A variance analysis of angular momentum in stellar and planetary atmospheres

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ABSTRACT

For purposes of studying the dynamics of stellar and planetary atmospheres, a system of balance equations is derived for the square of the absolute angular momentum, for the square of its zonal average and for its variance along the length of closed latitude circles. Pressure and frictional forces are included, but magnetic effects are not considered at present. As discussed by the first author previously, limitations on possible axisymmetric flows lead to criteria defining forbidden regimes. Physical pictures of the manner in which nonzonal eddies alter the variance of angular momentum are outlined for certain systems. An argument is made to suggest that thermal forcing of the pure eddy type (perhaps like Rayleigh convection) tends to establish an equatorial acceleration in stellar and certain planetary atmospheres, through negative eddy viscous effects. A significantly more general set of balance equations is derived in an appendix.

Introduction

The aim of the material below is to present a logical extension and amplification of the mode of reasoning applied to cosmic masses of rotating fluid by Starr (1971*b*), a reference hereafter designated as SVP. By-and-large the same general (physical) model is used. Although new additional results are arrived at, the previous ones are also regained, so that the discussion now given is complete in itself.

In considering the absolute angular momentum, the treatment includes not only the balance of this primitive quantity directly, but also its square, its zonal variance and the square of its zonal mean. As might be expected, the zonal variance of the angular momentum is closely related to the zonal eddy kinetic energy, while the square of its zonal average bears a similarity to the (absolute) mean zonal kinetic energy. For a discussion of this last quantity it is presently convenient to refer to Starr & Gaut (1969), henceforth denoted as S & G in what follows.

Except as noted further on, the basic physical object of study is the one used in SVP, namely a gaseous or fluid mass in space, isolated from external torques, and possessed of a mean rotation about an axis passing through the center of mass. As in S & G we assume the density surfaces to be figures of revolution to a certain degree of accuracy, and that the density ϱ is not a function of time t. This serves purposes of convenience in formulating several quantities and, in contrast, these are assumptions not encountered in SVP. However, due to the importance of making a minimum number of approximations for certain possible applications, one form of the equations to be developed, free from all restrictions on the density, is included in the Appendix. Inside of this mass we may suppose (though only in particular cases specifically indicated later) that there is a, perhaps rotating, solid core having a surface of revolution as its top boundary (e.g., a spheroidal one). In any case its axis of rotation remains coincident with that of the composite mass, zonal symmetry of density being assumed also in the core. Quantities are referred to a nonrotating spherical polar system of coordinates (R, λ, ϕ), the symbols in order denoting radius, sidereal longitude and (geocentric, heliocentric, etc.) latitude. The direction of increasing λ is taken to be in the direction of mean rotation of the fluid part.

The conclusion was reached in SVP that certain symmetrical regimes of motion were internally inconsistent and therefore not possible. Since such distributions are actually observed as zonal averages, one question we now treat is how the inclusion of the added eddy components of motion renders this possible. Since the development of the subject has much in common with that given in the former paper, the treatment here tends to brevity for the sake of saving space.

Although appeals are here made to more general facts concerning actual observed systems, no extensive compilation of observational statistics are included in this paper. This needed extension is deferred for future investigation because of its inherent diversities and consequent magnitude of effort required.

Definitions and generalities

The absolute angular momentum per unit mass of fluid about the polar axis is denoted by M, so that

$$M = ru; \quad u = rD\lambda/Dt \tag{1}$$

where r is the perpendicular distance to the polar axis, $R \cos \phi$. The coordinates, as stated, are nonrotating, so that if t is time, $D\lambda/Dt$ is the individual rate of increase of absolute longitude for a fluid particle. Thus u is therefore the absolute linear particle velocity eastward, i.e., in the direction of the mean rotation of the fluid mass.

We next define the zonal average along circles of latitude at a given value of R by square brackets, so that for any quantity ( ) continuously distributed,

$$[(\ )] \equiv \frac{1}{2\pi} \oint (\ )\, d\lambda \tag{2}$$

The integration over λ is over a range of 2π, thus tacitly implying that in applications such as the earth's atmosphere we assume that the surface of the planet is smooth, without orographic barriers. Surface friction is nonetheless taken to be present. Using (2) we now define a deviation from the zonal mean by a prime, i.e.,

$$(\ ) \equiv [(\ )] + (\ )' \tag{3}$$

the deviation thus being a function of all the independent variables (R, λ, ϕ, t).

According to customary statistical usage we may write that

$$\sigma^2(M) \equiv [M^2] - [M]^2 \tag{4}$$

is the variance of M along the length of a closed latitude circle. By use of (3) it follows that this zonal variance may be written

$$\sigma^2(M) \equiv [(M')^2] \tag{5}$$

We may also note that from (1) it now is evident that

$$\sigma^2(M) = [r^2(u')^2] \tag{6}$$

a quantity akin to, but not identical with, the zonal eddy kinetic energy per unit mass. For our aims the quantity M is simpler to work with than is the kinetic energy derived from the absolute particle velocity. Because of its prominence in subsequent arguments to be made, we set down the simple result of eliminating $\sigma^2(M)$ between (4) and (5), namely

$$[M^2] \equiv [M]^2 + [(M')^2] \tag{7}$$

which is obviously related to the inequality of Schwarz.

To the above kinematic relationships we may add the equation of continuity and three others derived from it. Thus we have

$$\frac{\partial \varrho}{\partial t} + \nabla \cdot \varrho \mathbf{v} = 0 \tag{8}$$

$\mathbf{v}$ being the vector velocity. The continuity equation (8), when combined with Euler's relation between substantial and partial time derivatives gives us the theorem that

$$\varrho \frac{D(\ )}{Dt} = \frac{\partial \varrho(\ )}{\partial t} + \nabla \cdot \varrho(\ )\mathbf{v} \tag{9}$$

Finally, use of (8) enables us to write the theorems that

$$\varrho(\ )\mathbf{v} \cdot \nabla(\ ) = \nabla \cdot \varrho \frac{(\ )^2}{2}\mathbf{v} + \frac{(\ )^2}{2}\frac{\partial \varrho}{\partial t} \tag{10}$$

and

$$(\ )\nabla \cdot \varrho(\ )\mathbf{v} = \nabla \cdot \varrho \frac{(\ )^2}{2}\mathbf{v} - \frac{(\ )^2}{2}\frac{\partial \varrho}{\partial t} \tag{11}$$

Since most often in our problem $\partial\varrho/\partial t = 0$, (10) and (11) are further simplified, and we may then avail ourselves of the useful fact that

$$\varrho(\ )\mathbf{v}\cdot\nabla(\ ) = (\ )\nabla\cdot\varrho(\ )\mathbf{v} \tag{12}$$

The proofs of (10) and (11) are so simple that they are left to the interested reader to supply.

To the kinematic principles expressed in the twelve equations given above, we adjoin the dynamical equation governing M, on the assumption that the only real forces acting are frictional and pressure forces. Thus

$$\varrho\frac{DM}{Dt} = -\frac{\partial\varrho}{\partial\lambda} + rF \tag{13}$$

where p is pressure and F is the zonal frictional force per unit volume assumed positive in the λ direction. Equation (13) may be written in the form

$$\frac{\partial\varrho M}{\partial t} + \nabla\cdot\varrho M\mathbf{v} = -\frac{\partial p}{\partial\lambda} + rF \tag{14}$$

by virtue of (9).

Conservation of angular momentum squared

Equation (14) expresses the conservation of M for the fluid. In the absence of frictional and pressure torques, the time change of M at a point results only from an exchange with the surroundings by gross motions. More generally, it is true that the frictional and pressure torques when present are also exerted by the environment, and hence by Newton's second law comprise exact exchanges of M with the environment whatever the details may be.

We next investigate the balance equation for $[M^2]$, which is the first term in (7). Multiplying (13) by M, we get

$$\varrho\frac{D}{Dt}\left\{\frac{M^2}{2}\right\} = -M\left(\frac{\partial p}{\partial\lambda} - rF\right) \tag{15}$$

or with the aid of (9)

$$\frac{\partial}{\partial t}\left\{\varrho\frac{M^2}{2}\right\} + \nabla\cdot\varrho\frac{M^2}{2}\mathbf{v} = -M\left(\frac{\partial p}{\partial\lambda} - rF\right) \tag{16}$$

Alternatively, through the multiplication of (14) by M,

$$M\frac{\partial\varrho M}{\partial t} + M\nabla\cdot\varrho M\mathbf{v} = -M\left(\frac{\partial p}{\partial\lambda} - rF\right) \tag{17}$$

which is equivalent to (16) by virtue of the (exact) use of theorem (11). The equation (16) is quite analogous to (14) and expresses the same kind of physical facts, except as follows. We now integrate it spatially over the entire volume of our system—let us say up to the surface $R = R_1$, within which the mass of the system is enclosed. Thus,

$$\frac{d}{dt}\int_T \varrho\frac{[M^2]}{2}dT = -\int_T M\left(\frac{\partial p}{\partial\lambda} - rF\right)dT \tag{18}$$

where T is volume. Had this same integration been performed on (14), thus securing the integrated balance equation for $[M]$, it is obvious that the pressure term would vanish because of the cyclic partial integration with respect to λ contained in the volume integral over T, the volume element dT being $R^2\cos\phi\, dR\, d\lambda\, d\phi$. This does not happen in (18) because of the possibility of a nonzero covariance between M and $\partial p/\partial\lambda$ along the length of each latitude circle. We thus have a pressure source term for $[M^2]$ whereas none exists for $[M]$.

Along the same lines, it is to be noted that in writing (18) we have purposely used a slightly redundant notation to achieve a desired form of the first term. It is thus obvious that, ϱ being independent of λ, we have

$$\int_T \varrho\frac{M^2}{2}dT = \int_T \varrho\frac{[M^2]}{2}dT$$

Such consequences of the cyclic integration with respect to λ recur in what follows.

Conservation of mean angular momentum squared

To obtain the balance equation for $[M]^2$, which is the second term in (7), we multiply (14) by $[M]$. Noting that $M = [M] + M'$, the result is

$$\frac{\partial}{\partial t}\left\{\varrho\frac{[M]^2}{2}\right\} + [M]\frac{\partial}{\partial t}\{\varrho M'\} + [M]\nabla\cdot\varrho M\mathbf{v}$$

$$= -[M]\left(\frac{\partial p}{\partial\lambda} - rF\right) \tag{19}$$

As before, we integrate over the volume T obtaining

$$\frac{d}{dt}\int_T \varrho\frac{[M]^2}{2}dT + \int_T [M]\,\nabla\cdot\varrho M\mathbf{v}\,dT = \int_T [M]\,rF\,dT \tag{20}$$

It is to be noted that like the balance equation for $[M]$ itself, were we to write it from (14) in integral form, the balance equation for $[M]^2$ contains no pressure torque term. However, (20) contains a new feature, namely the second term on the left, which is a source term due to gross motions. More will be said concerning this later.

Conservation of the variance of angular momentum

In this instance we again follow the usual pattern of procedure and multiply (14) by M', and like before, using the fact that $M = [M] + M'$ we find that

$$M'\frac{\partial}{\partial t}\{\varrho[M]\} + \frac{\partial}{\partial t}\left\{\varrho\frac{(M')^2}{2}\right\} + M'\nabla\cdot\varrho M\mathbf{v} = -M'\left(\frac{\partial p}{\partial\lambda} - rF\right) \tag{21}$$

Rearranging the third term by use of equations (3) and (11) we have, upon integrating over the volume T,

$$\frac{d}{dt}\int_T \varrho\frac{(M')^2}{2}dT - \int_T [M]\,\nabla\cdot\varrho M\mathbf{v}\,dT = -\int_T M'\left(\frac{\partial p}{\partial\lambda} - rF\right)dT \tag{22}$$

As previously, the integration of the first term of (21) yields zero because of the cyclic partial integration with respect to λ. It is to be noted that, as in (18), the pressure term does not drop out because of the λ-dependence of M' and $\partial p/\partial\lambda$.

Properties of the system of conservation equations

In order to illuminate, so far as may easily be done, the nature of the friction effects, we consider them in the form of torques due to proper parts of the stress tensor. This procedure still exempts us from committal to any assumption concerning the relation of these stresses to the motion field, although eventually something has to be said concerning this subject. As has been explained in SVP, the right hand side of (13), for example, may be written as the divergence of an angular momentum transport vector $\boldsymbol{\tau}$, having components $r\boldsymbol{\tau}_{R\lambda}$, $r\boldsymbol{\tau}_{\lambda\lambda}$ and $r\boldsymbol{\tau}_{\phi\lambda}$ in the R, λ, ϕ directions respectively, these components being the ordinary linear momentum transport stresses weighted by the torque arm r. If we identify the component $r\boldsymbol{\tau}_{\lambda\lambda}$ with the pressure effect and keep it separate as in equation (13), the remainder of $\boldsymbol{\tau}$ is then a two-dimensional, purely frictional, transport vector which lies in the meridional plane. Let us call it $\boldsymbol{\tau}_m$. By use of this formalism we now may rewrite (18), (20) and (22) as follows:

$$\frac{d}{dt}\int_T \varrho\frac{[M^2]}{2}dT = -\int_T M'\left(\frac{\partial p}{\partial\lambda}\right)'dT - \int_T M\nabla\cdot\boldsymbol{\tau}_m\,dT \tag{23}$$

$$\frac{d}{dt}\int_T \varrho\frac{[M]^2}{2}dT = -\int_T [M]\,\nabla\cdot\varrho M\mathbf{v}\,dT - \int_T [M]\,\nabla\cdot\boldsymbol{\tau}_m\,dT \tag{24}$$

$$\frac{d}{dt}\int_T \varrho\frac{[(M')^2]}{2}dT = +\int_T [M]\,\nabla\cdot\varrho M\mathbf{v}\,dT - \int_T M'\left(\frac{\partial p}{\partial\lambda}\right)'dT - \int_T M'\nabla\cdot\boldsymbol{\tau}_m\,dT \tag{25}$$

Equations (23), (24) and (25) do not form a system of three independent relations because they are connected by (7), so that any one of them may be obtained from the other two. We could have written $[\boldsymbol{\tau}_m]$ and $\boldsymbol{\tau}_m'$ in (24) and (25) respectively, in place of $\boldsymbol{\tau}_m$, but this is not a mathematical necessity in view of the cyclic character of the partial integration with respect to λ implied in dT. Some other important features of the three equations as now written are the following, remembering that the application is to a body which is entirely fluid.

(*a*) If the state of motion is steady, the left hand members are zero, although the converse is not necessarily true. However, let us assume that the left hand members vanish, either instantaneously or at least statistically over some appropriate time period when averaged temporally. In effect then, also the right hand sides of the three equations are each zero.

(*b*) The three frictional integrals may each be zero, positive or negative. Since the spatially averaged value of $\nabla \cdot \boldsymbol{\tau}_m$ over the volume T must be zero for an isolated body, the sign of the integral corresponds to the sign of the spatial linear correlation of (in order) M, $[M]$, M' with $\nabla \cdot \boldsymbol{\tau}_m$. Any one of the integrals may be zero if the appropriate correlation is zero.

(*c*) Equations (23) and (25) contain a pressure term which accounts for either a negative or positive (or zero) generation of the integrated variance of M. Since the mean value of $(\partial p/\partial \lambda)'$ over the region of integration must be zero, the sign of this integral depends upon the linear correlation of M and $\partial p/\partial \lambda$ over the volume. If these quantities are uncorrelated, the integral has a zero value.

(*d*) Equations (24) and (25) contain an identical conversion integral with opposite signs, transforming the mass integral of $[M]^2$ into that of the $(M')^2$, if it is positive. Again since the divergence of angular momentum transported by gross motions, $\nabla \cdot \varrho M \mathbf{v}$ must integrate to zero, the sign of the conversion integral depends upon the correlation between this divergence and $[M]$. It should be observed that a nonzero value of the transformation integral can arise only through the presence of M', since if $M = [M]$, it reduces to zero according to (11).

(*e*) By the same token in a steady, zonally symmetric regime the friction term in (24) must be zero, i.e., $[M]$ and $\nabla \cdot \boldsymbol{\tau}_m$ must be uncorrelated over each and every individual meridional plane. If now negative viscous effects are excluded from the friction term, by-and-large maxima of $\nabla \cdot \boldsymbol{\tau}_m$ will correspond to maxima of angular velocity $[u]/r$. As explained in SVP, this leads to the identification of certain forbidden regimes of zonally symmetric flow. Thus, although a certain part of the argument made in this regard in SVP is here recaptured, the more detailed physical and geometric reasoning of the former paper provides a sharper tool, enabling us to exclude certain cases which on the basis of the present, purely integral, considerations would not be classed as being forbidden.

(*f*) It is of course obvious that the frictional terms play a most important part in our conception of the operation of dynamic balance equations. Without it every term in (23), (24) and (25) could be zero, leading to an indeterminate situation. On the other hand, very little is known as to how the angular momentum flux vector $\boldsymbol{\tau}_m$ is related to the motion field, if indeed there is any simple enough systematic connection. One might say that there should be such a connection if only molecular effects are included in the friction terms. In that case, however, all eddy effects down to but not including the molecular scale must be included in the other terms present, so that practically speaking no obvious advantage is gained. Also, the molecular effect on the larger scales of the motion field in actual cosmic-size systems is extremely small (see, e.g., Rosen, 1971).

Under these circumstances it is perhaps best not to seek any but the most loose and general relation between $\boldsymbol{\tau}_m$ and the more gross motion field. We might thus say that this transport vector includes in some proper sense, small scale eddy effects, but not those larger ones which might cause a flux of angular momentum up the gradient of angular velocity; that is, we avoid negative viscous eddy processes, preferring to deal with them as part of the other actions. We may then say that what we class as friction always causes a divergence of angular momentum out of the regions of closed maxima of angular velocity, as was done in SVP, without parameterization in terms of a viscosity coefficient. Of course this latter may be done for illustrative and expository purposes in an approximate way (see SVP also for this), but conclusions reached which are true without such detailed an assumption are more to be trusted, and hence to be sought after when possible.

Application to the case of an equatorial jet

In SVP it was pointed out how an inconsistency arises if we consider the hypothetical instance of a steady zonally symmetric regime with an equatorial acceleration. To counteract the diffusion of angular momentum out of the maximum of angular velocity by true friction, it is necessary that gross motions bring in a continuing new supply. This the mean meridional circulation cannot do, and hence the hypothetical scheme breaks down.

It is nevertheless true that from the point of view of zonal averages, the atmospheres of, e.g., Jupiter and of the sun do show an equatorial acceleration. Barring complications from magnetic effects in the solar case, the conclusion is indicated that the main mechanism of these two (quasi-steady) circulations cannot be symmetrical, but must involve nonzonal perturbations in the gross motions, as emphasized recently by Hide (1970). How, in greater detail, does this eddy process operate—especially within the context of the present discussion? The problem is not alone one of accounting for a net convergence of angular momentum by gross motions into the mean jet in accordance with the continuity equation in the form of (14) for this momentum. In addition, we must describe how individual particles at certain longitudes increase their angular momenta in approaching the mean jet, in terms of equation (13).

In view of the fact that the nonsymmetric regime retains the effect of the zonal eddy pressure term in (13) and also in (25), whereas the pressure effect drops out in the symmetric case, this added force is now the means for generating high eddy momenta for one set of individual particles and low eddy momenta for another set. Thus although the eddy zonal pressure force cannot act to generate mean zonal momentum directly, it can generate a variance of zonal momentum $[(M')^2]$. If now there should exist a sorting mechanism causing those particles which acquire high momenta to migrate selectively into the mean jet, and those which acquire low momenta to migrate away from it, then we have arrived at the needed arrangement. Let us consider some evidence for the existence of such a sorting mechanism for the case of the sun.

What we have outlined amounts to saying that along individual latitude circles bordering the jet for some distance poleward on each side, there should be a positive correlation between the zonal velocity (u) and the equatorward meridional particle velocity. Such a particle velocity correlation is precisely what was found by Ward through the use of the Greenwich sunspot data, originally in 1964 (see Starr, 1968, for a bibliography and results for virtually the entire Greenwich record).

Since $(\partial p/\partial\lambda)'$ in (25) measures in essence the meridional particle velocity according to the geostrophic concept (to use a meteorological terminology), and since this concept no doubt finds application to the larger scale solar motions, it follows that the pressure term in (25), namely,

$$-\int_T M'\left(\frac{\partial p}{\partial \lambda}\right)' dT \tag{26}$$

should be positive, contributing toward an increase in the variance $[(M')^2]$. Thus the observed correlation in the velocity components assures us of three things for the photospheric region. It implies a convergence of angular momentum into the jet to balance friction and so fulfill (14); it gives us the information that the generation term due to pressure in (25) is positive; finally it tells us that the sorting process is in fact present, bringing particles with high momenta into the core of the (statistical) jet—something that could not happen in the symmetrical case as explained more fully in SVP.

In an article by Starr (1969) and by Starr & Fischer (1972), there is presented an inferred picture of the probable structure of photospheric eddies. Fig. 1 which is an idealized picture constructed by the same methods may be used to give a more concrete idea of what has been said. Let us suppose, for ease of discussion, that the streamlines are stationary relative to the Greenwich rotating coordinates, although this doubtless is not the case, as explained in the above articles. For the present this assumption does no harm. The paths of particles now coincide with the streamlines, and the flow is approximately geostrophic (except immediately near the equator) so that the regions in the vicinity of the sloping heavy lines are ridges of high pressure, since the streamlines are now also approximate isobars.

The patterns of eddy motion thus envisaged have a correlation of velocity components so as to carry momentum equatorward to satisfy (14), since the velocities u are larger in the regions to the right of the ridges, these being indicated by the heavy segments of the sample latitude circle, where the meridional eddy velocity components are directed equatorward.

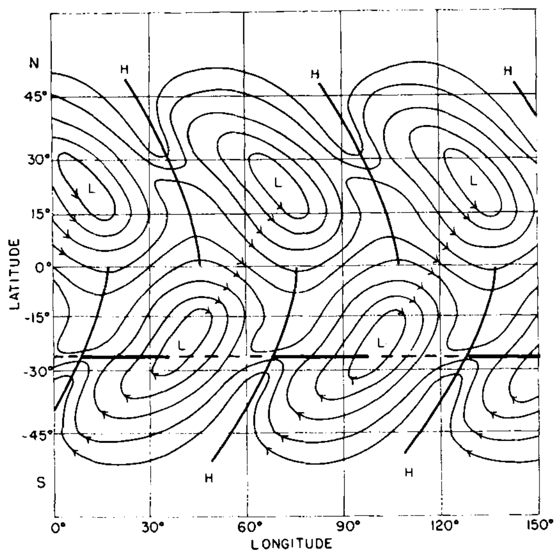

Fig. 1. Hypothetical idealized streamline pattern thought to exist in the solar photosphere. The figure depicts the maintenance of an equatorial acceleration through eddy processes, as explained in the text.

The pressure force on individual particles is seen to be to the right along these heavy portions of the latitude circle, and to the left along the dashed portions, thus ensuring a positive effect from the pressure term in (25), the higher latitudes probably causing only minor effects. Finally due to the distribution of meridional velocity components, the sorting action operates in the proper sense.

We have at our disposal observational information of only a much more general nature concerning the Jovian atmosphere. Since the planet shows spots, it is known from following these that an equatorial acceleration is present. However, no study concerning the statistics of the eddy motions comparable to that by Ward for the sun has as yet been made. It is hoped, however, that the considerable effort now being made at the Massachusetts Institute of Technology to study spot motions will in the near future result in the printing of an ephemeris of such data. The plan is that the volume should contain, among other quantities, the positions in terms of latitude and longitude of each spot relative to rotating coordinates, for a period of six (terrestrial) months. On each of the approximately 300 photographs being used there are about 75 visible spots distributed over the disk. Once available, this information can in principle be used to obtain wind velocities comparable to meteorological winds, as was done by Ward for the solar case from the Greenwich sunspot tabulations.

Application to the terrestrial atmosphere

The crucial circumstance making the examining of the earth's atmosphere important is the fact that it alone among the cosmic systems is one for which voluminous and detailed observational data are at our disposal, concerning the field of motion and various other quantities. This is not to say that still more statistics, especially of certain kinds, are not needed, but relatively speaking we are quite well off in regard to the wealth of recorded measurements. In the following pages it is not our intention, however, to make use of extensive calculations performed specifically for the present purposes, this being too large an order for now. Nevertheless some general descriptive facts derived from data are considered.

Unlike the case of Jupiter or the sun, our atmosphere consists of a shell of gas which is thin in the sense that the bulk of its mass is found within an interval of height which is small compared to the radius of the denser, mostly solid, part. We are therefore confronted with the necessity of studying the ideal case of a fluid surrounding a solid core—a case which we have tacitly assumed would not contain special difficulties for what was said about the relatively deep and more massive Jovian atmosphere.

By-and-large, the integral relations such as (23), (24) and (25) apply to the system now under consideration, if we assume that the solid core is a figure of revolution about the polar axis—let us say spherical. The only modes of interaction between the solid and the gas is through pressure and friction. These forces exerted by the gas on the solid may cause the latter to change its total angular momentum with time either in magnitude or direction, periodically or otherwise. Thus an Eulerian nutation and other kindred effects may be excited occasionally or continuously, but we shall not now stop to study these phenomena. Assuming that the magnitudes involved are like those for the earth, we may then suppose that we are dealing with a quasi-steady regime in which the angular momentum of the solid and of the gas is in each case steady on the average, in magnitude and direction relative to our polar coordinates, and that each of the two vectors coincides with the polar axis.

So viewed, there must exist in the system a balance of torques due to stresses at the surface of the solid. These can be exerted only through frictional (i.e., tangential) stresses at the solid-gas interface. The frictional torque per unit volume may be written in the form $-\nabla \cdot \boldsymbol{\tau}_m$ as was done in equation (23), for example. Placing this in (14) and integrating over T,

$$\frac{d}{dt}\int_T \varrho M\, dT = -\int_T \nabla \cdot \boldsymbol{\tau}_m\, dT = \int_{S_0} r\boldsymbol{\tau}_{R\lambda}\, dS_0 = 0 \qquad (27)$$

where S_0 is the surface area of the solid. Thus, whereas in the instance of a completely fluid mass the flux of angular momentum has to be zero at every point of the free boundary, now due to the presence of the interface this need not be so. So long as positive and negative contributions balance, making the surface torque integral zero, the necessities for a steady state are taken into proper account.

As is well known, there is in the actual atmosphere a large upward flux by ground friction of absolute angular momentum into the atmosphere, roughly between 30° latitude N and S. On the other hand, the flux out of this region across the latitude walls at 30° N and S is accomplished by relatively large eddies which produce the necessary negative viscous action. It follows that there is a net frictional convergence of momentum in the half of the atmosphere between these latitudes. This belt is also characterized by large positive values of $[M]$ due almost entirely to the large linear speed of the earth's rotation, the effect of the relative winds being minor. From the point of view of the friction term in (24), these circumstances dominate the picture giving the contribution of the entire integral a positive value, since in the remaining polar cap regions $[M]$, although still positive, is much smaller so that the compensating frictional divergence is weighted much less heavily. We thus see that were this action of friction unopposed, it would lead to an increase of the $[M]^2$ integral on the left of (24).

Since we may assume that on the average the left hand side is near zero, it follows that the remaining transformation term on the right of (24) must produce an effect equal and opposite to that of the friction term. That is, it must tend to decrease the $[M]^2$ integral on the left. Through the use of the facts and the kind of arguments just made in regard to the frictional convergence of momentum between 30° N and S, we now see that we have in this belt a divergence of angular momentum by the larger motions. This now confirms the inference that the transformation term tends to reduce the integral on the left.

In equation (25) the transformation term must now tend to increase the $[(M')^2]$ integral on its left. This transformation action must be offset in (25) by the pressure and friction terms, if conditions are steady. We may examine the pressure term by the same method as was used in the solar case. Since on geostrophic grounds the integrand measures the poleward eddy transport of angular momentum, and since the latter is known to be positive from about 40° N to 40° S latitude, the effect of the pressure term in (25) is in the sense of decreasing the $[(M')^2]$ integral. The likelihood is that the eddy friction term also tends to decrease this integral, but by our current procedures there is no simple way of ascertaining this. Be that as it may, pending quantitative evaluations, the combination of the pressure and friction effects balance the contribution of the transformation term.

Turning to equation (23), which is the sum of the other two, the friction effect on $[M^2]$ balances the eddy pressure term already discussed. Since the latter decreases the $[M^2]$ integral, it follows that the friction term must balance this effect, tending to produce an increase. The friction terms in (24) and (25) add to give the one in (23), hence if as we have said, the one in (24) is accelerative, any retarding action of that in (25) cannot be too large. Further assessment of the several quantities in the balance equations must await fairly complete quantitative computations from hemispheric or global meteorological data—an extensive project, even today with all available measurements.

Tentatively, the gross picture may be

described as follows. The action of surface friction on the mean zonal easterlies and westerlies is to tend to increase the $[M]^2$ integral for the entire atmosphere, according to (24). This frictional generation is counteracted by the transformation term which converts the output into the variance quantity $[(M')^2]$ in (25). Finally, in this eddy form, it is destroyed by the eddy pressure term and (presumably) also by the eddy friction term. The cycle could equally well have been started at some other point, in view of the fact that no cause and effect relations are necessarily implied—only a description of the observed processes.

It may be noted that in the solar case the eddy pressure term creates zonal variance of angular momentum which is then fed into the zonally averaged zonal flow—the eddies being thus required in order to avoid a dynamic inconsistency as stated in SVP. In our atmosphere the pressure term acts to destroy the zonal variance of angular momentum, and this variance must now be resupplied through a reversal of the sign of the transformation term which drains it out of the mean zonal flow. The action of the eddy pressure term thus permits particles to arrive at the latitude of the jet in each hemisphere without the appearance of inordinately large zonal velocities relative to the earth—a concept not new to meteorology (see Lorenz, 1967).

It would of course be possible to construct a diagram on the plan of Fig. 1, but with changes to make it appropriate for the earth's atmosphere at the elevation of the upper troposphere where the strongest winds are encountered. The picture would then contain two mean jet stream regions around 35° N and S. The troughs and ridges of the eddy structures would transport momentum from low latitudes poleward, and the eddy pressure effect would reduce the angular momentum of particles moving toward the mean jets. Likewise it would increase the eddy momentum of the particles moving equatorward away from the jets, but the net effect would, as has been said, be to destroy the total variance for the atmosphere.

One may note an added feature. Poleward of the mean jets there exists in our atmosphere an equatorward eddy transport of angular momentum as a secondary smaller effect (see, e.g., Starr et al., 1970, also Obasi, 1963, for S.H.). This phenomenon in the polar cap regions is comparable in a way to the solar regime, except that in the sun it becomes dominant while the two mean jets become merged into a single one at the equator. As indicated in the references cited, seasonal changes produce considerable oscillations about the mean condition in various quantities.

Supplementary comments and remarks

(1) In writing (23), (24) and (25) use was made of the absence of dependence of the density ϱ upon the independent variables λ and t. Any limiting effects of this assumption disappear in discussions of the special case of steady symmetric flow.

(2) The integrals in (23), (24) and (25) extend over the entire fluid system, so that no free boundaries are involved. These relations are capable of being rewritten so as to cause boundary integrals to appear. There is a problem in doing this, if these added surface integrals are intended to represent the complete physical transports of the three quantities at these surfaces (and at the solid-fluid interface if one is present). The corresponding problem in the instance of the mean zonal kinetic energy equation is discussed by Starr & Sims (1970).

(3) As was done in S & G for the kinetic energy counterpart of (24), our system of three equations could be written in component form with a view to the utilization of observational data for the evaluation of various terms. This might be of especial pertinence for the meteorological case where extensive observations are at our disposal. However, for the sake of simplicity this has not been done in the present paper. Practically speaking, the question of time averaging is also to be considered in treating long statistical time periods.

(4) In SVP and also to some extent above, we have discussed the case of a mass of fluid having, as a special case, no eddy effects but only an entirely symmetric motion. Let us now begin by considering what amounts to a still more specialized instance, namely that the mass has zonal symmetry and that the motion consists of a solid rotation at a uniform angular velocity Ω. In the absence of disturbing factors, such a regime can continue as a steady condition, with pressure and gratitational forces in

proper balance. All the terms in our system of equations (23)–(25) are then zero, the friction vanishing because of the absence of angular shear.

As a thought experiment, let it be imagined that an eddy disturbance is now introduced through some nonmechanical means, perhaps by the appearance of a purely eddy heating and cooling, i.e., without the presence of a mean zonal component. The departures of the density from complete symmetry thus generated will lead to nonzero eddy components of the pressure p'. Thus eddy components of the pressure force $-(\partial p/\partial \lambda)'$ will arise and will tend to generate horizontal motions (together with, in general, similar effects in the meridional direction). If the eddy velocities are not too great, these eddy motions will presently become approximately geostrophic, the Rossby number being sufficiently small. What can we now say as to the further details of the behavior of the system?

Considering equation (25), in order that there should take place a building up of the eddy motions $[(M')^2]$ by the pressure term, this term must be positive in its effect (the more so when we consider the action of the friction term as a damping influence). An accelerative action of the pressure term by definition implies a negative correlation between M' and $(\partial p/\partial \lambda)'$. We may note that this must be true even before any tendency in the direction of a geostrophic behavior becomes sensible, and possibly for some small-scale effects for which the relative accelerations are and remain large in comparison to the Coriolis terms in the horizontal equations of motion. (Thus, for example, the convection represented by the granulation in the solar photosphere does not appear to be sensibly affected by the solar rotation—at least in the present context.)

However, for the larger convective cells, the horizontal convergence and divergence involved in their structure must at once start to be affected by the basic rotation in accordance with the Bjerknes circulation theorem, as explained and illustrated for example by Starr (1959). It therefore follows that even long before any close approximation to the geostrophic condition is reached, $(\partial p/\partial \lambda)'$ must be proportional to $\varrho v'$. Thus we may expect that a negative (positive) correlation between $\varrho v'$ and M' will be present in the northern (southern) hemisphere, already somewhat before the development of quasi-geostrophic horizontal flow and, as we shall presently see, also afterwards. It follows that by-and-large there should be a simultaneous eddy convergence of angular momentum horizontally in to the equatorial regions of T, let us say between the latitude walls at $\phi = 30°$ N and S. Likewise, this gives, by already familiar arguments, a proper sign to the conversion term, so that the $[M]^2$ quantity is increased in (24) at the expense of $[(M')^2]$ in (25). The general initial result, then, is that of generating an equatorial acceleration of some type depending upon further particulars. The more complete picture would no doubt depend also upon the latitudes and levels at which the eddy heating is introduced and other such matters.

It might be noted that a continued generation of momentum variance by the pressure term presumably continues after a closer attainment to geostrophic flow in order to compensate for the losses through the conversion term which must supply $[M]^2$ momentum to maintain the differential rotation against friction (in the present case) and perhaps other losses. No doubt friction also tends to destroy the momentum variance directly. Of course the motions must average somewhat subgeostrophic to permit a convective release of potential energy (see Starr, 1959).

In addition to what has just been said, it should be noted that the tendency toward a geostrophic character of the motions enables us to predict the direction of the horizontal momentum transport only, at least in our formulation as far it goes. Various factors might then lead to vertical transports of angular momentum, let us say downward in the more equatorial latitudes, perhaps to generate a faster rotating interior, again through eddy transports as suggested by Starr (1971*a*) and in SVP. As stated above, in order to build up the eddy motions, the correlation of M' and $(\partial p/\partial \lambda)'$ should be negative. However, this need not be so at all latitudes and levels in T, so long as the predominating effect is negative, especially after an increased store of $[(M')^2]$ has been established. At deeper levels where forced eddies might be present, there could thus arise a compensating geostrophic poleward convergence of angular momentum with an upward flux of this quantity in the polar cap regions, in order to close the cycle.

The upshot of the speculations under the present heading is that the development of an equatorial acceleration in our system is not to be unexpected on physical grounds, as a consequence of something like Rayleigh-type convection due to vertical instability, the cells being probably distorted by angular shear and otherwise. They would, however, have to be of proper size for the horizontal motions to tend toward the geostrophic. Moreover, even if the Rayleigh convection is being driven by a uniform, symmetric heat source, it is required that the convective regime be unstable to perturbations of moderate wave number so that eddy heating can occur. The contention made here is in general agreement with the study of Davies-Jones and Gilman (1971).

Except for their importance in connection with the internal convective phenomena, the density variations which we have tacitly neglected in the remainder of the argument advanced are probably not important. An appeal can be made to the more exact equations (38), (39) and (40) in the Appendix. The presence of the time derivative terms containing ϱ' in (40) is probably of small significance except for extremely violent eddy actions, and would vanish for a quasi-steady eddy regime.

(5) The general type of reasoning which we have employed to make plausible the idea that Rayleigh-type convection would lead to a regime with an equatorial acceleration cannot in any obvious way be used as a model to yield a terrestrial form of general circulation regime—so far as the writers can see at present. Possible inferences which may be made are that (*a*) the earth's atmosphere is not driven by Rayleigh convection, but by the so-called baroclinic type of instability involving a species of convection due to meridional temperature gradients between the equator and poles, and that (*b*) our atmosphere is a thin shell with a large quasi-solid core, making surface friction a strong determining factor for the gross form of the global circulation (comp. Jeffreys 1926, as a matter of historical interest). Actually, because of surface friction, the conversion term acts to transform $[M]^2$ into $[(M')^2]$ according to (24) and (25), as noted previously. However, the small Equatorward transient momentum fluxes at some locations, as well as the persistent negative standing eddy fluxes at low latitudes and elsewhere, found by Starr et al (1970) are perhaps weak convective effects due to tropical disturbances and to flow of cold air over warm water.

(6) With the limitation that magnetic forces and such effects as zonal gravitational components are still neglected, the system of more general equations given in the Appendix are applicable to various hypothetical systems departing considerably in over-all properties from the ones here dealt with. Were one able to include some of the added features mentioned, the system thus amplified yet further might be useful in studying, at least in principle, angular momentum considerations in various more odd shaped and perhaps variable cosmic systems that are found in nature.

(7) The planet Jupiter, as is well known, exhibits a differential rotation characterized by an equatorial acceleration which, however, is not so marked as that of the sun. If we accept, for argument's sake, the correctness of the contentions outlined in item (4) above, we are tempted to assume that this equatorial mean jet owes its existence to the prevalence of Rayleigh convection in the interior of the Jovian atmosphere. Thus, although the planet may have a solid core, its gaseous envelope is relatively deep, and the arguments which were made could apply nevertheless. In view of these circumstances it is of interest to note that some scientific opinion holds that Jupiter possesses an internal heat source (see Hide, 1971) which for our purposes could lead to the maintenance of a convective mantle, as is the case in the sun.

Appendix

Though the system of equations (23)–(25) is sufficiently general to permit its application to a wide variety of fluid systems, it seems desirable to present here a still more general set of analogous equations arrived at when no assumptions are made concerning the spatial or temporal distribution of density. That is to say, in what follows we shall treat $\varrho = \varrho(R, \lambda, \phi, t)$.

Our point of departure from the derivations given in the main text is equation (14). We note that no special assumptions concerning ϱ have been made in writing this equation. Before proceeding further however, we pause to note the following useful identity

$$\frac{\partial}{\partial t}\left\{\varrho\frac{(\ )^2}{2}\right\} \equiv (\ )\frac{\partial\varrho(\ )}{\partial t} - \frac{(\ )^2}{2}\frac{\partial\varrho}{\partial t} \tag{28}$$

where, for that matter, ϱ could be replaced by any function. The proof of (28) is straightforward calculus and so is omitted here. Continuing now, our immediate objective is to obtain a set of differential equations governing M^2, $[M]^2$ and $(M')^2$, similar to (16), (19) and (21), but with ϱ now taken to be arbitrary. The first such differential equation is obtained through multiplication of (14) by M, thereby reproducing (16) with the aid of (9), exactly:

$$\frac{\partial}{\partial t}\left\{\varrho\frac{M^2}{2}\right\} + \nabla\cdot\varrho\frac{M^2}{2}\mathbf{v} = -M\left(\frac{\partial p}{\partial\lambda} - rF\right) \tag{16}$$

We next seek a differential equation governing the mean angular momentum squared. This may be accomplished by letting $M = [M] + M'$ in (14) and then multiplying by $[M]$. The result is

$$[M]\frac{\partial}{\partial t}(\varrho[M]) + [M]\frac{\partial}{\partial t}(\varrho M') + [M]\nabla\cdot\varrho[M]\mathbf{v} + [M]\nabla\cdot\varrho M'\mathbf{v} = -[M]\left(\frac{\partial p}{\partial\lambda} - rF\right) \tag{29}$$

Use is next made of theorem (11) to rewrite the third term on the left hand side of (29), so that it becomes

$$[M]\frac{\partial}{\partial t}(\varrho[M]) + \nabla\cdot\varrho\frac{[M]^2}{2}\mathbf{v} - \frac{[M]^2}{2}\frac{\partial\varrho}{\partial t} + [M]\frac{\partial}{\partial t}(\varrho M') + [M]\nabla\cdot\varrho M'\mathbf{v} = -[M]\left(\frac{\partial p}{\partial\lambda} - rF\right) \tag{30}$$

The identity (28) may be applied to combine the first and third terms on the left hand side of (30), and the relationship (9) may be used to combine the fourth and fifth terms. The final result is

$$\frac{\partial}{\partial t}\left\{\varrho\frac{[M]^2}{2}\right\} + \nabla\cdot\varrho\frac{[M]^2}{2}\mathbf{v} + \varrho[M]\frac{DM'}{Dt} = -[M]\left(\frac{\partial p}{\partial\lambda} - rF\right) \tag{31}$$

The last differential equation we seek is that governing the variance of angular momentum. The procedure is identical to that used in arriving at (31), except that now (14) is multiplied by M' after having again let $M = [M] + M'$, so that we have

$$M'\frac{\partial}{\partial t}(\varrho[M]) + M'\frac{\partial}{\partial t}(\varrho M') + M'\nabla\cdot\varrho[M]\mathbf{v} + M'\nabla\cdot\varrho M'\mathbf{v} = -M'\left(\frac{\partial p}{\partial\lambda} - rF\right) \tag{32}$$

As before, theorem (11) is again used but now to rewrite the fourth term on the left hand side on (32). Both (28) and (9) are again used to simplify the resulting equation, so that we may finally write

$$\frac{\partial}{\partial t}\left\{\varrho\frac{(M')^2}{2}\right\} + \nabla\cdot\varrho\frac{(M')^2}{2}\mathbf{v} + \varrho M'\frac{D[M]}{Dt} = -M'\left(\frac{\partial p}{\partial\lambda} - rF\right) \tag{33}$$

Thus (16), (31) and (33) represent our desired set of differential equations which we shall next integrate over our volume T. Before doing so, however, let us note the following relationships, a, b being arbitrary,

$$[ab] = [a][b] + [a'b'] \tag{34}$$

$$(M^2)' = \{(M')^2\}' + 2M'[M] \tag{35}$$

$$\int_T \varrho M'\frac{D[M]}{Dt}\,dT + \int_T \varrho[M]\frac{DM'}{Dt}\,dT = \int_T \frac{\partial}{\partial t}(\varrho[M]M')\,dT \tag{36}$$

The proof of (34) is obvious. Equation (35) is also easily gotten if it is noticed that $M^2 \equiv [M^2] + (M^2)'$ by virtue of (3), but that also $M^2 = \{[M] + M'\}^2$. Hence one has that $(M^2)' = (M')^2 + 2M'[M] - [(M')^2]$, through the additional use of (7). Since $[(\ )]' \equiv 0 \equiv [(\ )']$ and $(\ )'' \equiv (\ )'$, (35) is reproduced. Lastly, to obtain (36) we note that its left hand side may be rewritten as

$$\int_T \varrho\frac{D[M]M'}{Dt}\,dT$$

which may in turn be reformulated using (9), as

$$\int_T \frac{\partial}{\partial t}(\varrho[M]\,M')\,dT + \int_T \nabla\cdot\varrho[M]\,M'\mathbf{v}\,dT$$

Since T represents a closed system for our purposes, the volume integral of a divergence vanishes, and therefore (36) is reproduced.

Let us now proceed to get the desired integral equations. Upon integrating the left hand side of (16) over T, the divergence term drops out, and we are left with

$$\int_T \frac{\partial}{\partial t}\left(\varrho\frac{M^2}{2}\right)dT$$

Since time differentiation and zonal averaging are commutable, this integral may be rewritten as

$$\int_T \frac{\partial}{\partial t}\left[\varrho\frac{M^2}{2}\right]dT$$

But now (34) may be employed to show that

$$\int_T \frac{\partial}{\partial t}\left[\varrho\frac{M^2}{2}\right]dT = \int_T \frac{\partial}{\partial t}\left([\varrho]\frac{[M^2]}{2}\right)dT + \int_T \frac{\partial}{\partial t}\left[\varrho'\frac{(M^2)'}{2}\right]dT \qquad (37)$$

Leaving the result of integrating the right hand side of (16) over T the same as that given in the main text, and dropping the now extraneous brackets around ϱ in the first term on the right hand side of (37), we may write our final result of integrating (16) as

$$\frac{d}{dt}\int_T \varrho\frac{[M^2]}{2}dT = -\int_T \frac{\partial}{\partial t}\left[\varrho'\frac{(M^2)'}{2}\right]dT - \int_T M'\left(\frac{\partial p}{\partial\lambda}\right)'dT - \int_T M\nabla\cdot\boldsymbol{\tau}_m\,dT \qquad (38)$$

We next perform an integration over T upon (31). Again the divergence term vanishes, and by (34),

$$\left[\varrho\frac{[M]^2}{2}\right] = [\varrho]\frac{[M]^2}{2} \quad \text{since} \quad \{[M]^2\}' = 0.$$

Hence we have simply that

$$\frac{d}{dt}\int_T \varrho\frac{[M]^2}{2}dT = -\int_T \varrho[M]\frac{DM'}{Dt}dT - \int_T [M]\,\nabla\cdot\boldsymbol{\tau}_m\,dT \qquad (39)$$

Finally we integrate (33) over T. We note that

$$\left[\varrho\frac{(M')^2}{2}\right] = [\varrho]\frac{[(M')^2]}{2} + \left[\varrho'\frac{\{(M')^2\}'}{2}\right]$$

by virtue of (34), and that on further use of (35), we have

$$\left[\varrho\frac{(M')^2}{2}\right] = [\varrho]\frac{[(M')^2]}{2} + \left[\varrho'\frac{(M^2)'}{2}\right] - [\varrho' M'[M]]$$

This last relationship may be applied in connection with the integral of the first term on the left hand side of (33). The second term on the left hand side of (33) vanishes upon integrating over T. Finally, the integral of the last term on the left hand side of (33) may be rewritten, by virtue of (36) as

$$-\int_T \varrho[M]\frac{DM'}{Dt}dT + \int_T \frac{\partial}{\partial t}(\varrho[M]\,M')\,dT$$

Thus, noting that integrating the right hand side of (33) achieves the same result as integrating the right hand side of (21) in the main text, we may write at last

$$\frac{d}{dt}\int_T \varrho\frac{[(M')^2]}{2}dT = \int_T \varrho[M]\frac{DM'}{Dt}dT - \int_T \frac{\partial}{\partial t}\left[\varrho'\frac{(M^2)'}{2}\right]dT - \int_T M'\left(\frac{\partial p}{\partial\lambda}\right)'dT - \int_T M'\nabla\cdot\boldsymbol{\tau}_m\,dT \qquad (40)$$

The system (38), (39) and (40) is the one we have been seeking, and is analogous to (23), (24) and (25), but with ϱ taken now to be completely arbitrary in space and time. For convenience, we reproduce (38)–(40) in the form below

$$\frac{d}{dt}\int_T \varrho\frac{[M^2]}{2}dT = -\int_T \frac{\partial}{\partial t}\left[\varrho'\frac{(M^2)'}{2}\right]dT - \int_T M'\left(\frac{\partial p}{\partial\lambda}\right)'dT - \int_T M\nabla\cdot\boldsymbol{\tau}_m\,dT \qquad (38)$$

$$\frac{d}{dt}\int_T \varrho\frac{[M]^2}{2}dT = -\int_T \varrho[M]\frac{DM'}{Dt}dT - \int_T [M]\nabla\cdot\boldsymbol{\tau}_m\,dT \tag{39}$$

$$\frac{d}{dt}\int_T \varrho\frac{[(M')^2]}{2}dT = \int_T \varrho[M]\frac{DM'}{Dt}dT - \int_T \frac{\partial}{\partial t}\left[\varrho'\frac{(M^2)'}{2}\right]dT - \int_T M'\left(\frac{\partial p}{\partial \lambda}\right)' dT - \int_T M'\nabla\cdot\boldsymbol{\tau}_m\,dT \tag{40}$$

It may be easily shown that indeed the above three equations reduce to (23), (24) and (25), respectively, under the condition that ϱ be independent of λ and t. For in such a case, the first term on the right hand side of (38) which is also the second term on the right hand side of (40), vanishes. Furthermore, by (9), we have

$$\int_T \varrho[M]\frac{DM'}{Dt}dT = \int_T [M]\frac{\partial\varrho M'}{\partial t}dT + \int_T [M]\nabla\cdot\varrho M'\mathbf{v}\,dt \tag{41}$$

But since we now assume $\varrho = [\varrho]$, the first term on the right hand side of (41) vanishes on integrating around a closed latitude circle. As for the second term on the right hand side of (41), we may make use of the fact that $M' = M - [M]$ so that

$$\int_T [M]\nabla\cdot\varrho M'\mathbf{v}\,dT = \int_T [M]\nabla\cdot\varrho M\mathbf{v}\,dT - \int_T [M]\nabla\cdot\varrho[M]\,\mathbf{v}\,dT \tag{42}$$

But the second term on the right hand side of (42) vanishes by virtue of theorem (11) and our present assumption that $\partial\varrho/\partial t = 0$. Hence by letting $\partial\varrho/\partial\lambda = \partial\varrho/\partial t = 0$ in (38)–(40), we recover (23)–(25), which we recall had been directly derived with such restrictions on ϱ.

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АНАЛИЗ ИЗМЕНЕНИЙ МОМЕНТА КОЛИЧЕСТВА ДВИЖЕНИЯ В ЗВЕЗДНЫХ И ПЛАНЕТНЫХ АТМОСФЕРАХ

Для изучения динамики звездных и планетных атмосфер выводится система уравнений баланса для квадрата абсолютного момента количества движения, для корреляции его зонально усредненной величины и для его изменения вдоль замкнутого круга широт. Учтены силы давления и трения, но магнитные эффекты здесь не рассматриваются. Как уже обсуждалось ранее первым автором, ограничения на возможные осесимметричные течения приводят к критериям, определяющим запрещенные режимы. Для некоторых систем намечена физическая картина того, каким образом незональные вихри меняют корреляцию момента количества движения. Приводятся соображения, что термическое возбуждение чисто вихревого типа (возможно, подобное рэлеевской конвекции) стремится установить экваториальное ускорение в звездных и в определенных планетных атмосферах благодаря эффекту отрицательной вязкости. Значительно более общий набор уравнений баланса выводится в приложении.