SHORTER CONTRIBUTION

On the energy spectra of a random field of internal waves

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(Manuscript received September 10, 1971; revised version December 9, 1971)

Introduction

Several authors (e.g. Phillips, 1966; Voorhis, 1968; Fofonoff, 1969; Frankignoul, 1970) have attempted to interpret the energy spectra observed in the deep ocean as resulting from a random field of linear internal waves propagating in all directions.

Further studies of erratic motions in the deep ocean by methods of weak turbulence theory (Nihoul, 1971) seem to suggest that, in fact, for frequencies smaller than the Brunt-Väisälä frequency, the cogent factor responsible for the shape of the spectra may be the interaction of random internal waves with a weak mean shear.

In order to examine this suggestion, one investigates here the simple situation of a random field of linear internal waves superimposed on a mean current which varies slowly (and, to be specific, in a linear way) with depth under the assumption that the Brunt-Väisälä frequency is constant.

Mathematical analysis

The mean current is assumed to be in the x_1 direction and of the form

$$\mathbf{V} = \mathbf{V}_0(1 + \alpha x_3) \tag{1}$$

In axes moving with the average mean velocity $\mathbf{V}_0$, the Boussinesq equations for the inviscid motion of a stratified fluid may be written, neglecting Coriolis forces (e.g. Nihoul, 1971).

$$\nabla_i w_i = 0 \tag{2}$$

$$\frac{\partial w_i}{\partial t} + a x_3 \mathbf{V}_0 \cdot \nabla w_i + \alpha V_0 w_3 \delta_{i1} + \nabla_i q + N(w_3 \delta_{i4} - w_4 \delta_{i3}) = 0 \tag{3}$$

where, in a convenient four-dimensional space

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, 0\right) \tag{4}$$

$$W = (w_1, w_2, w_3, w_4) \tag{5}$$

w_1, w_2, w_3 are the components of the waves induced velocity, w_4 is a "buoyancy velocity" defined by

$$w_4 = -\frac{g\hat{\varrho}}{\varrho_0 N} \tag{6}$$

N is the Brunt-Väisälä frequency, g the acceleration of gravity, ϱ_0 the reference constant density, $\hat{\varrho}$ the fluctuation of density due to wave motion, q the deviation from hydrostatic pressure divided by ϱ_0

Let

$$\begin{pmatrix} w_i \\ q \\ x_3 \end{pmatrix} = \int_{-\infty}^{\infty} \begin{pmatrix} W_i \\ Q \\ X \end{pmatrix} e^{i(\boldsymbol{\varkappa}\cdot\mathbf{x} - \omega t)} d\boldsymbol{\varkappa} \tag{7}$$

One has. δ denoting the Dirac function, $\mathring{\delta}$ its derivative

$$X(\varkappa) = i\delta(\varkappa_1)\,\delta(\varkappa_2)\,\mathring{\delta}(\varkappa_3)$$

and

$$\int_{-\infty}^{\infty} X(\boldsymbol{\varkappa}')\, W_i(\boldsymbol{\varkappa} - \boldsymbol{\varkappa}')\, d\boldsymbol{\varkappa} = i\frac{\partial W_i}{\partial \varkappa_3} \tag{9}$$

Thus, taking the Fourier transform of (5) and eliminating the pressure using the continuity equation, one obtains

$$-i\mu W_i - \beta\varkappa_1 \frac{\partial W_i}{\partial \varkappa_3} + \beta W_3 \left(\delta_{i1} - \frac{2\varkappa_i \varkappa_1}{\varkappa^2}\right) + W_3 \delta_{i4} - W_4 \left(\delta_{i3} - \frac{\varkappa_i \varkappa_3}{\varkappa^2}\right) = 0 \tag{10}$$

where

$$\mu = \frac{\omega}{N}; \quad \beta = \frac{\alpha V_0}{N} \tag{11}$$

β is the inverse square root of the Richardson number and may be estimated in typical situations as $O(10^{-1})$. Hence, although the β terms represent a small correction, they dominate the contributions of the neglected terms that arise from the Coriolis effect or eventual variation of the Brunt-Väisälä frequency with depth (this is usually small beneath the subsurface layer (Frankignoul, 1970)).

Introducing spherical coordinates (k, ψ, e; $\sigma = \sin e$) in k space, one has

$$\varkappa \frac{\partial}{\partial \varkappa_3} = \sqrt{1-\sigma^2}\left(\varkappa \frac{\partial}{\partial \varkappa} - \sigma \frac{\partial}{\partial \sigma}\right) \tag{12}$$

Let

$$F_{ij} = E_{ij} + iG_{ij} = \int_0^\infty \varkappa^2 W_i(\boldsymbol{\varkappa})\, W_j(-\boldsymbol{\varkappa})\, d\boldsymbol{\varkappa} \tag{13}$$

From (10), one obtains, using (12), multiplying by $\varkappa^2$ and integrating

$$\begin{aligned} &\beta\sigma^2\sqrt{1-\sigma^2}\cos\psi \frac{\partial F_{ij}}{\partial \sigma} + 3\beta\sigma\sqrt{1-\sigma^2}\cos\psi F_{ij} \\ &+ \beta F_{3j}\left(\delta_{i1} - 2\frac{\varkappa_i \varkappa_1}{\varkappa^2}\right) + \beta F_{i3}\left(\delta_{j1} - 2\frac{\varkappa_j \varkappa_1}{\varkappa^2}\right) \\ &= -F_{3i}\delta_{i4} - F_{i3}\delta_{j4} + F_{4j}\left(\delta_{i3} - \frac{\varkappa_i \varkappa_3}{\varkappa^2}\right) \\ &+ F_{i4}\left(\delta_{j3} - \frac{\varkappa_j \varkappa_3}{\varkappa^2}\right) \end{aligned} \tag{14}$$

The system of equations (14) can be integrated neglecting second order terms in β. Constants of integration appear in the zeroth order and first order terms. The zeroth order constants are determined next in terms of one of them using the relations

$$\mu G_{ij} + E_{3j}\delta_{i4} - E_{4j}\left(\delta_{i3} - \frac{\varkappa_i \varkappa_3}{\varkappa^2}\right) \sim 0(\beta) \tag{15}$$

$$-\mu E_{ij} + G_{3j}\delta_{i4} - G_{4j}\left(\delta_{i3} - \frac{\varkappa_i \varkappa_3}{\varkappa^2}\right) \sim 0(\beta) \tag{16}$$

obtained from (10).

The compatibility (15) and (16) yields in addition

$$\mu = \sigma \tag{16}$$

The final result for the energy spectra is, using (17)

$$\varepsilon_h \equiv \frac{E_{11} + E_{22}}{C} = \frac{1-\mu^2}{\mu^2} \tag{18}$$

$$\varepsilon_{33} \equiv \frac{E_{33}}{C} = 1 \tag{19}$$

$$\varepsilon_{44} \equiv \frac{E_{44}}{C} = \frac{1}{\mu^2} \tag{20}$$

where C is the remaining constant of integration.

In (18), (19), and (20), only the zeroth order terms have been kept and no direct influence of the shear is apparent. One verifies that the

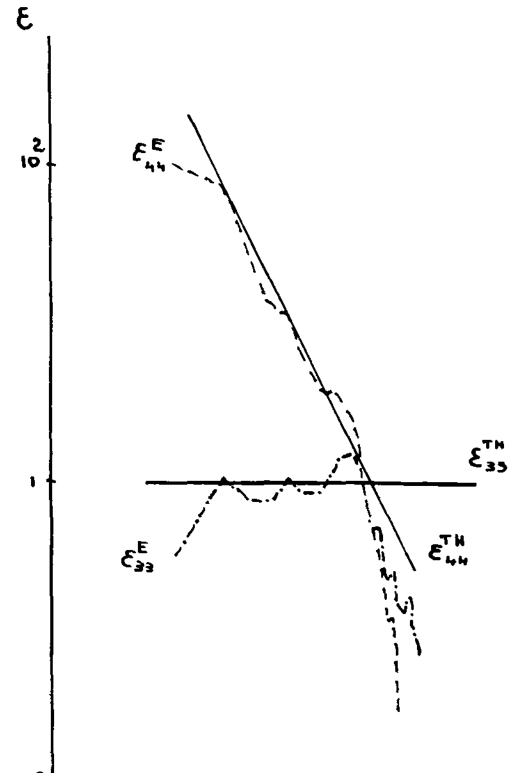

Fig. 1

ratios of kinetic horizontal and vertical energies to potential energy are the same as one would have obtained from (10) or (14) neglecting the shear altogether. However, in the absence of the weak mean shear, (10) and (14) are systems of homogeneous algebraic equations allowing only the determination of these ratios while, here, the requirement that the differential equations (14) be satisfied at the first order in β removes the indetermination over the zeroth order spectral components which are obtained not in terms of one of them as for $\beta = 0$ but entirely, except only for the constant C.

Discussion

Fig. 1 shows a comparison between the theoretical predictions for the spectra of the reduced vertical energy ε_{33} and the reduced potential energy ε_{44} with the observations made by Voorhis (Voorhis, 1968) at a depth of 725 meters relative to a drifting frame of reference at site D (39°20′ N, 70° W). For frequencies smaller than the Brunt-Väisälä frequency ($\mu \leqslant 1$) where the theoretical model is valid, there appears to be an excellent agreement between observed and predicted spectra.

The comparison of horizontal energy spectra is less straightforward as these spectra are usually measured in a fixed frame of reference. For instance, corresponding to Voorhis's experiments, one disposes only of eulerian spectra at 511 and 1 013 meters. Interpolating between those and correcting to lagrangian spectra as well as possible from estimations of the mean velocity, one finds a fair agreement between theoretical and observed values except in the near vicinity of the Brunt-Väisälä frequency where the theoretical curve goes to zero while the experimental one tends on the contrary to flatten, perhaps because of the appearance of non-linear effects.

As these interpolation and correction are rather crude with the available data, the comparison is not reported here. The discussion of eulerian horizontal kinetic energy spectra, extending beyond the Brunt-Väisälä frequency, by methods of weak ocean turbulence theory (Nihoul, 1970, 1971) will be reported in a forthcoming paper.

REFERENCES

Fofonoff, N. P. 1969. Spectral characteristics of internal waves in the ocean. *Deep Sea Res.*, Suppl. to vol. 16, 59–71.

Frankignoul, C. J. 1970. The effect of weak shear and rotation on internal waves. *Tellus 22*, 194–204.

Nihoul, J. C. J. 1970. Small scales erratic motions in the deep ocean. *Proc. Second Colloquium on Ocean Hydrodynamics. Liége, March 17–20, 1970*, 1–19.

Nihoul, J. C. J. 1971. Proper and improper turbulence in the sea, *Proc. Convegno sulla Theoria della Turbolenza, Rome, 26–29 aprile 1971.*

Phillips, O. M. 1966. *The dynamics of the upper ocean.* Cambridge University Press.

Voorhis, A. D. 1968. Measurements of vertical motion and the partition of energy in the New England slope water. *Deep Sea Res. 15*, 599–608.