The steepening of long, internal waves

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ABSTRACT

An investigation is made of the deformation of long waves in a stratified fluid as influenced by the curvature of the vertical profile of density, by the terms neglected in the Boussinesq approximation and by the vertical shear. In the continuous case these are taken to be small effects and the results for waves with an even number of nodal surfaces are as follows: (1) If the density profile is concave upward, a wave of elevation tends to break in the direction of the propagation, i.e. the forward part of the wave steepens; if the profile is convex upward, it tends to break backward. (2) The terms neglected in the Boussinesq approximation cause a wave of elevation to tend to break backward. (3) If the wave is propagating toward negative x and the fluid speed along the x-axis increases with height, the wave of elevation tends to break forward. To the first order there is no tendency for breaking if the waves have an odd number of nodal surfaces. An analysis is also made of a two-fluid system. The same tendencies exist in (1) above provided the concave-upward profile corresponds to a shallower lower fluid. In (2) and (3) above the tendencies for breaking in the continuous system and in the two-fluid system are opposite.

1. Introduction

Water waves of finite amplitude tend to break forward[1] (in the direction of propagation) if the waves are sufficiently long. This was first discussed by Airy (Lamb, 1932) who showed that a given wave height η' above the undisturbed height h propagates at a speed, which, to first order, is

$$\sqrt{gh}\left[1+\frac{3}{2}\frac{\eta'}{h}\right]+0(\alpha^2) \tag{1}$$

where $\alpha = a/h$ and a is representative of the wave amplitude. Thus the higher parts travel faster than the lower parts and the wave tends to break forward. It is not obvious from the discussion in Lamb, but it may be shown that (1) implies that $\mu \ll \alpha$ where $\mu = h^2/\lambda^2$ and λ is representative of the length of the disturbance. The basic purpose of this paper is to examine the steepening of long waves in a stratified fluid. In one section, however, the method is extended to the case of Rossby waves on a β-earth.

At first glance, it would seem natural to investigate the stratified problem by considering long, finite-amplitude waves in a model in which (1) there is a linear vertical density gradient; (2) the Boussinesq approximation is made; (3) the fluid is basically at rest. However, we may show that long, finite-amplitude waves of arbitrary form may propagate without change of shape in such a model. Thus, assuming a uniform current, under these conditions the steady-state disturbance streamfunction ψ' satisfies the differential equation (Long, 1953)

$$\frac{\partial^2\psi'}{\partial z'^2}+\sigma^2\psi'=0$$

where z' is the vertical coordinate and σ is the constant ratio of the Brunt–Väisälä frequency to the speed of the current. This has a solution

$$\psi' = f(x')\sin n\pi\frac{z'}{H}, \quad \sigma H = n\pi$$

where n is an integer, x' is the horizontal coordinate, and we have assumed rigid surfaces at $z'=0$ and $z'=H$. Notice that $f(x')$ is arbitrary.

[1] In the text the term "wave-breaking" is used to denote a tendency for a long wave to steepen either in its forward or rear portions as it propagates. Whether the wave actually breaks in a manner similar to the breaking of water waves will depend on effects of higher order and on non-hydrostatic pressures.

Thus, this model is rather special and steepening will depend on (1) curvature of the density profile, (2) terms neglected in the Boussinesq approximation, or (3) vertical shear. The importance of these factors for the existence of the solitary wave in stratified fluids was pointed out by Long (1965) and Benjamin (1966).

In the following sections, we investigate wave-steepening as affected by these factors. Section 2 is concerned with a two-fluid system while Sections 3, 4 and 5 contain discussion of the above three factors in a continuously stratified fluid. Section 6 derives similar results for long waves on a β-earth. In the final section we mention briefly the effect of non-hydrostatic pressures.

2. Wave deformation in a two-fluid system

Long (1956) derived some results related to wave-steepening in a system of two superimposed liquids, but the investigation was rather special and it is instructive to derive a general result by another technique. The total depth is H, the lower fluid has an undisturbed depth h_1, the height of the disturbance above h_1 is η_1' and subscripts "1" and "2" are used to denote the lower and upper layers, respectively. The basic equations, assuming hydrostatics, are

$$\frac{\partial u_1'}{\partial t'} + (U_1 + u_1')\frac{\partial u_1'}{\partial x'} = -g'\frac{\partial \eta_1'}{\partial x'} - \frac{1}{\varrho_1}\frac{\partial \pi'}{\partial x'} \tag{2}$$

$$\frac{\partial u_2'}{\partial t'} + (U_2 + u_2')\frac{\partial u_2'}{\partial x'} = -\frac{1}{\varrho_2}\frac{\partial \pi'}{\partial x'} \tag{3}$$

$$\frac{\partial \eta_1'}{\partial t'} + (U_1 + u_1')\frac{\partial \eta_1'}{\partial x'} + (h_1 + \eta_1')\frac{\partial u_1'}{\partial x'} = 0 \tag{4}$$

$$-\frac{\partial \eta_1'}{\partial t'} - (U_2 + u_2')\frac{\partial \eta_1'}{\partial x'} + (H - h_1 - \eta_1')\frac{\partial u_2'}{\partial x'} = 0 \tag{5}$$

where we have assumed rigid surfaces at $z' = 0$ and $z' = H$; π' is the pressure at the upper rigid surface, g' is modified gravity,

$$g' = \frac{(\varrho_1 - \varrho_2)}{\varrho_1} g \tag{6}$$

and U_1, U_2 are the basic currents in the two fluids.

We now scale Eqs. (2)–(5) using the following considerations: (1) The zero-order terms are those corresponding to steady, long waves of infinitesimal amplitude; (2) The time-dependent tendency for steepening arises from the effect of the non-linear terms in these equations. Specifically, we assume that the first and third terms in Eq. (2) are of the same order of magnitude. We are thus led to the following definitions:

$$u_1' = U_1 \alpha u_1,\ u_2' = U_1 \alpha u_2,\ \eta_1' = a\eta_1,\ \pi' = \varrho_2 U_1^2 \alpha \pi$$

$$x' = \lambda x,\ z' = h_1 z,\ t' = \frac{\lambda}{U_1 \alpha} t,\ \gamma = \frac{g' h_1}{U_1^2},\ s = \frac{\varrho_2}{\varrho_1}$$

$$r = \frac{U_2}{U_1},\ R = \frac{H - h_1}{h_1},\ \alpha = \frac{a}{h_1} \tag{7}$$

where α is considered to be small, but s, r and R are of order one, Eqs. (2)–(5) become

$$\alpha\frac{\partial u_1}{\partial t} + (1 + \alpha u_1)\frac{\partial u_1}{\partial x} = -\gamma\frac{\partial \eta_1}{\partial x} - s\frac{\partial \pi}{\partial x} \tag{8}$$

$$\alpha\frac{\partial u_2}{\partial t} + (r + \alpha u_2)\frac{\partial u_2}{\partial x} = -\frac{\partial \pi}{\partial x} \tag{9}$$

$$\alpha\frac{\partial \eta_1}{\partial t} + (1 + \alpha u_1)\frac{\partial \eta_1}{\partial x} + (1 + \alpha\eta_1)\frac{\partial u_1}{\partial x} = 0 \tag{10}$$

$$-\alpha\frac{\partial \eta_1}{\partial t} - (r + \alpha u_2)\frac{\partial \eta_1}{\partial x} + (R - \alpha\eta_1)\frac{\partial u_2}{\partial x} = 0 \tag{11}$$

We now assume expansions in α, for example,

$$u_1 = u_1^{(0)} + \alpha u_1^{(1)} + 0(\alpha^2) \tag{12}$$

The zeroth-order equations may be integrated directly. If we assume that the disturbance vanishes at great distances along the x-axis, we obtain

$$u_1^{(0)} + \gamma\eta_1^{(0)} + s\pi^{(0)} = 0 \tag{13}$$

$$r u_2^{(0)} + \pi^{(0)} = 0 \tag{14}$$

$$\eta_1^{(0)} + u_1^{(0)} = 0 \tag{15}$$

$$-r\eta_1^{(0)} + Ru_2^{(0)} = 0 \tag{16}$$

Solutions are

$$\eta_1^{(0)} = f(x,t) \tag{17}$$

$$u_1^{(0)} = -f(x,t) \tag{18}$$

$$u_2^{(0)} = \frac{r}{R} f(x,t) \tag{19}$$

$$\pi^{(0)} = -\frac{r^2}{R} f(x,t) \tag{20}$$

$$\gamma - 1 - \frac{sr^2}{R} = 0 \tag{21}$$

where $f(x,t)$ is arbitrary. The equations of the next approximation yield

$$\frac{\partial u_1^{(1)}}{\partial x} + \gamma \frac{\partial \eta_1^{(1)}}{\partial x} + s\frac{\partial \pi^{(1)}}{\partial x} = f_t - ff_x \tag{22}$$

$$r\frac{\partial u_2^{(1)}}{\partial x} + \frac{\partial \pi^{(1)}}{\partial x} = -\frac{r}{R} f_t - \frac{r^2}{R^2} ff_x \tag{23}$$

$$\frac{\partial \eta_1^{(1)}}{\partial x} + \frac{\partial u_1^{(1)}}{\partial x} = 2ff_x - f_t \tag{24}$$

$$-r\frac{\partial \eta_1^{(1)}}{\partial x} + R\frac{\partial u_2^{(1)}}{\partial x} = f_t + \frac{2r}{R} ff_x \tag{25}$$

These require that

$$f_t + \sigma ff_x = 0, \quad \sigma = \frac{3\left(\frac{sr^2}{R^2} - 1\right)}{2\left(1 + \frac{sr}{R}\right)} \tag{26}$$

This non-linear partial differential equation has the general solution

$$f = G(x - \sigma ft) \tag{27}$$

where G is arbitrary, or, approximately,

$$\eta_1 = \eta_1(x - \sigma\eta_1 t) \tag{28}$$

If σ is non-zero, this yields a general tendency for breaking. As examples, if $s = 0$, $\sigma = -(3/2)$, and we obtain the result in Eq. (1). If $s = 1$ (Boussinesq approximation) and $r = 1$ (zero shear),

$$\sigma = \frac{3(1-R)}{2R}$$

It may be shown that this transforms to the result of Long (1956). It reveals that waves of elevation on a shallow lower fluid tend to break forward as they propagate ($\sigma < 0$), while waves of elevation on a deep lower fluid tend to break backward ($\sigma > 0$). In case the two fluids are of equal depth, there is no tendency to this order for the waves to break. This case of (1) equal depth, (2) assumed Boussinesq approximation and (3) zero shear, is analogous to the special model discussed in Section 1, i.e. a system of two fluids of equal depth corresponds, as we might expect, to a fluid with linear density gradient in the continuous case.

Two other cases may be discussed. In the first, we drop the Boussinesq approximation and assume $s = 1 - \varepsilon$ where ε is positive and small. Then for equal depth and zero shear

$$\sigma = -\tfrac{3}{4}\varepsilon \tag{29}$$

so that waves of elevation tend to break forward as they propagate. In the second case, we assume a slight increase of speed with height so that $r = 1 + \varepsilon$. Then with equal depth and with the Boussinesq approximation,

$$\sigma = \frac{3\varepsilon}{2} \tag{30}$$

so that waves of elevation tend to break backward as they propagate. With a decrease of speed with height, *they tend to break forward.*

3. A curved density profile in a continuously stratified liquid

We simplify the analysis by assuming a *slightly* curved profile of density in the form

$$\varrho = \varrho_0 - \frac{N^2\varrho_0}{g} z' - \frac{C\varrho_0}{g} z'^2 + \frac{\varrho_0}{g}\xi'(x,z,t) \tag{31}$$

where ξ' is related to the density perturbation. The constant N^2 must be positive but C may be of either sign. With the Boussinesq approximation and a uniform current U, the equations are

$$\frac{\partial u'}{\partial t'}+U\frac{\partial u'}{\partial x'}+u'\frac{\partial u'}{\partial x'}+w'\frac{\partial u'}{\partial z'}=-\frac{\partial P'}{\partial x'} \quad (32)$$

$$\frac{\partial w'}{\partial t'}+U\frac{\partial w'}{\partial x'}+u'\frac{\partial w'}{\partial x'}+w'\frac{\partial w'}{\partial z'}=-\frac{\partial P'}{\partial z'}-\xi' \quad (33)$$

$$\frac{\partial \xi'}{\partial t'}+U\frac{\partial \xi'}{\partial x'}+u'\frac{\partial \xi'}{\xi x'}+w'\frac{\partial \xi'}{\partial z'}-w'(N^2+2Cz')=0 \quad (34)$$

$$\frac{\partial u'}{\partial x'}+\frac{\partial w'}{\partial z'}=0 \quad (35)$$

where

$$P'=p'/\varrho_0+gz'-N^2\frac{z'^2}{2}-C\frac{z'^3}{3} \quad (36)$$

and p' is the pressure.

We non-dimensionalize Eqs. (32)–(35) by taking as zero-order terms those yielding a steady wave of infinitesimal amplitude in a current U for a fluid with linear density gradient. As we have mentioned, such a steady solution remains an exact solution of the full, non-linear equations of the special model of Section 1 even when the amplitude of the disturbance is finite. As a result of this fact, we also assume that the time-dependent term in (34) is of the same order as the curvature term in this equation. We take a to be representative of the amplitude of the disturbance and λ and h to be representative of the horizontal and vertical scales. Then the above scaling assumptions suggest the following definitions:

$$\xi'=aN^2\xi,\ u'=aNu,\ w'=\frac{aU}{\lambda}w,\ x'=\lambda x,$$

$$z'=hz,\ t'=\frac{\lambda N^3}{CU^2}t$$

$$P'=aNUP,\ \varepsilon=CUN^{-3},\ h=UN^{-1},\ \alpha=\frac{a}{h},$$

$$\mu=\frac{h^2}{\lambda^2},\ m=\frac{CU^2}{aN^4} \quad (37)$$

Eqs. (32)–(35) become

$$m\alpha u_t+u_x+\alpha uu_x+\alpha wu_z=-P_x \quad (38)$$

$$m\alpha\mu w_t+\mu w_x+\alpha\mu uw_x+\alpha\mu ww_z=-P_z-\xi \quad (39)$$

$$m\alpha\xi_t+\xi_x+\alpha u\xi_x+\alpha w\xi_z-w(1+2m\alpha z)=0 \quad (40)$$

$$u_x+w_z=0 \quad (41)$$

We begin the solution by assuming that α is small, that m is of order one and that $\mu<\alpha^2$. Thus we expand as[1]

$$u=u^{(0)}+\alpha u^{(1)}+\alpha^2u^{(2)}+0(\alpha^3) \quad (42)$$

The zero-order solutions are governed by

$$u_x^{(0)}+P_x^{(0)}=0 \quad (43)$$

$$P_z^{(0)}+\xi^{(0)}=0 \quad (44)$$

$$u_x^{(0)}+w_z^{(0)}=0 \quad (45)$$

$$\xi_x^{(0)}-w^{(0)}=0 \quad (46)$$

In terms of the streamfunction, an appropriate solution is

$$\psi^{(0)}=f_x(x,t)\sin z,\ \ \xi^{(0)}=f_x(x,t)\sin z \quad (47)$$

where $f(x, t)$ is an arbitrary function of x and t. This solution satisfies the kinematic condition at the bottom of the channel, $z'=0$. At the top, $z'=H$, we must have $\psi^{(0)}=0$ so that

$$\frac{HN}{U}=n\pi \quad (48)$$

This is the condition U must satisfy for the waves to remain stationary in the current. Conversely, in a coordinate system in which the current is zero, the waves propagate to the left at the speed U.

The equations of the next approximation may be reduced to the following if we make the assumption that the disturbance is localized or that the fluid is undisturbed at $|x|=\infty$:

$$\psi_{zz}^{(1)}+\psi^{(1)}=2mf_t\sin z-2mzf_x\sin z \quad (49)$$

Eq. (49) and the kinematic condition at the bottom are satisfied by

$$\psi^{(1)}=-mf_tz\cos z+\tfrac{1}{2}mz^2f_x\cos z-\tfrac{1}{2}mf_xz\sin z+g(x,t)\sin z \quad (50)$$

[1] We may mention here a paper by Benney (1966) which contains a general discussion of long waves in a stratified fluid. Benney emphasized waves of permanent form and did not expand his solutions to high enough order to obtain the phenomena discussed here.

The condition at the top yields the form

$$f = f(x + \tfrac{1}{2}n\pi t) \tag{51}$$

so that to the zeroth-order the disturbance propagates toward the left without change of shape at the dimensional speed $U(1 - \frac{1}{2}n\pi m\alpha)$. To obtain a tendency for steepening, we must obtain information about $g(x, t)$ and this requires one more term in the approximation.

Using (47), (50) and the following equation for $\xi^{(1)}$,

$$\xi^{(1)} = +\tfrac{3}{4}mf'z \sin z - \tfrac{1}{4}mn\pi f' \sin z - \tfrac{1}{2}mn\pi f'z \cos z + \tfrac{1}{2}z^2 mf' \cos z + g \sin z \tag{52}$$

the equations of the next approximation are

$$\psi^{(2)}_{zzx} + \psi^{(2)}_x = \tfrac{3}{2}m^2 n\pi f''z \sin z - \tfrac{1}{4}m^2n^2\pi^2 f'' \sin z - \tfrac{1}{2}m^2n^2\pi^2 f''z \cos z + \tfrac{3}{2}m^2z^2 n\pi f'' \cos z - m^2z^3 f'' \times \cos z + m^2z^2 f'' \sin z + 2m(1 - \cos 2z) f'f'' + 2mg_t \sin z - 2zg_x m \sin z \tag{53}$$

with solution

$$\psi^{(2)}_x = -mg_t z \cos z - 2mg_x S_2 + 2mf'f'' + \tfrac{2}{3}mf'f'' \cos 2z + \tfrac{1}{8}m^2n^2\pi^2 f''z \cos z + \tfrac{3}{2}m^2 n\pi f'' S_2 - \tfrac{1}{2}m^2n^2\pi^2 f'' C_2 + \tfrac{3}{2}m^2 n\pi f'' C_3 - f''m^2 C_4 + m^2 f'' S_3 + h(x, t) \sin z + k(x, t) \cos z \tag{54}$$

where

$$S_2 = -\frac{z^2}{4}\cos z + \frac{z}{4}\sin z$$

$$C_2 = \frac{z^2}{4}\sin z + \frac{z}{4}\cos z$$

$$S_3 = -\frac{z^3}{6}\cos z + \frac{z^2}{4}\sin z + \frac{z}{4}\cos z$$

$$C_3 = \frac{z^3}{6}\sin z + \frac{z^2}{4}\cos z - \frac{z}{4}\sin z$$

$$C_4 = \frac{z^4}{8}\sin z + \frac{z^3}{4}\cos z - \frac{3}{8}z^2 \sin z - \frac{3}{8}z \cos z \tag{55}$$

Imposing the two conditions at top and bottom, we obtain

$$g_t - \tfrac{1}{2}n\pi g_x = M(x + \tfrac{1}{2}n\pi t) \tag{56}$$

where

$$M = +\frac{8}{3}\frac{f'f''}{n\pi}[(-1)^n - 1] + \frac{5}{8}mf'' - \frac{1}{6}mn^2\pi^2 f'' \tag{57}$$

Eq. (56) has a solution

$$g = tM \tag{58}$$

We may interpret the solution by finding first an expression for the elevation of the wave. Thus we may write

$$\xi' = N^2(z' - z_0') + C(z'^2 - z_0'^2) \tag{59}$$

where z_0' is the height of the density surface in the undisturbed region. This may be rearranged to yield

$$\delta \equiv \frac{(z' - z_0')}{h} = \alpha\xi - 2mz\alpha^2\xi + 0(\alpha^3) \tag{60}$$

This is of the form

$$\delta = \alpha f' \sin z + \alpha^2 P(x + \tfrac{1}{2}n\pi t, z) + \alpha^2 tM \sin z + 0(\alpha^3) \tag{61}$$

Defining the speed of propagation of a given height δ by the equation

$$\frac{\partial\delta}{\partial t} + c\frac{\partial\delta}{\delta x} = 0$$

we obtain,

$$c = -\tfrac{1}{2}n\pi - \frac{\alpha M}{f''} + 0(\alpha^2) \tag{62}$$

or

$$c = \text{constant} - \frac{8}{3}\frac{\alpha f'}{n\pi}[(-1)^n - 1] + 0(\alpha^2) \tag{63}$$

The dimensional speed of propagation c' is given approximately by

$$c' = \text{constant} - \frac{8}{3}\frac{CaH}{n^2\pi^2 N}f'[(-1)^n - 1]$$

Thus there is no tendency for breaking if n is even. If n is odd, the waves will tend to break.

The direction in which they break depends on the curvature. If the density profile is concave upward, waves of elevation tend to break forward. If the density profile is convex upward, they tend to break backward. As we have mentioned, this is exactly the behavior of the two-fluid model if we associate the curvature of the density profile with the depth difference of the two fluids.

Notice that the approximation we have obtained is valid only for small times, i.e. for

$$t' < \frac{3\lambda N n^2 \pi^2}{16\, Cah}$$

It may be useful to introduce values of the various constants appropriate to disturbances in a laboratory channel. We envisage a total fluid depth, $H = 30$ cm, an amplitude $a = 1$ cm, and a wave length $\lambda = 300$ cm. With $n = 1$, we then have $h \simeq 10$ cm. The conditions $\mu < \alpha^2$, $\alpha < 1$ are satisfied by these values. With density variations of the order of ten percent over the fluid depth, reasonable values of N^2 and C may be $N^2 = 3$ sec^{-2}, $C = 10^{-1}$cm^{-1} sec^{-2}. Then $U \simeq 17$ cm sec^{-1} and $\varepsilon = \frac{1}{3}$. We compute that the solution is valid for $t' < 10^3$ sec, approximately. The distance from ridge to trough changes by ten percent in a time interval of 7–8 sec.

4. The effect of terms neglected in the Boussinesq approximation

In this section we assume a zero shear and a linear density gradient (Eq. (31) without the curvature term), but we do not make the Boussinesq approximation. The scaled differential equations are now

$$(1 - m\alpha z + m\alpha^2\xi)(m\alpha u_t + u_x + \alpha u u_x + \alpha w u_z) = -P_x \quad (64)$$

$$(1 - m\alpha z + m\alpha^2\xi)(m\alpha\mu w_t + \mu w_x + \alpha\mu u w_x + \alpha\mu w w_z) = -P_z - \xi \quad (65)$$

$$m\alpha\xi_t + \xi_x + \alpha u\xi_x + \alpha w\xi_z - w = 0 \quad (66)$$

$$u_x + w_z = 0 \quad (67)$$

where

$$u' = aNu,\ w' = \frac{aU}{\lambda} w,\ \xi' = aN^2\xi,\ x' = x\lambda,$$

$$z' = zh,\ t' = \frac{\lambda g}{UN^2h} t$$

$$P' = aN^2hP,\ \alpha = \frac{a}{h},\ h = UN^{-1},\ \mu = \frac{h^2}{\lambda^2},$$

$$\varepsilon = \frac{N^2h}{g},\ \frac{N^2h^2}{ga} = m \quad (68)$$

Again, we consider α small, m of order one and $\mu < \alpha^2$.

The procedure is now identidal to that of Section 3 and we will omit the details. The zeroth-order approximation is the same:

$$\psi^{(0)} = f' \sin z,\ \xi^{(0)} = f' \sin z \quad (69)$$

where now

$$f = f(x + \tfrac{1}{4}n\pi t) \quad (70)$$

and again

$$\frac{HN}{U} = n\pi \quad (71)$$

The first-order approximation yields

$$\psi^{(1)} = -\frac{n\pi}{4} mf'z\cos z + m\frac{f'}{4} z\sin z + m\frac{f'}{4} z^2\cos z + g(x,t)\sin z \quad (72)$$

$$\xi^{(1)} = -\frac{n\pi}{4} mf'z\cos z + m\frac{f'}{4} z\sin z + m\frac{f'}{4} z^2\cos z - \tfrac{1}{4} n\pi mf'\sin z + g(x,t)\sin z \quad (73)$$

The second-order approximation yields

$$\psi_x^{(2)} = h(x,t)\sin z + k(x,t)\cos z - mg_t z\cos z - m^2 f'' z\cos z\left(\tfrac{1}{8} - \tfrac{3}{32}n^2\pi^2\right) + m\frac{g_x}{2} z\sin z - \frac{n\pi}{8} m^2 f'' z\sin z - mg_x S_2 + \frac{m}{2} f'f''\cos 2z + \tfrac{1}{2}mf'f''$$

$$+\left(\frac{7}{4}-\frac{n^2\pi^2}{8}\right)f''m^2C_2+\tfrac{7}{8}n\pi m^2f''S_2$$

$$+\tfrac{3}{8}m^2f''n\pi C_3-\tfrac{3}{2}m^2f''S_3-\tfrac{1}{4}m^2f''C_4 \quad (74)$$

The two conditions at top and bottom now yield

$$g_t-\frac{n\pi}{4}g_x=M \quad (75)$$

so that

$$g=Mt \quad (76)$$

where M is

$$M=\frac{f'f''}{n\pi}[(-1)^n-1]+mf''\left(\frac{1}{32}+\frac{n^2\pi^2}{8}\right) \quad (77)$$

The wave speed is now

$$c=-\frac{n\pi}{4}-\frac{\alpha M}{f''}+0(\alpha^2) \quad (78)$$

or

$$c=\text{constant}-\frac{\alpha f'}{n\pi}[(-1)^n-1]+0(\alpha^2) \quad (79)$$

Again, if n is even there is no tendency to this order for the waves to break. If n is odd, however, the waves tend to break backward as they progress. Surprisingly, this is the opposite of the behavior in a two-fluid system. In both cases the variable component of the dimensional speed is proprtional to

$$U\frac{\Delta\varrho}{\varrho_0}\frac{a}{h}$$

and we would have expected a similar behavior for the wave steepening.

5. The effect of shear

We now investigate the effect of shear. We adopt the Boussinesq approximation and assume a linear, basic density gradient (Eq. (31) without the curvature term). The basic velocity is $U+Cz$ where C is of either sign. The scaled equations become

$$m\alpha u_t+u_x+m\alpha u_x z+\alpha uu_x+\alpha wu_z+m\alpha w=-P_x \quad (80)$$

$$m\alpha\mu w_t+\mu w_x+\mu m\alpha w_x z+\alpha\mu uw_x+\alpha\mu ww_z=-P_z-\xi \quad (81)$$

$$m\alpha\xi_t+\xi_x+m\alpha z\xi_x+\alpha u\xi_x+\alpha w\xi_z-w=0 \quad (82)$$

$$u_x+w_z=0 \quad (83)$$

where

$$u'=\frac{aN^2h}{U}u,\; w'=\frac{aN^2h^2}{U\lambda}w,\; P'=aN^2hP,$$

$$\xi'=aN^2\xi,\; x'=x\lambda,\; z'=zh,\; t'=\frac{\lambda}{Ch}t,$$

$$h^2=\frac{U^2}{N^2},\; \mu=\frac{h^2}{\lambda^2},\; \varepsilon=\frac{Ch}{U},\; \alpha=\frac{a}{h},\; m=\frac{Ch^2}{Ua} \quad (84)$$

Again the procedure is identical to that of Section 3 and we omit the details. The zeroth-order approximation is the same:

$$\psi^{(0)}=f'\sin z,\; \xi^{(0)}=f'\sin z \quad (85)$$

where

$$f=f\left(x-\frac{n\pi}{2}t\right) \quad (86)$$

and again

$$\frac{HN}{U}=n\pi \quad (87)$$

The first approximation yields

$$\psi^{(1)}=\frac{mn\pi}{2}f'z\cos z-\tfrac{1}{2}f'm(z^2\cos z-z\sin z)$$
$$+g(x,t)\sin z \quad (88)$$

$$\xi^{(1)}=\frac{mn\pi}{2}f'z\cos z-\tfrac{1}{2}mf'z\sin z-\tfrac{1}{2}mf'z^2\cos z$$
$$+\frac{mn\pi}{2}f'\sin z+g(x,t)\sin z \quad (89)$$

The second-order approximation yields

$$\psi_x^{(2)}=h(x,t)\sin z+k(x,t)\cos z-mg_t z\cos z$$
$$+2mg_xS_2-\frac{n^2\pi^2}{2}f''m^2C_2+\tfrac{5}{2}n\pi f''m^2S_2$$
$$+\tfrac{3}{8}n^2\pi zf''m^2z\cos z+\tfrac{3}{2}n\pi f''m^2C_3$$
$$-2m^2f''S_3-\frac{m}{2}f'f''\cos 2z-\tfrac{3}{2}mf'f''$$
$$-m^2f''C_4 \quad (90)$$

The two conditions at top and bottom now yield

$$g_t + \frac{n\pi}{2} g_x = M \tag{91}$$

$$g = Mt \tag{92}$$

where M is

$$M = \frac{2}{n\pi} f'f''[1-(1)^n] + \tfrac{1}{12} m f'' n^2\pi^2 - \tfrac{1}{8} f'' m \tag{93}$$

The wave speed is

$$c = \frac{n\pi}{2} - \frac{\alpha M}{f''} + 0(\alpha^2) \tag{94}$$

or

$$c = \text{constant} - \frac{2\alpha f'}{n\pi}[1-(-1)^n] + 0(\alpha^2) \tag{95}$$

Again if n is even there is no tendency to this order for the waves to break, but if n is odd and the fluid speed increases with height, the wave tends to break forward. As in Section 4, this behavior is opposite to that found for a two-fluid system.

6. Waves on a β-earth

In this section, we discuss the tendency for wavebreaking for Rossby waves on a β-earth (Rossby, 1939). These waves are different physically from the internal gravity waves but the mathematical similarity is close. The single differential equation that governs this motion is

$$\left[\frac{\partial}{\partial t'} + (U + bz')\frac{\partial}{\partial x'} - \frac{\partial \psi'}{\partial z'}\frac{\partial}{\partial x'} + \frac{\partial \psi'}{\partial x'}\frac{\partial}{\partial z'}\right] \times \left[\frac{\partial^2 \psi'}{\partial x'^2} + \frac{\partial^2 \psi'}{\partial z'^2}\right] + \beta \frac{\partial \psi'}{\partial x'} = 0 \tag{96}$$

where we have introduced a basic shear in the west-wind current. As in the stratified case, finite-amplitude waves satisfy the exact nonlinear equations if the shear is neglected. The scaled equation is

$$\psi_x + \psi_{zzx} + m\alpha\psi_{zzt} + m\alpha z\psi_{zzx} + m\alpha\mu\psi_{xxt} + m\alpha z\mu\psi_{xxx} + \mu\psi_{xxx} - \alpha\psi_z(\psi_{zzx} + \mu\psi_{xxx}) + \alpha\psi_x(\psi_{zzz} + \mu\psi_{xxz}) = 0 \tag{97}$$

where

$$\psi' = Ua\psi,\ x' = \lambda x,\ z' = hz,\ t' = \frac{\lambda t}{bh},\ \beta h^2 = U$$

$$\frac{bh}{U} = \varepsilon,\ \alpha = \frac{a}{h},\ \mu = \frac{h^2}{\lambda^2},\ m = \frac{bh^2}{Ua} \tag{98}$$

The zeroth-order approximation, assuming rigid walls at two latitude circles a distance H apart, is

$$\psi^{(0)} = f_x(x,t)\sin z \tag{99}$$

where

$$f = f\left(x - \frac{n\pi}{2}t\right) \tag{100}$$

$$\frac{\beta^{\frac{1}{2}} H}{U^{\frac{1}{2}}} = n\pi \tag{101}$$

The first-order approximation is

$$\psi^{(1)} = \frac{m}{4} f' n\pi z \cos z - \frac{mf'}{4}(z^2\cos z - z\sin z) + g(x,t)\sin z \tag{102}$$

The second-order approximation is

$$\psi_x^{(2)} = -\frac{m}{2} f'f'' - \frac{m}{6} f'f''\cos 2z + \frac{m^2}{8} n^2\pi^2 f'' z\cos z - \frac{n^2\pi^2}{8} m^2 f'' C_2 + \tfrac{7}{8} m^2 n\pi f'' S_2 + m g_x S_2 + \tfrac{3}{8} m^2 f'' n\pi C_3 - \tfrac{3}{4} m^2 f'' S_3 - \frac{m^2}{4} f'' C_4 - \tfrac{1}{2} m g_t z\cos z + h(x,t)\sin z + k(x,t)\cos z \tag{103}$$

The conditions at the two walls yield

$$g_t + \frac{n\pi}{2} g_x = M,\quad g = Mt \tag{104}$$

where

$$M = \frac{4}{3n\pi} f''f'[1-(1)^n] + \tfrac{1}{16} m n^2\pi^2 f'' - \tfrac{3}{16} m f'' \tag{105}$$

The displacement function is

$$\delta \equiv \frac{z' - z'_0}{h} = \alpha\psi - m\alpha^2 z\psi + 0(\alpha^3) \qquad (106)$$

The speed of propagation of a given value of δ is

$$c = \text{const.} - \frac{4\alpha f'}{3n\pi}[1 - (-1)^n] + 0(\alpha^2) \qquad (107)$$

with a dimensional form which is, approximately,

$$c' = \text{const.} - \frac{4ba}{3n\pi} f'[1 - (-1)^n] \qquad (108)$$

Then, for example, if the wind decreases with latitude, as is normally the case, the crests steepen in their eastern portions when n is odd.

7. Effects of vertical accelerations

In the previous sections we have completely neglected vertical accelerations by assuming that the wave lengths of the disturbances are very long. Although the assumption may be justified for the small times for which the above analyses are valid, tendencies for steepening involve a shortening of the wave length and an increasing importance of non-hydrostatic pressures as time goes on. The disturbance may then lose its tendency to steepen and develop into a solitary or cnoidal wave or some other stable configuration. The problem is very difficult, but some insight may be obtained from the corresponding water-wave problem. This was studied long ago by Boussinesq (1872) and Korteweg & deVries (1895), but a simpler version of the governing differential equation is discussed by Cole (1968).[1] The result is

$$\varphi_{\bar{t}} + \tfrac{3}{4}\varphi_{\bar{x}}^2 = -\frac{K^2}{6}\varphi_{\bar{x}\bar{x}\bar{x}} \qquad (109)$$

where $\varphi(\bar{x}, \bar{t})$ is the first approximation to the velocity potential at the bottom of the channel, $K = \mu/\alpha$, and

$$\bar{t} = \alpha t$$

$$\bar{x} = x - t \qquad (110)$$

where x and t are non-dimensional distance and time. If K is taken to be zero, the pressure is hydrostatic and we obtain breaking waves with a first approximation to the wave speed as given in Eq. (1). However, it is evident that the steepening process will eventually make the right-hand side of (109) important for any K.

We may discuss an analogous approach for the problem of Section 3. If we let $\mu = K\alpha$, the first approximation is of the same form as (47):

$$\psi^{(0)} = f(x, t)\sin z, \quad \xi^{(0)} = f(x, t)\sin z \qquad (111)$$

where $f(x, t)$ is arbitrary. The next approximation is also straight-forward. It is

$$\psi_x^{(1)} = -\frac{z\cos z}{2}(2mf_t - Kf_{xxx}) + \frac{mf_x}{2} \times (z^2\cos z - z\sin z) + g\sin z \qquad (112)$$

where $g(x, t)$ is arbitrary and we have imposed the kinematic condition at the lower boundary. The condition at the upper boundary yields

$$f_t - \frac{n\pi}{2} f_x = \varepsilon f_{xxx}, \quad \varepsilon = \frac{K}{2m} \qquad (113)$$

If we use new, independent variables similar to those in Eq. (110),

$$\bar{t} = t$$

$$\bar{x} = x + \tfrac{1}{2}n\pi t \qquad (114)$$

we get

$$f_{\bar{t}} = \varepsilon f_{\bar{x}\bar{x}\bar{x}} \qquad (115)$$

For initial conditions $f = f_0(\bar{x})$, the solution is

$$f = f_0(\bar{x}) + \varepsilon\bar{t} f_0^{(\mathrm{iii})} + \frac{\varepsilon^2\bar{t}^2}{2!} f_0^{(\mathrm{vi})} + \dots \qquad (116)$$

For a single Fourier component of the initial disturbance, this becomes

$$f = A\cos[kx + (\tfrac{1}{2}n\pi k - \varepsilon k^3)t] \qquad (117)$$

which signifies that the effect of non-hydrostatic pressure is to change the speed of propagation without any tendency, to this order, for steepening. For a superposition of waves, the effect is to make the system dispersive.

[1] Cole's equation (5.2.124) contains an arithmetical error. An equation similar to (109) is integrated numerically by Zabusky & Galvin (1971) and the results compared with experiment. There is substantial agreement.

The next approximation may be carried through for arbitrary f_0. The results are complicated but the importance of the non-hydrostatic terms is undeniable. Indeed, one may demand that the disturbance does not suffer a change of form. The analysis then yields the solitary and cnoidal waves.

The effects of vertical accelerations in a two-fluid system, bounded above and below by rigid horizontal planes, may be investigated in a manner similar to the analogous problem of long water waves. With the same notation as in Section 2, we non-dimensionalize as follows:

$$\frac{\varphi_i'}{\sqrt{g'h_1^3}} = \varphi_i,\ \frac{x'}{h_1} = x,\ \frac{y'}{h_1} = y,\ t'\sqrt{\frac{g'}{h_1}} = t, R = \frac{H-h_1}{h_1}$$

$$\frac{U^2}{g'h_1} = F^2,\ \frac{\eta'}{h_1} = \eta,\ S = \frac{\varrho_2}{\varrho_1},\ g' = \frac{\varrho_1-\varrho_2}{\varrho_1}g \quad (118)$$

where φ_i' is the velocity potential in either layer, η' is the height of the interface above the undisturbed level h_1 at $|x'| = \infty$, and we have assumed the upper layer has a basic velocity U in the x'-direction. We move our coordinate system so that the lower fluid is basically at rest.

A combination of the two Bernoulli equations at the interface yields

$$-\varphi_{1t} + S\varphi_{2t} + \tfrac{1}{2}(\varphi_{1x}^2 + \varphi_{1y}^2) - \frac{S}{2}(\varphi_{2x}^2 + \varphi_{2y}^2) + \eta = -\frac{S}{2}F^2 \quad (119)$$

and the two kinematic conditions at the interface yield

$$\varphi_{1y} + \eta_t - \varphi_{1x}\eta_x = 0 \quad (120)$$

$$\varphi_{2y} + \eta_t - \varphi_{2x}\eta_x = 0 \quad (121)$$

We now assume expansions of the velocity potentials similar to that used by Rayleigh in his study of the solitary wave (Lamb, 1932):

$$\varphi_1 = F_1 - \frac{y^2}{2!}F_{1xx} + \frac{y^4}{4!}F_{1xxxx} + \dots \quad (122)$$

$$\varphi_2 = -Fx + F_2 - \frac{(y-1-R)^2}{2!}F_{2xx} + \frac{(y-1-R)^4}{4!}F_{2xxxx} + \dots \quad (123)$$

where F_1 and F_2 are functions of x and t. These satisfy Laplace's equation and the kinematic conditions at the top and bottom of the channel. In addition these expansions are inherently suitable for our problem since we assume that the disturbances are long. Then higher derivatives in the x-direction are small. Specifically, we assume

$$\frac{\partial}{\partial x} \sim \frac{\partial}{\partial t} \sim \alpha^{\frac{1}{2}} \quad (124)$$

where $\alpha = a/h_1$, and a is the maximum amplitude of the disturbance. We assume, of course, that α is small. Since $\eta \sim \alpha$, it follows from the Bernoulli equation that φ_t and, therefore, φ_x are also of order α.

Using (122)–(124), equations (119)–(121) become

$$-F_{1t} + SF_{2t} + \tfrac{1}{2}F_{1xxt} - \frac{S}{2}R^2F_{2xxt} + \tfrac{1}{2}F_{1x}^2 - \frac{S}{2}(F_{2x}^2 - 2FF_{2x} + R^2FF_{2xxx}) + \eta = 0(\alpha^3) \quad (125)$$

$$-\eta F_{1xx} - F_{1xx} + \tfrac{1}{6}F_{1xxxx} + \eta_t - F_{1x}\eta_x = 0(\alpha^{7/2}) \quad (126)$$

$$-\eta F_{2x} + RF_{2xx} - \frac{R^3}{6}F_{2xxxx} + \eta_t - (F_{2x} - F)\eta_x = 0(\alpha^{7/2}) \quad (127)$$

The equations of the first approximation may be combined to yield

$$\eta_{xx}\left(1 - \frac{SF^2}{R}\right) - 2\frac{SF}{R}\eta_{xt} - \eta_{tt}\left(1 + \frac{S}{R}\right) = 0(\alpha^3) \quad (128)$$

We choose the solution,

$$\eta = \eta(x - bt) + 0(\alpha^2) \quad (129)$$

where

$$b = \frac{\dfrac{SF}{R} + \sqrt{1 + \dfrac{S}{R} - \dfrac{F^2S}{R}}}{\left(1 + \dfrac{S}{R}\right)} \quad (130)$$

It is now convenient to choose new independent variables,

$$\bar{x} = x - bt$$
$$\bar{t} = t \tag{131}$$

Eqs. (125)–(127) become

$$\eta - F_{1\bar{t}} + bF_{1\bar{x}} + SF_{2\bar{t}} - bSF_{2\bar{x}} - \frac{b}{2}F_{1\bar{x}\bar{x}\bar{x}}$$
$$+ \frac{SR^2}{2}bF_{2\bar{x}\bar{x}\bar{x}} + \tfrac{1}{2}F_{1\bar{x}}^2 - \frac{S}{2}(F_{2\bar{x}}^2 - 2FF_{2\bar{x}}$$
$$+ R^2FF_{2\bar{x}\bar{x}\bar{x}}) = 0(\alpha)^3 \tag{132}$$

$$-\eta F_{1\bar{x}\bar{x}} - F_{1\bar{x}\bar{x}} + \tfrac{1}{6}F_{1\bar{x}\bar{x}\bar{x}\bar{x}} + \eta_{\bar{t}} - b\eta_{\bar{x}} - F_{1\bar{x}}\eta_{\bar{x}} = 0(\alpha^{7/2}) \tag{133}$$

$$-\eta F_{2\bar{x}\bar{x}} + RF_{2\bar{x}\bar{x}} - \frac{R^3}{6}F_{2\bar{x}\bar{x}\bar{x}\bar{x}} + \eta_{\bar{t}} - b\eta_{\bar{x}}$$
$$-\eta_{\bar{x}}(F_{2\bar{x}} - F) - 0(\alpha^{7/2}) \tag{134}$$

From (129), $\eta_{\bar{t}} = 0(\alpha^{5/2})$. Then the integrals of the first approximations to (133) and (134) yield

$$b\eta = -F_{1\bar{x}} + 0(\alpha^2) \tag{135}$$

$$(b - F)\eta = RF_{2\bar{x}} + 0(\alpha^2) \tag{136}$$

Using these relations in the higher order terms of Eqs. (133) and (134), we obtain

$$F_{1\bar{x}\bar{x}} = 2b\eta\eta_{\bar{x}} - \frac{b}{6}\eta_{\bar{x}\bar{x}\bar{x}} + \eta_{\bar{t}} - b\eta_{\bar{x}} + 0(\alpha^{7/2}) \tag{137}$$

$$RF_{2\bar{x}\bar{x}} = \frac{2(b-F)}{R}\eta\eta_{\bar{x}} + \frac{R^2}{6}(b-F)\eta_{\bar{x}\bar{x}\bar{x}}$$
$$-\eta_{\bar{t}} + b\eta_{\bar{x}} - F\eta_{\bar{x}} + 0(\alpha^{7/2}) \tag{138}$$

We now differentiate (132) with respect to $\bar{x}$, using (130) and (135)–(138). The result, neglecting higher order terms, is

$$l\eta_{\bar{t}} + m\eta\eta_{\bar{x}} + n\eta_{\bar{x}\bar{x}\bar{x}} = 0 \tag{139}$$

where

$$l = 2b + \frac{2S}{R}(b - F) \tag{140}$$

$$m = 3b^2 - \frac{3S}{R^2}(b - F)^2 \tag{141}$$

$$n = \frac{b^2}{3} + \tfrac{1}{3}SR(b - F)^2 \tag{142}$$

This may be put in the form,

$$\eta_{\bar{t}} + \eta\eta_{\bar{x}} + K^2\eta_{\bar{x}\bar{x}\bar{x}} = 0 \tag{143}$$

by appropriate changes of variables. Eq. (143) is the form of the Korteweg–deVries equation used by Zabusky and Galvin (1971). In the form (139) we see that the non-linear term may vanish under certain circumstances, in particular when $S = 1$ (Boussinesq approximation), $R = 1$ and $F = 0$. We saw in Section 2 that there is no tendency for steepening in this case. If the non-linear term does not vanish, the phenomena should resemble those discussed by Zabusky and Galvin. On the other hand, if the last term in (139) is neglected, we obtain the results of Section 2.

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ИЗМЕНЕНИЕ КРУТИЗНЫ ПРОФИЛЯ ДЛИННЫХ ВНУТРЕННИХ ВОЛН

Изучена деформация длинных волн в стратифицированной жидкости благодаря кривизне вертикального профиля плотности, благодаря членам, пренебрегаемым в приближении Буссинеска и благодаря вертикальному сдвигу скорости. В случае непрерывной стратификации эти эффекты предполагаются малыми. Для волн с четным числом узловых поверхностей получены следующие результаты: (1) Если профиль плотности вогнут вверх, волна возвышения стремится обрушиться в направлении распространения, т. е. передняя часть волны становится круче; если профиль выпуклый вверх, то волна стремится обрушиться назад. (2) Члены, которыми пренебрегают в приближении Буссинеска, приводят к обрушению волны назад. (3) Если волна распространяется в отрицательном направлении x, а скорость жидкости в том же направлении растет с высотой, то волна стремится обрушиться вперед. Если волны имеют нечетное число узловых поверхностей, то в первом приближении нет тенденции к обрушению. Проведен также анализ для двухслойной жидкости. Те же тенденции существуют при условии (1), если вогнутый вверх профиль соответствует более мелкому нижнему слою жидкости. В упомянутых выше условиях (2) и (3) тенденции к обрушению в непрерывных системах и в двухслойных системах противоположны.