{\partial p}{\partial z} \qquad (2.10\,b)^*$$

$$\left(\delta^2 \frac{\partial^2}{\partial t^2} + 1\right) u = -\left(\frac{\partial p}{\partial y} + \delta \frac{\partial^2 p}{\partial x\,\partial t}\right) \qquad (2.10\,c)^*$$

$$\left(\delta^2 \frac{\partial^2}{\partial t^2} + 1\right) v = -\left(-\frac{\partial p}{\partial x} + \delta \frac{\partial^2 p}{\partial y\,\partial t}\right) \qquad 2.10\,d)^*$$

$$\frac{\partial}{\partial t}(p - p_a) = \left(\frac{C_0}{fL}\right)^2 (w + w_\tau) \quad \text{at } z = 0 \qquad (2.11\,a)^*$$

$$\delta(w + w_E) = -\left(u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y}\right) \quad \text{at } z = -H \qquad (2.11\,b)^*$$

C_B and C_0 in equation (2.10b)* and (2.11a)* are defined by

$$C_B = \sqrt{(Q_s/\varrho_m)\,gH_0}, \quad C_0 = \sqrt{gH_0} \qquad (2.12\,a, b)$$

Thus C_B and C_0 are of order the internal and external long wave velocities respectively.

3. The response in the main part of the basin

The character of the solution to equations (2.10)* and (2.11)* is strongly influenced by the relative as well as the absolute magnitudes of the parameters C_B/fL, C_0/fL and δ. In the Baltic the stratification is characterized by a seasonal thermocline (depth ~ 20 m and temperature vaiation $\sim 10°$C) and a permanent halocline (depth ~ 50–100 m and salinity change ~ 4 ‰). The density variation between the top and the bottom is of order $2 \cdot 10^{-3}$ and the mean depth is ~ 60 m. Thus, for the Baltic we have

$$Q_s/\varrho_m \sim 2 \cdot 10^{-3},\ H_0 = 60\ \text{m},\ C_0 \sim 25\,m/s,\ C_B \sim 1\,m/s$$

Furthermore, the horizontal dimensions of the Baltic are of order 200 km and the coriolis parameter $f \sim 10^{-4} s^{-1}$. Thus we obtain

$$C_0/fL \sim 1, \quad C_B/fL \sim 0.05$$

Consequently it seems reasonable to study the properties of our system on the basis of the assumptions:

$$C_B/fL \ll 1, \quad C_0/fL \sim 1, \qquad (3.1)$$

where L represents the horizontal dimension of the main part of the basin.

If we require that the characteristic scales of the motion are those used in the nondimensionalization (i.e. that the horizontal scale is not very much smaller than L) and by making use of (3.1), we obtain from (2.10b)* the lowest approximation:

$$\frac{\partial}{\partial t}\frac{\partial p}{\partial z}=0$$

and for the time-dependent part in which we are interested

$$\frac{\partial p}{\partial z}=0 \tag{3.2)*}$$

Thus the response to forcing, with the scaling characteristics we have imposed (i.e. (2.8) and (3.1)) is barotropic. From (3.2)* and (2.10a)* it now follows that $\partial w/\partial z$ must be independent of z. Consequently w is completely determined from the boundary conditions and we have

$$\frac{\partial w}{\partial z}=\frac{1}{H(x,y)}[w(z=0)-w(z=-H)] \tag{3.3)*}$$

Introducing the boundary condition (2.11)* in (3.3)* and eliminating w from (2.10 a)* we obtain

$$\left(\delta^2\frac{\partial^2}{\partial t^2}+1\right)\left[\left(\frac{c_0}{fL}\right)^{-2}\frac{\partial}{\partial t}(p-p_a)-(w_\tau-w_E)+\right.$$
$$\left.+\delta^{-1}\left(u\frac{\partial H}{\partial x}+v\frac{\partial H}{\partial y}\right)\right]=$$
$$=H(x,y)\frac{\partial}{\partial t}\nabla_1^2 p \tag{3.4)*}$$

In the general case when $\delta=0(1)$ we cannot proceed very much further analytically because of the complicated coupling between (3.4)* and (2.10c and d)*. The response consists of a complicated pattern of barotropic inertio-gravitational waves scattered around by the topography. However, the most important part of the forcing is definitely on time-scales large compared with the inertial period i.e. approximately contained within the assumption

$$\delta<1 \tag{3.5}$$

In view of (3.1) this means that (3.4)* to lowest order in δ reduces to the constraint

$$u\frac{\partial H}{\partial x}+v\frac{\partial H}{\partial y}=0 \tag{3.6)*}$$

which, by the way, could have been obtained directly from the boundary condition (2.11b)*. It should be emphazised that to obtain (3.6)* we must of course require $\nabla H\gtrsim 1$. Since H has been scaled with the mean depth this amounts to a requirement that the slope of the bottom is at least of order H_0/L; a very modest requirement.

Thus we conclude that if the forcing varies sufficiently slowly for (3.5) to be satisfied, the main part of the motion will be along the depth contours. To lowest order, therefore, the flow may be described as a function of time and one suitably chosen space variable, which should have a unique value on each depth contour. It is now possible to attack equation (3.4)* with an expansion procedure in δ. The transformation to a single suitable space variable is then achieved through an integration over the horizontal surface inside a particular closed depth contour. In the next subsection however we will follow a somewhat more direct and, possibly, more illuminating procedure.

3.1. *The prognostic equation for the barotropic response when $\delta<1$*

The following relation for the rate of change of the circulation around any horizontal closed contour is easily obtained from (2.1a and b)

$$\frac{\partial}{\partial t}\oint_S u_l\,dl=-f\oint_S u_n\,dl \tag{3.7}$$

where $\oint$dl represents a line integral around the closed contour S, u_l and u_n are the horizontal velocities parallel to and perpendicular to S; u_n is defined positive inwards and u_l in a clockwise direction. To lowest order in δ we obtain from (2.10c and d)* the geostrophic relation (here written in dimensional form).

$$u_l=\frac{1}{f\varrho_m}\frac{\partial p}{\partial n} \tag{3.8}$$

where $\partial p/\partial n=\nabla p\cdot\mathbf{n}$, and $\mathbf{n}$ is the inward normal to S.

Inserting (3.8) in (3.7) we obtain

$$\frac{\partial}{\partial t}\oint_S\frac{\partial p}{\partial n}dl=-f^2\varrho_m\oint_S u_n\,dl \tag{3.9}$$

Considering mass continuity for the region inside the vertical cylinder through S and also assuming S to coincide with a line of constant depth, we obtain

$$-H\oint_S u_n\,dl - \oint_S (\mathbf{M}_\tau + \mathbf{M}_E)\cdot\mathbf{n}\,dl + \\ + \int_A (w(z=0)+w_\tau)\,dA = 0 \qquad (3.10)$$

where $\mathbf{M}_\tau$ and $\mathbf{M}_E$ are the boundary layer transports created by wind-stress and bottom friction respectively and $\int_A dA$ represents an integral over the horizontal surface A enclosed by S.

Expressing $w(z=0)+w_\tau$ in terms of p by use of (2.3 a) and eliminating $\oint u_n\,dl$ from (3.9) and (3.10) we obtain

$$\frac{\partial}{\partial t}\left[gH\oint_S \frac{\partial p}{\partial n}dl + f^2\int_A (p-p_a)\,dA\right] = \\ = g\varrho_m f^2 \oint_S (\mathbf{M}_\tau + \mathbf{M}_E)\cdot\mathbf{n}\,dl \qquad (3.11)$$

Let us now define a new independent variable h such that $h = h(x, y)$ has a unique constant value on each contour of constant depth but is otherwise unspecified. (A possible choice of $h(x, y)$ would be a function having at each point a value equal to the area enclosed by the depth contour running through the point.) In view of (3.8) the result expressed by (3.6)*, which is valid for small δ, and the more general result (3.2)*, may now be summerized as

$$p = p(h, t) \qquad (3.12)$$

Furthermore, from the definition of h we have

$$A = A(h),\; H = H(h)$$

We also need an expression for the transport $\mathbf{M}_E$ caused by bottom friction. In accordance with the Ekman boundary layer theory (Ekman, 1905), (applicable when $\delta \ll 1$) we have

$$\mathbf{n}\cdot\mathbf{M}_E = -\frac{\varepsilon}{2}H_0 u_l = -\frac{\varepsilon H_0}{2\varrho_m f}\frac{\partial p}{\partial n} \qquad (3.13)$$

where ε is a nondimensional parameter characterizing the bottom boundary layer. (According to the Ekman theory $\varepsilon H_0 \sim$ the thickness of the boundary layer. Since, by assumption, the top and bottom boundary layers are thin compared with H_0, we necessarily have $\varepsilon \ll 1$.) ε is related to the "effective" kinematic viscosity ν through the relation $\varepsilon H_0 = \sqrt{2\nu/f}$. As usual, a good deal of arbitrariness is involved when choosing a numerical value for ν.

Introducing h as independent variable in (3.11), making use of (3.13), and differentiating once with respect to h, we obtain after some manipulation

$$\frac{\partial}{\partial t}\left[\frac{\partial}{\partial h}a(h)\frac{\partial p}{\partial h} + b(h)\,p\right] + \frac{\partial}{\partial h}c(h)\frac{\partial p}{\partial h} = g(h, t) \qquad (3.14\,\text{a})$$

where a, b and c are known functions of h defined by

$$a(h) = gH(h)\oint_S \frac{\partial h}{\partial n}dl \qquad (3.14\,\text{b})$$

$$b(h) = f^2\frac{d}{dh}A(h) \qquad (3.14\,\text{c})$$

$$c(h) = fgH_0\frac{\varepsilon}{2}\oint_S \frac{\partial h}{\partial n}dl \qquad (3.14\,\text{d})$$

and $g(h, t)$ is a forcing function determined by $\mathbf{M}_\tau$ and p_a according to

$$g(h, t) = f^2\frac{\partial}{\partial h}\int_A \frac{\partial p_a}{\partial t}dA + \frac{\partial}{\partial h}g\varrho_m f^2\oint_S \mathbf{M}_\tau\cdot\mathbf{n}\,dl \qquad (3.14\,\text{e})$$

A solution to (3.14) is determined by

(i) an initial condition on p,

(ii) a regularity condition on p at points of extreme depth (where $\oint \partial h/\partial n dl = 0$ and therefore (3.14) degenerates),

(iii) a condition on p at the lateral boundary of the region, and

(iv) information on the forcing functions $\mathbf{M}_\tau$ and p_a.

Evidently the lateral boundary (S^B) of the region must be chosen as a vertical surface along a contour of constant depth. Furthermore, since the scale analysis leading to (3.14) requires that the depth is everywhere of order H_0, S^B must be located at a certain distance from the coast, as illustrated in Fig. 1. Applying mass continuity to the region R^B between S^B and the coastline we obtain

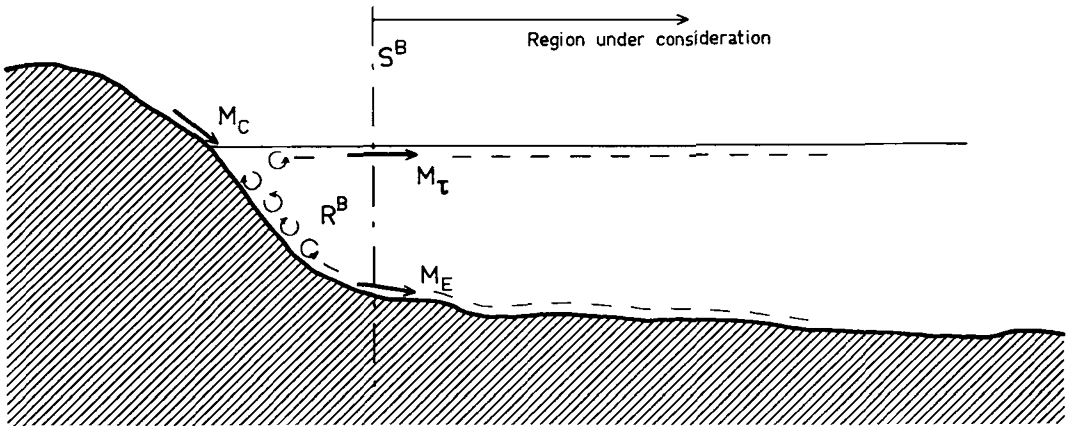

Fig. 1. Illustration of the position of the artificial lateral boundary S^B. The small curved arrows illustrate the mixing in the near shore region which influences the vertical distribution of the "inviscid" flow through S^B

$$\oint_{S^B} (M_\tau + M_E + H^B u_n - M_c)\, dl = 0 \quad \text{at } S^B \qquad (3.15)$$

where $M_\tau = \mathbf{M}_\tau \cdot \mathbf{n}$, $M_E = \mathbf{M}_E \cdot \mathbf{n}$, H^B is the depth at S^B, and M_c is the discharge from the coast per unit length of coastline. In deriving (3.15) we have ignored the influence of the free surface, an approximation which is valid if R^B covers only a small fraction of the area of the basin.

Making use of (3.13) and introducing h as independent variable we obtain from (3.15) the the required boundary condition on p

$$\left(\frac{\partial}{\partial t} + \frac{\varepsilon}{2}\frac{H_0}{H^B} f\right)\frac{\partial p}{\partial h} = N(t) \quad \text{at } S^B \qquad (3.16\,\text{a})$$

where

$$N(t) = \left(H^B \oint_{S^B} \frac{\partial h}{\partial n}\, dl\right)^{-1} f^2 \varrho_m \oint_{S^B} (M_\tau - M_c)\, dl \qquad (3.16\,\text{b})$$

We can see that (3.16) depends on the total flux $\oint (M_\tau - M_c)\, dl$ We thus conclude that different distributions of $(M_\tau - M_c)$ along the coast give rise to identical pressure fields as long as $\oint (M_\tau - M_c)\, dl$ is the same. To avoid misunderstanding we emphasize that the distribution of the velocity component u_n (which carries the flux $(M_\tau - M_c)$ into the basin) nevertheless depends on how $(M_\tau - M_c)$ is distributed along the coast. Since $u_n < u_l$, the associated differences in the pressure field are of second order and thus do not show up in (3.14) and (3.16) which describe the pressure field to lowest order only.

4. A boundary layer approach to the coastal region

When deriving the boundary condition at the lateral boundary S^B in section 3, we considered only the total mass flux through S^B and ignored that we are dealing with a stratified water body. From the more complete discussion of the lateral boundary condition given below, it will become clear that a realistic boundary condition at S^B in general requires a baroclinic response, in contradiction with the analysis of section 3 leading to a barotropic lowest order pressure field. To cope with this situation we will try a boundary layer type solution to the problem formulated in section 2. Thus, we will assume that the solution consists of a barotropic part, being a solution to (3.14), and a baroclinic part non-zero only in a narrow region close to the coasts. We will begin, however, with a discussion of how a physically reasonable lateral boundary condition should be formulated.

4.1. *The lateral boundary condition*

We will now extend the discussion of the boundary condition at S^B to include the vertical distribution of the flow through S^B.

Suppose there is a water supply M^B (per unit length of coastline) with density ϱ^B to the region R^B (see Fig. 1). (M^B may be identified with any one of the fluxes $-M_\tau$, $-M_E$ or M_c which are *not* included in the "inviscid" velocity u_n.)

Since we are dealing with a stratified waterbody this essentially means a supply of water to a layer where $\varrho_s \simeq \varrho^B$. Because of mixing with water of other densities this supply is necessarily distributed over a certain range of densities, i.e. over a layer of finite thickness. To create this water with density slightly different from ϱ^B, some water from other levels will in general be consumed, i.e. there will occur a slight withdrawal of water outside the layer where the main supply occurs. If $M^B < 0$, i.e. when dealing with a withdrawal of water of a certain density (or from a certain level), the discussion of the influence of mixing needs some modification. The qualitative result is however similar; the effective withdrawal is from a layer of finite thickness.

If R^B is sufficiently narrow the supply M^B will create a flux out of R^B with a density distribution similar to that of the supply. Ignoring transports parallel to the coast in R^B (which is also justified if R^B is "narrow") we thus obtain

$$u_n = M^B f^B_{(z)} \quad \text{at } S^B$$

where

$$\int_{-H^B}^{0} f^B \, dz = 1,$$

and $f^B_{(z)}$ is small everywhere except close to the depth where $\varrho_s = \varrho^B$.

From considerations *a posteriori* it may be shown that this expression for u_n is valid if the width of the region R_B is small compared with the width of the region in which the density field is perturbed (i.e. the width of the baroclinic boundary layer discussed in 4.2). Since this condition is often poorly satisfied and since the mixing conditions in the near shore region are not wellknown, our knowledge of the boundary condition on S^B will always be approximate. In particular, any results depending on the small scale features of the boundary condition should not be taken too seriously.

The flux $-M_\tau$ or $-M_E$ may be considered as equivalent to a supply M^B with ϱ^B equal to $\varrho_s(z=0)$ and $\varrho_s(z=-H^B)$ respectively. For simplicity we assume that the discharge from the coast has a density close to the surface density $\varrho_s(z=0)$. We thus obtain

$$u_n = (M_c - M_\tau) f_1(z) + (-M_E) f_2(z) \quad \text{at } \xi = 0 \qquad (4.1)$$

where

$$\int_{-H^B}^{0} f_1(z)\, dz = \int_{-H^B}^{0} f_2(z)\, dz = 1,$$

while $f_1(z)$ and $f_2(z) \ll (H^B)^{-1}$ except close to $z=0$ and $z=-H^B$ respectively and ξ is a coordinate measuring distance from S^B along $\mathbf{n}$ (see figure 3).

The qualitative features of u_n with weak and

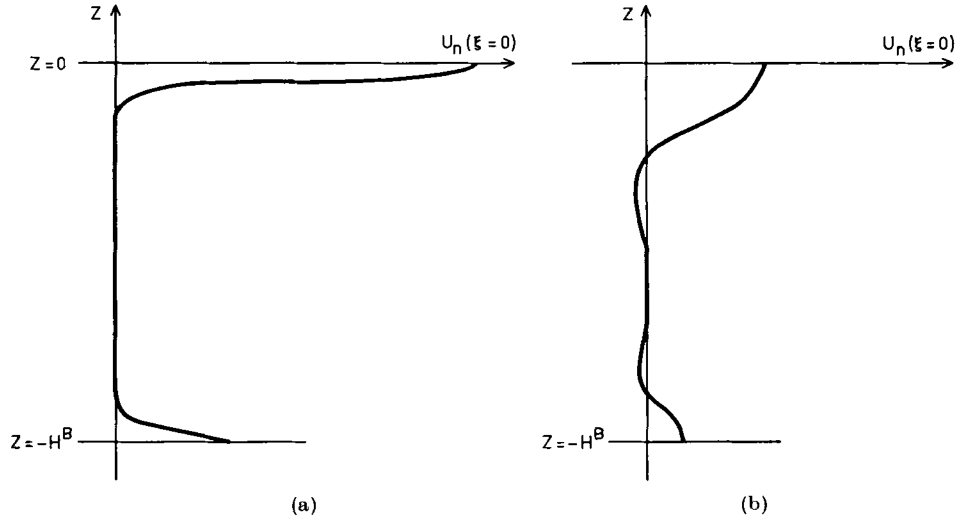

Fig. 2. (*a*) Illustration of the vertical distribution of the inviscid flow perpendicular to the lateral boundary S^B caused by the boundary layer fluxes (M_τ, M_E) and the discharge M_c. In the case illustrated we have $(M_c - M_\tau, -M_E) > 0$. (*b*) Same as (*a*) but with somewhat more mixing in the near shore region. Note the return flows at mid depth required by heat (or salt) continuity.

more intense nearshore mixing is illustrated in Figs. 2*a* and *b*.

4.2. *The baroclinic "boundary layer" equations*

Let us now assume that the solution to equations (2.10)* and (2.11)* are built up of two parts according to

$$\varphi = \varphi^I + \varphi^c, \tag{4.2a}$$

where φ represents any of the dependent variables. φ^I is assumed to be of the type discussed in section 3, while φ^c is assumed to be nonzero only in a relatively narrow region of width L_c along the coast. Thus by definition

$$\lim_{\xi/L_c \to \infty} \varphi^c = 0 \tag{4.2b}$$

The boundary conditions (2.3) should of course be satisfied by the combined solution $\varphi^I + \varphi^c$.

The width L_c has to be such that equations (2.10)* allow a baroclinic solution when L is replaced by L_c. This is possible only if both sides of (2.10b)* are of equal magnitude or if

$$C_B/fL_c \sim 1 \tag{4.3a}$$

Comparing with condition (3.1) we obtain also

$$L_c/L < 1, \tag{4.3b}$$

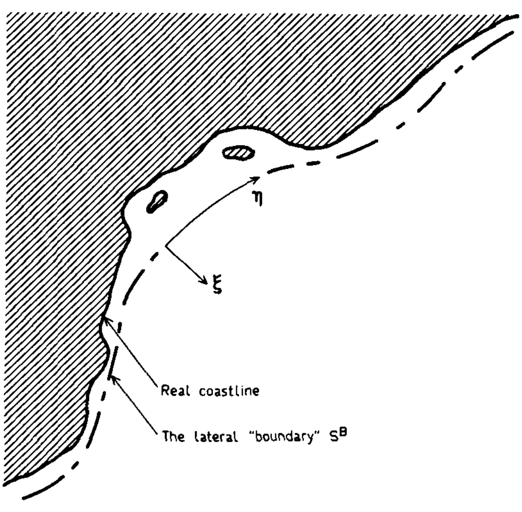

Fig. 3. Illustration of the artificial lateral boundary S^B and the coordinates ξ and η, used in the description of the baroclinic coastal response.

$$C_0/fL_c \gg 1 \tag{4.3c}$$

Equation (4.3b) expresses the anticipated boundary layer character of the baroclinic response. We assume that the form of the coastline, as well as the topography, is characterized by the length scale L. Then equation (4.3b) and the scaling properties of φ^c are equivalent to

$$\frac{\partial \varphi^c}{\partial \xi} \sim \frac{L}{L_c}\frac{\partial \varphi^c}{\partial \eta} \gg \frac{\partial \varphi^c}{\partial \eta} \tag{4.3d}$$

where η measures distance along S^B (see Fig. 3).

We also assume

$$\delta < 1, \tag{4.3e}$$

in consistency with our analysis of the barotropic response.

Applying (4.3) to the non-dimensional equations (2.10)* with φ replaced by φ^c and L by L_c, we obtain to lowest order in L_c/L and δ the following equations for φ^c (in dimensional form).

$$f^2 \frac{\partial w^c}{\partial z} = \frac{1}{\varrho_m}\frac{\partial}{\partial t}\frac{\partial^2 p^c}{\partial \xi^2} \tag{4.4a}$$

$$g\frac{\partial \varrho_s}{\partial z} w^c = \frac{\partial}{\partial t}\frac{\partial p^c}{\partial z} \tag{4.4b}$$

$$f^2 u^c = -\frac{1}{\varrho_m}\left(f\frac{\partial p^c}{\partial \eta} + \frac{\partial^2 p^c}{\partial \xi \, \partial t}\right) \tag{4.4c}$$

$$f^2 v^c = -\frac{1}{\varrho_m}\left(-f\frac{\partial p^c}{\partial \xi}\right) \tag{4.4d}$$

where u^c and v^c are the horizontal components of $\mathbf{v}^c$ in the direction of increasing ξ and η respectively. Observe that the last term in (4.4c) although neglected in a pure quasi-geostrophic approximation ($\delta < 1$), cannot in general be neglected in this case unless $\delta < L_c/L$ which we are *not* assuming.

We next introduce the variable

$$\phi = \int_z^0 p^c \, dz \tag{4.5}$$

in (4.4), since it turns out that the boundary conditions are best formulated in terms of ϕ.

Substituting (4.5) into (4.4a, b and c) ((4.4d) will not be needed) after (4.4a and c) have been integrated once with respect to z we obtain

$$f^2(w^c(z=0)-w^c(z)) = \frac{1}{\varrho_m}\frac{\partial}{\partial t}\frac{\partial^2\phi}{\partial\xi^2} \qquad (4.6\,\mathrm{a})$$

$$g\frac{\partial\varrho_s}{\partial z}w^c(z) = -\frac{\partial}{\partial t}\frac{\partial^2\phi}{\partial z^2} \qquad (4.6\,\mathrm{b})$$

$$f^2\int_z^0 u^c dz = -\frac{1}{\varrho_m}\left(f\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\xi\,\partial t}\right)\phi \qquad (4.6\,\mathrm{c})$$

5. The separation of the boundary conditions

If, as we assume, the forcing functions w_τ and p_a in (2.11)* are smooth functions of position, the barotropic solution φ^I is capable of satisfying the complete boundary conditions at $z=0$ and $z=-H$. The boundary condition on φ^c will thus be the homogeneous counterpart of (2.11)*. Applying conditions (4.3) on (2.11)* with φ replaced by φ^c, L by L_c and $w_\tau = p_a = 0$, we thus obtain to lowest order

$$w^c = 0 \quad \text{at } z=0, \qquad (5.1\,\mathrm{a})$$

$$w^c + w_E^c = 0 \quad \text{at } z=-H^B, \qquad (5.1\,\mathrm{b})$$

where it should be noted that we have neglected the term

$$u\frac{\partial H}{\partial\xi}+v\frac{\partial H}{\partial\eta}$$

in (5.1b). To justify this we are forced to assume that the depth just outside S^B (i.e. in the boundary layer region within which the baroclinic part of the solution differs from zero) varies with the length scale L, typical for the main part of the basin, despite the proximity of the coast. Using the depth scale H_0 (as in (2.11)*) and length scale L_c for ξ we thus have in nondimensional form

$$\frac{\partial H}{\partial\xi}\sim\frac{L_c}{L} \qquad (5.2\,\mathrm{a})^*$$

Furthermore the smoothness of the topography and the definition of η and ξ implies

$$\frac{\partial H}{\partial\eta}\sim\frac{L_c}{L}\frac{\partial H}{\partial\xi}\sim\left(\frac{L_c}{L}\right)^2 \qquad (5.2\,\mathrm{b})^*$$

To estimate u and v we have to study (2.10c and d)* with L replaced by L_c but, in doing so, remember that

$$\frac{\partial}{\partial\eta}\sim\frac{L_c}{L}\frac{\partial}{\partial\xi}$$

We obtain

$$v\sim 1,\ u\sim\mathrm{Max}\left(\frac{L_c}{L},\delta\right) \qquad (5.2\,\mathrm{c})^*$$

From (5.2)* it is easily proved that the depth variation may actually be neglected in (5.1b) as proposed.

Let us now formulate (5.1) in terms of ϕ. The upper boundary condition (5.1a) is taken care of by simply introducing $w(z=0)=0$ in (4.6a). However, from the definition of ϕ (4.5) we obtain instead

$$\phi = 0 \quad \text{at } z=0. \qquad (5.3)$$

According to the Ekman theory, we have, in view of (4.3d),

$$w_E^c = \frac{\partial}{\partial\xi}(\mathbf{n}\cdot\mathbf{M}_E^c) = -\frac{\varepsilon H_0}{2\varrho_m f}\left(\frac{\partial^2 p^c}{\partial\xi^2}\right)_{z=-H^B} = \frac{\varepsilon H_0}{2\varrho_m f}\left(\frac{\partial^2}{\partial\xi^2}\frac{\partial\phi}{\partial z}\right)_{z=-H^B} \qquad (5.4)$$

Expressing (5.1b) in terms of ϕ with the aid of (4.6a) and (5.4) we obtain

$$-\frac{1}{\varrho_m f^2}\frac{\partial}{\partial t}\frac{\partial^2}{\partial\xi^2}\phi+\frac{\varepsilon H_0}{2\varrho_m f}\frac{\partial^2}{\partial\xi^2}\frac{\partial\phi}{\partial z}=0 \quad \text{at } z=-H^B$$

which in view of (4.2b) integrates to

$$\left(\frac{\partial}{\partial t}-\frac{\varepsilon H_0 f}{2}\frac{\partial}{\partial z}\right)\phi=0 \quad \text{at } z=-H^B \qquad (5.5)$$

Having found the proper form (5.3 and 5.5) of the boundary conditions on φ^c (i.e. on ϕ) at $z=0$ and $z=-H^B$ we now turn our attention to the lateral boundary condition. Introducing (4.2) into (4.1) we obtain

$$u_n^I + u^c + M_E^I f_2(z) + M_E^c f_2(z) = (M_c - M_\tau)f_1(z) \qquad (5.6\,a)$$

where, in accordance with (3.13),

$$M_E^I = -\frac{\varepsilon H_0}{2\varrho_m f}\frac{\partial p^I}{\partial\xi} \qquad (5.6\,\mathrm{b})$$

$$M_E^c = -\frac{\varepsilon H_0}{2\varrho_m f}\left(\frac{\partial p^c}{\partial\xi}\right)_{z=-H^B} = \frac{\varepsilon H_0}{2\varrho_m f}\frac{\partial}{\partial\xi}\left(\frac{\partial\phi}{\partial z}\right)_{z=-H^B} \qquad (5.6\,\mathrm{c})$$

A priori it is not at all obvious how separate conditions on φ^c and φ^I can be obtained from (5.6). Let us first look at the much easier case with no bottom friction, i.e. when $\varepsilon = 0$. In this case the lower boundary condition (5.5) simplifies to

$$\phi = 0 \quad \text{at } z = -H^B \tag{5.7a}$$

which as seen from (4.6c) implies

$$\int_{-H^B}^{0} u^c dz = 0 \tag{5.7b}$$

Thus when there is not bottom friction u^c is purely baroclinic (as might be expected).

Equation (5.7b) is valid for any ξ and in particular at the boundary $\xi = 0$. Thus when (5.6a) is integrated vertically we obtain the condition (remembering that $M_E = 0$ when $\varepsilon = 0$)

$$H^B u_n^I = M_c - M_\tau \quad \text{at } \xi = 0$$

which is the proper boundary condition on u_n^I. Inserting this expression in (5.6a) we obtain a separate boundary condition on u^c, i.e.

$$u^c = (M_c - M_\tau)\left(f_1(z) - \frac{1}{H^B}\right)$$

We next carry through essentially the same procedure in the more complicated case $\varepsilon \neq 0$.

Let us first integrate (5.6a) with respect to z. We obtain

$$-z u_n^I + \int_z^0 u^c dz + M_E^I F_2(z) + M_E^c F_2(z) = $$
$$= (M_c - M_\tau) F_1(z) \tag{5.8}$$

where

$$F_1(z) = \int_z^0 f_1(z)\,dz \quad \text{and} \quad F_2(z) = \int_z^0 f_2(z)\,dz$$

As in the previous case the condition on u^c that determines how to separate (5.8) should be obtained from the boundary condition (5.5). A useful form is for example

$$\int_{-H^B}^{0} u^c dz = -\frac{1}{\varrho_m f}\left(\frac{\partial \phi}{\partial \eta}\right)_{-H^B} - M_E^c \tag{5.9}$$

which results when (4.6c), (5.6c) and (5.5) are combined. Introduce (5.9) into (5.8), put $z = -H^B$, and recall that $F_1(-H^B) = F_2(-H^B) = 1$. We obtain

$$-H^B u_n^I = -\frac{1}{\varrho_m f}\left(\frac{\partial \phi}{\partial \eta}\right)_{-H^B} + M_E^I + M_\tau - M_c \tag{5.10}$$

As discussed in section 3.1, the pressure distribution p^I is completely determined from the integral of u_n^I around the whole boundary S^B. Since ϕ has to be continuous we thus find

$$H^B \oint_{S^B} u_n^I\, d\eta = \oint_{S^B} (M_c - M_\tau - M_E^I)\, d\eta \tag{5.11}$$

which is identical with equation (3.15) already derived in section 3. From (5.11) we may determine p^I independently of φ^c as discussed in section 3, despite the fact that u_n^I at S^B as given by (5.10) is affected by φ^c. *When solving for φ^c we may thus consider p^I as known.*

Eliminating u_n^I from (5.8) with the aid of (5.10) and at the same time expressing $\int_z^0 u^c dz$ and M_E^c in terms of ϕ with the aid of (4.6c), (5.6c) and (5.5), we obtain finally the required boundary condition on ϕ. (Note that (5.5) enables us to express M_E^c in two alternative ways.)

$$-\frac{\partial \phi}{\partial \eta} - f^{-1}\frac{\partial \phi}{\partial \xi\,\partial t} - \frac{z}{H^B}\left(\frac{\partial \phi}{\partial \eta}\right)_{z=-H^B} + F_2(z) f^{-1}\left(\frac{\partial \phi}{\partial \xi\,\partial t}\right)_{z=-H^B} = G(z,\eta,t) \quad \text{at } \xi = 0 \tag{5.12a}$$

where

$$G(z,\eta,t) = f\varrho_m\left[(M_c - M_\tau)\left(F_1(z) + \frac{z}{H^B}\right) - M_E^I\left(F_2(z) + \frac{z}{H^B}\right)\right] \tag{5.12b}$$

$G(z, \eta, t)$ is to be considered as a known forcing function. The qualitative features of the functions

$$f_1(z),\ f_2(z),\ F_1(z) + \frac{z}{H^B} \quad \text{and} \quad F_2(z) + \frac{z}{H^B}$$

are illustrated in Fig. 4.

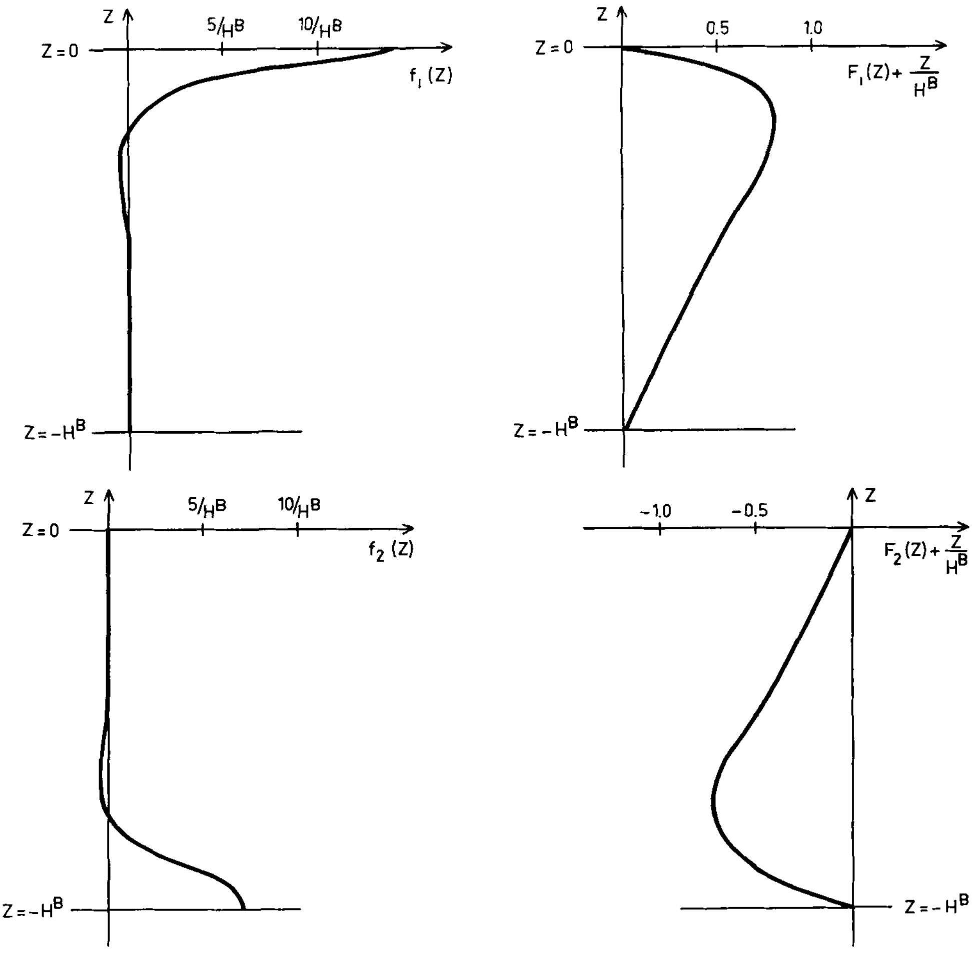

Fig. 4. Illustration of the functions $f_1(z)$, $F_1(z)+z/H^B$, $f_2(z)$ and $F_2(z)+z/H^B$, which determine the vertical distribution of the forcing function $G(z, \eta, t)$ in the lateral boundary condition (5.12). The precise form of these functions depend on mixing in the near shore region, which is not adequately known. By definition we have $F_1(z)=\int_z^0 f_1(z)\,dz$, $F_2(z)\,dz=\int_z^0 f_2(z)\,dz$, $F_1(-H^B)=F_2(-H^B)=1$.

6. Separated solutions to the coastal region problem

Let us now discuss the properties of φ^c as described by the baroclinic boundary layer equations (4.6) and the boundary conditions derived in section 5. Eliminating w^c from (4.6a and b), remembering that $w^c(z=0)=0$ (5.1a), we obtain

$$\frac{\partial}{\partial t}\frac{\partial^2\phi}{\partial z^2}=\frac{g}{\varrho_m f^2}\frac{\partial \varrho_s}{\partial z}\frac{\partial}{\partial t}\frac{\partial^2\phi}{\partial \xi^2} \tag{6.1}$$

We note that (6.1) contains no derivative with respect to η. Thus the η-dependence comes in through the boundary condition at $\xi=0$ only, a feature which derives from the boundary layer approximation we have made.

6.1. *Free waves*

Let us first consider free motions i.e. assume that dissipation as well as forcing is negligible. Thus with $\varepsilon=M_E=M_\tau=0$ we obtain from (5.3), (5.5), (5.12) and (4.2b)

$$\phi = 0 \quad \text{at } z = 0 \tag{6.2a}$$

$$\phi = 0 \quad \text{at } z = -H^B \tag{6.2b}$$

$$\left(f\frac{\partial}{\partial \eta} + \frac{\partial}{\partial \xi\, \partial t}\right)\phi = 0 \quad \text{at } \xi = 0 \tag{6.2c}$$

$$\lim_{\xi/L_c \to \infty} \phi = 0 \tag{6.2d}$$

A general solution to (6.1) and (6.2) may be written

$$\phi = \sum_{\nu} \phi_\nu(\eta + c_\nu t)\, \psi_\nu(z) \exp(-\alpha_\nu \xi) \tag{6.3a}$$

where

$$c_\nu f^{-1} = \alpha_\nu^{-1}, \quad (c_\nu, \alpha_\nu) > 0 \tag{6.3b}$$

and (α_ν^2, ψ_ν) are determined from the eigen value problem

$$\left(\frac{d^2}{dz^2} + a^2(z)\,\alpha_\nu^2\right)\psi_\nu(z) = 0 \tag{6.4a}$$

where

$$\psi_\nu(z) = 0 \quad \text{at } z = 0 \quad \text{and } z = -H^B \tag{6.4b}$$

and

$$a^2(z) = -\frac{g}{\varrho_m f^2}\frac{\partial \varrho_s}{\partial z} \tag{6.4c}$$

We require that the basic density distribution $\varrho_s(z)$ is gravitationally stable, i.e. $\partial\varrho_s/\partial z \leqslant 0$ and $a^2 \geqslant 0$. Equation (6.4) is then of the Sturm-Liouville type (Courant–Hilbert 1953, page 292), and it is easily proved that ψ_ν, assumed properly normalized, satisfies the orthogonality relation

$$\int_{-H^B}^{0} a^2 \psi_\nu \psi_\mu \, dz = \begin{cases} 0 & \text{if } \mu \neq \nu \\ 1 & \text{if } \mu = \nu \end{cases} \tag{6.4d}$$

and that α_ν^2 is a real positive number. In view of (6.2d) α_ν should be chosen as the root with positive real part. In this case α_ν is real and the decay with ξ of the normal mode solutions are purely exponential.

When $\partial\varrho_s/\partial z$ (i.e. a^2) is constant we have from (6.4a and b)

$$\psi_\nu = \sin \nu \frac{\pi}{H^B} z, \quad \alpha_\nu = \nu\pi(aH^B)^{-1}$$

This last equation is valid as an estimate for α_ν even when ϱ_s is not a linear function of z. Combining with (6.4c) and (6.3b) we then obtain the following estimates for the propagation velocities

$$c_\nu \sim \frac{1}{\pi\nu}\left(\frac{\Delta\varrho_s}{\varrho_m} g H^B\right)^{\frac{1}{2}} \tag{6.5}$$

where $\Delta\varrho_s$ may be taken as the total variation of ϱ_s between $z = 0$ and $z = -H^B$.

It is clear from (6.5) that c_ν may be identified as the propagation velocities of the various modes of long internal gravity waves. In fact we are dealing with a superposition of internal Kelvin waves traveling along the coast in a counter clockwise direction.

The arbitrary functions ψ_ν are completely determined by an initial condition on ϕ at $\xi = 0$ only. Thus if ϕ is prescribed for all ξ our problem becomes over-determined. The explanation is that the response to an arbitrary initial baroclinic pressure distribution consists of two parts; (i) a time dependent part of the type (6.3) which is determined by the initial condition at $\xi = 0$ and (ii) a steady part (i.e varying slowly in time compared with ϕ) which has been excluded from the present analyses by the assumpton $\partial\varphi/\partial t \neq 0$.

The existence of the free wave solution (6.3) strongly influences the response when forcing and bottom friction is included. In fact we expect to find a resonant type of response which strongly favours waves of the Kelvin type.

6.2. *Forced waves: formal aspects*

Let us now discuss how our system may be handled with forcing and bottom friction included. A general solution to (6.1) satisfying (5.3), (5.5) and (4.2b) may be written

$$\phi = \sum_{\nu,\omega,l} A_\nu^{\omega,l} \exp(il\eta + i\omega t)\, \psi_\nu^\omega(z) \exp(-\alpha_\nu^\omega \xi) \tag{6.6}$$

where $(\psi_\nu^\omega, (\alpha_\nu^\omega)^2)$ are solutions to the eigenvalue problem

$$\left[\frac{d^2}{dz^2} + a^2(z)\,(\alpha_\nu^\omega)^2\right]\psi_\nu^\omega = 0 \tag{6.7a}$$

$$\psi_\nu^\omega = 0 \quad \text{at } z = 0 \tag{6.7b}$$

$$\left(1 + \frac{i}{2}\varepsilon f \omega^{-1} H_0 \frac{d}{dz}\right)\psi_\nu^\omega = 0 \quad \text{at } z = -H^B \tag{6.7c}$$

and the coefficients $A_\nu^{\omega,l}$ are at our disposal to satisfy the boundary condition (5.12). Observe that while ω and l are defined as real quantities, the coefficients α_ν^ω are in general complex. Note that we obtain a particular sequence of eigenfunctions φ_ν^ω for each value of the frequency ω. This of course depends on the presence of the time derivative in the lower boundary condition (5.5).

The eigenvalue problem (6.7) is not of standard type because of the imaginary coefficient in (6.7c). It is easily proved, however, that ψ_ν^ω (for a fixed ω) in fact satisfies the same orthogonality relation (6.4d) as ψ_ν. From (6.7c) we find that the importance of the frictional term is dependent upon ω; i.e. the lower the frequency the larger is the difference between ψ_ν^ω and ψ_ν.

In fact the two terms in (6.7c) are of equal magnitude when

$$\varepsilon f \omega^{-1} \sim \varepsilon \delta^{-1} \sim 1$$

where it is recalled that ε is the ratio of the Ekman boundary layer thickness and the mean depth H_0.

In the case $a^2(z) = \text{const.}$ we obtain from (6.7a and b)

$$\psi_\nu^\omega = \sin mz, \quad m^2 = (\alpha_\nu^\omega)^2 a^2$$

which, when combined with (6.7c), gives

$$\operatorname{tg} mH^B = imH^B r, \quad r = \frac{H_0}{H^B} \varepsilon f \omega^{-1}$$

When r is small we may expand around the solution valid for $r = 0$ obtaining

$$m = \frac{\nu\pi}{H^B}(1 + ir), \quad \alpha_\nu^\omega = \frac{\nu\pi}{H^B a}(1 + ir)$$

Within the same approximation we obtain

$$\psi_\nu^\omega = \sin \frac{\nu\pi}{H^B} z \cosh \frac{\nu\pi}{H^B} rz + i \cos \frac{\nu\pi}{H^B} z \sinh \frac{\nu\pi}{H^B} rz$$

We now turn our attention to the problems connected with the determination of the coefficients $A_\nu^{\omega,l}$. Inserting our solution (6.6) in the lateral boundary condition (5.12), multiplying with $\exp(-i\omega' t - il'\eta)$ and integrating with respect to η around the whole basin and with respect to t over a time period long compared with the time-scales under consideration, we obtain, dropping the primes,

$$\sum_\nu \left(A_\nu^{\omega,l} iz_\nu^{\omega,l} \psi_\nu^\omega(z) - \frac{z}{H^B} Q - F_2(z) R\right) = G_{\omega,l}(z) \tag{6.8 a}$$

where

$$z_\nu^{\omega,l} = (\omega f^{-1} \alpha_\nu^\omega - l) \tag{6.8 b}$$

$$Q = \sum_\nu A_\nu^{\omega,l} il\psi_\nu^\omega(z = -H^B) \tag{6.8 c}$$

$$R = \sum_\nu A_\nu^{\omega,l} i\omega f^{-1} \alpha_\nu^\omega \psi_\nu^\omega(z = -H^B) \tag{6.8 d}$$

and $G_{\omega,l}(z)$ is defined by

$$\sum_{\omega,l} G_{\omega,l}(z) \exp(i\omega t + il\eta) = G(\eta, z, t) \tag{6.8 e}$$

Multiplying (6.8 a) with $a^2(z)\, \psi_\mu^\omega$, integrating with respect to z, and using the orthogonality relation (6.4d), we obtain

$$A_\mu^{\omega,l} = \frac{-i}{z_\mu^{\omega,l}} \int_{-H^B}^{0} a^2(z) \psi_\mu^\omega \times \left(G_{\omega,l}(z) + \frac{z}{H^B} Q + F_2(z) R\right) dz \tag{6.9}$$

Since Q and R depend on all the $A_\mu^{\omega,l}$, the solution of (6.9) is non-trivial. We note, however, that Q and R are independent of μ which suggests the following procedure.

(i) Solve for $A_\mu^{\omega,l}$ in terms of Q and R (considering Q and R as known for the moment) which clearly may be done term by term.

(ii) Insert the result, $A_\mu^{\omega,l}(Q, R)$ in (6.8c and d).

(iii) Determine Q and R from these two equations.

(iv) Re-insert in $A_\mu^{\omega,l}$.

A successful performance of step (iii) above will require certain conditions for the series representations of Q and R to be convergent. These conditions will limit the choice of the prescribed functions in the boundary condition. If these conditions are not too restrictive, and we have no reason to believe they are, our solution will thus be able to satisfy reasonable lateral boundary conditions. We conclude that "the boundary layer approach" represented by (4.2a) is likely to be successful. This does not prove but makes it likely that the real response behaves approximately in accordance with assumptions (4.2). We thus feel justified to draw at least some qualitative geophysical conclusions from the properties of φ^I and φ^c respectively.

6.3. *Forced waves: qualitative properties*

Let us consider the response in φ^c to a forcing of the type

$$G(z, \eta, t) = G_0(z) \cos \omega_0 t \cos l_0 \eta, \quad (\omega_0, l_0) > 0 \tag{6.10a}$$

which may be written as the sum of two waves of equal amplitude travelling in opposite directions, i.e.

$$G(z, \eta, t) = \tfrac{1}{2} G_0(z)[\cos(\omega_0 t + l_0 \eta) + \cos(\omega_0 t - l_0 \eta)] \tag{6.10b}$$

The non-zero coefficients in a representation of G like (6.8e) become

$$G_{\pm\omega_0, \pm l_0}(z) = \tfrac{1}{4} G_0(z) \tag{6.10c}$$

It should be recognized that a spatial variation of $G(z, \eta, t)$ will occur as a result of bending of the coastline (G contains the normal component of the windstress) as well as spatial variation of the wind. Furthermore, the bending of the coastline is usually the more important factor. Thus we think of $2\pi\omega_0^{-1}$ and $2\pi l_0^{-1}$ as representing the characteristic time-scale of the wind system and the distance between consecutive large bays along the coast respectively.

The response to the forcing (6.10) as obtained from (6.6–6.9) can be written as

$$\begin{aligned}\phi = 2 \sum_\mu [&|A_\mu^{\omega_0, l_0}| \cos(\omega_0 t + l_0 \eta + \beta^+ + \gamma(\xi, z)) \\ &+ |A_\mu^{\omega_0, -l_0}| \cos(\omega_0 t - l_0 \eta + \beta^- + \gamma(\xi, z))] \\ &\times |\psi_\mu^{\omega_0} \exp - \alpha_\mu^{\omega_0} \xi|\end{aligned} \tag{6.11}$$

where β^+, β^- and $\gamma(\xi, z)$ are the arguments of the complex quantities $A_\mu^{\omega_0\, l_0}$, $A_\mu^{\omega_0, -l_0}$ and $\psi_\mu^{\omega_0} \exp(-\alpha_\mu^{\omega_0} \xi)$ respectively.

When the dissipation is weak, i.e. when

$$\varepsilon\delta^{-1} \ll 1 \tag{6.12a}$$

we have from (6.4) and (6.7)

$$(\psi_\mu^{\omega_0}, \alpha_\mu^{\omega_0}) \simeq (\psi_\mu, \alpha_\mu), \tag{6.12b}$$

and since ψ_μ and α_μ are real quantities

$$\gamma(\xi, z) \ll 1 \tag{6.12c}$$

Furthermore, as seen from (6.8), Q and R become small, since $\psi_\mu^\omega(z = -H^B) \simeq \psi_\mu(z = -H^B) = 0$, and we obtain from (6.9) to a first approximation

$$A_\mu^{\omega_0, \pm l_0} = \frac{-i}{\omega_0 f^{-1} \alpha_\mu^{\omega_0} \mp l_0} \int_{-H^B}^0 a^2(z) \psi_\mu \tfrac{1}{4} G_0(z)\, dz \tag{6.12d}$$

This expression clearly demonstrates the amplification of waves travelling in the *negative* η-direction and how the imaginary part of $\alpha_\mu^{\omega_0}$ associated with a small but finite dissipation limits the peak amplitude. The largest amplitude occurs if

$$\frac{\omega_0}{l_0} \sim f[\mathrm{Re}\,(\alpha_\nu^{\omega_0})]^{-1}$$

or, since $\alpha_\nu^{\omega_0} \simeq \mathrm{Re}\,\alpha_\nu^{\omega_0} \simeq \alpha_\nu$

$$\frac{\omega_0}{l_0} \sim f\alpha_\nu^{-1} = c_\nu \tag{6.13}$$

Thus the peak amplitude occurs roughly when ω_0/l_0 equals the propagation velocity of the free Kelvin wave. As will be discussed in connection with the presentation of some observations in the Baltic, Walin (1972), the "natural" length and time scales are frequently such that (6.12) is roughly satisfied, and we thus expect the resonance effect just discussed to be hydrographically important.

Let us now briefly discuss the behaviour of the response when ω_0/l_0 is very far from c_ν. Assuming again $\varepsilon\delta^{-1} \ll 1$ we obtain from (6.12d)

$$\beta^+ \simeq \beta^- \simeq -\frac{\pi}{2} \quad \text{when} \quad \frac{\omega_0}{l_0} \gg c_\nu$$

and

$$-\beta^+ \simeq \beta^- \simeq -\frac{\pi}{2} \quad \text{when} \quad \frac{\omega_0}{l_0} \ll c_\nu$$

Furthermore, in both cases

$$|A_\mu^{\omega_0, l_0}| \simeq |A_\mu^{\omega_0, -l_0}|$$

Making use of (6.12b and c) we thus find that ϕ becomes proportional to

$$\cos\left(\omega_0 t - \frac{\pi}{2}\right) \cos l_0 \eta \quad \text{when} \quad \frac{\omega_0}{l_0} \gg c_\nu$$

and

$$\cos \omega_0 t \cos\left(l_0 \eta + \frac{\pi}{2}\right) \quad \text{when} \quad \frac{\omega_0}{l_0} \ll c_\nu$$

Thus for high frequency forcing there is no phase shift along the coast but the maximum amplitude lags "90°" behind the forcing in time. In the opposite case of very low frequency forcing the maximum amplitude is shifted "90°" in the propagation direction of the free Kelvin wave to a point where the forcing vanishes.

It may be pointed out that the high frequency case

$$\omega_0/l_0 \gg c_\nu$$

corresponds to the approximation that the pressure gradient along the coast is ignored. In this case the mechanics is the same as in a strictly two-dimensional case and the response is completely determined from the local forcing. The dynamics is essentially the same as in the rotationally symmetric case discussed by Walin (1969).

The low frequency case

$$\omega_0/l_0 \ll c_\nu$$

on the other hand corresponds to a strictly geostrophic approximation.

7. Summary and concluding remarks

We have found that the time dependent response to the "synoptic" variable wind and pressure field over the Baltic (or a similar hydrographic system) may be divided into two parts with distinctly different properties: (i) a barotropic part non zero in the whole basin, (ii) a baroclinic part non zero only in a relatively narrow coastal region.

Under more restrictive conditions, as discussed below, the quantitative analyses of sections 3–5 is applicable. Thus the barotropic part of the motion may be predicted with the aid of equation (3.14) subject to the boundary condition (3.16), while the baroclinic coastal response may be found from equation (6.1) subject to conditions (5.3), (5.5), (5.12) and (4.2b) for example by the method of separation of variables as discussed in section 6.

7.1. *Conditions for validity of the results of sections 3–5*

From the properties of the linear response it is possible to estimate the lowest order influence of the nonlinearities neglected in (2.1). We thereby obtain additional conditions which will be presented below. We will however, firstly review the assumptions already adopted in sections 2–5.

The conditions behind our basic equations (besides the linearization) are summarized in connection with equation (2.1) and will not be reiterated here. The general scaling procedure of section (2.1) is performed under the condition

$$(Tf)^{-1} = \delta \lesssim 1 \tag{7.1}$$

which together with our basic scaling assumptions

$$C_B/fL \ll 1, \;\; C_0/fL \sim 1, \tag{7.2}$$

is enough to produce the lowest order result

$$\partial p^I/\partial z = 0 \tag{7.3}$$

for the large scale response. Observe that (7.3) alone implies that the baroclinic response is trapped in the neighbourhood of the lateral boundary.

The quantitative analysis of sections 3–5 was performed under the stronger condition

$$\delta \ll 1 \tag{7.4}$$

Furthermore, we made certain geometrical assumptions, most importantly

$$\nabla H(x, y) \sim \frac{H_0}{L} \tag{7.5}$$

If ∇H is locally smaller than H_0/L the topographic steering of the barotropic flow will be incomplete, we obtain "geostrophically free" regions. If, on the other hand, ∇H is large compared with H_0/L in the coastal region, the slope of the bottom must be taken into account in the analyses of the baroclinic response (e.g. see Csanady, 1971).

We should also remember the "smoothness" assumptions implicit in the whole scaling procedure; e.g. that the distribution of wind and pressure at the surface and the form of the basin do not have a much smaller length scale than the size of the basin.

Analysing the influence of the nonlinear contributions we find:

(i) Equation (7.3) may be derived if

$$R^I\delta^{-1} \lesssim 1 \tag{7.6}$$

where $R^I = U^I/fL$, U^I represents a characteristic velocity in the barotropic response and L represents the length scale of the basin.

(ii) The quantitative analyses of the barotropic response is valid if

$$R^I < 1 \tag{7.7}$$

Observe that we need not require that $R^I\delta^{-1} < 1$ which might be expected from an inspection of the complete nonlinear momentum equations. The reason is that the major nonlinear contribution cancels, when we integrate around a closed depth contour (i.e. in the derivation of equation (3.7)).

(iii) The quantitative analysis of the baroclinic response is valid if

$$R^c < 1 \tag{7.8a}$$

where $R^c = U^c/fL_c$, U^c is the baroclinic velocity scale *parallel* to the coast and L_c is the *width* of the boundary layer.

Condition (7.8) is obviously considerably stronger than (7.7) since $L_c < L$.

Condition (7.8) may also be written

$$U^{\perp}/fL_c\delta^{-1} < 1 \tag{7.8b}$$

where $U^{\perp}$ is the characteristic velocity perpendicular to the coast. Since $U^{\perp}$ is directly related to the strength of the forcing, i.e. to the windstress transport, we conclude that the quantitative description of the coastal region becomes poor for very small δ.

In this paper the whole quantitative analysis is carried out under the condition $\delta \ll 1$. This is necessary if we want a complete, consistent picture, since the analysis of the barotropic response depends in a vital way on this assumption. The coastal response, however, may be analysed in essentially the same way under the weaker condition $\delta < 1$. *Thus we conclude that the large scale flow is in principle more accessible to prediction the lower the frequency of forcing, while because of (7.8) the opposite is true for the baroclinic coastal response (i.e. witin the condition $\delta < 1$).*

7.2. *Hydrographical implications*

The question now arises as to what extent the results obtained in this paper may actually be verified, for example in the Baltic, and what significance they thus might have. Phenomena of types other than those considered here, such as tides and stationary currents, are weak in the Baltic. We thus expect the phenomena under consideration to be of major importance for large scale motion and eventually, when dissipated, also for the mixing conditions.

Introducing reasonable numerical values ($U^I = 10$ cm s^{-1}, $f = 10^{-4}s^{-1}$, $L = 200$ km) we obtain $R^I \lesssim 10^{-2}$. Thus, (7.7) as well as (7.2) (as discussed in section 3) are sufficiently well satisfied for a direct verification of our results concerning the large scale response to be possible. We should remember, however, that we can expect considerable energy on frequencies higher than described by our theory. We thus need a "current meter" which integrates over a sufficiently long time and preferably also measures a vertical mean value. We can, for example, imagine a float moving freely in the water, varying its own buoyancy in such a way that it travels up and down between the surface and the bottom. The mean displacement (during several days) of such a float would provide a suitable observable for the present purpose.

Turning now to the coastal response, we find that the conditions for our quantitative description to hold are much more stringent. In fact the topographical assumption (7.5) is frequently violated in the neighbourhood of the coast, and condition (7.8) is often poorly satisfied (with $U^c \sim 20$ cm s^{-1}, $f \sim 10^{-4}s^{-1}$, $L_c \sim 5$ km we obtain $R^c \sim 0.2$). Thus any detailed comparison between theory and observations can hardly be justified. We do, however, expect the qualitative features of our description to show up in the hydrographical observations. Most importantly we expect the perturbations on the temperature field to decay roughly as $\exp(-\alpha\xi)$ where $\alpha = c_1 f^{-1}$, c_1 being the maximum internal wave speed. Using values typical for the Baltic we find $\alpha^{-1} = 5$ km. We also expect the tendency for one-way wave propagation (counter-clockwise around the basin) to show up under favourable conditions.

As described in another paper (Walin, 1972) the temperature field on one section perpendicular to the Swedish coast was observed daily during the summer 1968. The observations clearly display the expected decay from the coast. In fact the temperature field at a distance of 10 km never deviated appreciably from the vertical structure typical for the season while closer to the coast the vertical excursions of the isotherms were sometimes comparable with the total depth. Some indications of southward transport of information along the coast was also obtained. Observations of the temperature field in the Great Lakes (e.g. Mortimer, 1963) show similar qualitative features.

The intense baroclinic motions in the coastal region will undoubtedly be a very important energy source for vertical mixing (e.g. Garvine (1971) and Wunsch (1970)). For example water from deeper layers occasionally brought up to the surface will be more or less diluted with surface water before finding its way back to an equilibrium position in the deep layers. It is a challenging task to estimate this coastal mixing and its importance for the whole hydrographic system.

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О ПЕРЕДАТОЧНОЙ ФУНКЦИИ МЕТЕОРОЛОГИЧЕСКИХ ВОЗМУЩЕНИЙ ДЛЯ БАЛТИЙСКОГО МОРЯ

Рассматривается передаточная функция внешних воздействий для гидрографической системы, подобной Балтийскому морю. Найдено, что при довольно общих условиях бароклинная передаточная функция ограничена узкой прибрежной областью, в то время как в основной части бассейна передаточная функция баротропна. Передаточная функция анализируется количественно при несколько более специальных условиях, причем используется подход типа пограничного слоя. Найдено, что баротропные крупномасштабные низкочастотные движения следуют изолиниям постоянной глубины. Эта особенность могла бы получить экспериментальную проверку. Бароклинная передаточная функция для Балтики ограничена прибрежной зоной шириной 5–10 *км*. Если диссипация достаточно слаба, то в прибрежной области могут возбуждаться внутренние волны Кельвина, распространяющиеся против часовой стрелки.