On the stability of a barotropic shear flow to nongeostrophic disturbances

By WILLIAM BLUMEN, *Astro-Geophysics Department, University of Colorado, Boulder, Colo, USA*

(Manuscript received December 14, 1970; revised version April 26, 1971)

ABSTRACT

The stability of a barotropic shear flow $\bar{u}^* = U \tanh (y^*/L)$ of a stably stratified fluid of depth H to three-dimensional disturbances, is investigated by a linear analysis. Both the basic-state and perturbed motion are hydrostatic and the Brunt-Väisälä frequency and Coriolis parameter are assumed constant. A neutral solution is found, under the condition that $1-(U/c_{gi})^2>0$, where c_{gi} denotes the phase speed of a gravity-inertia wave which is characterized by zonal and vertical wave numbers α^* and $n\pi/H$ respectively. With the aid of this solution, some characteristics of the unstable waves are found by numerical methods. It is then shown that, for values of the parameters characteristic of large-scale mid-latitude flow, the basic current is stable to the internal baroclinic modes ($n \neq 0$) but unstable to the barotropic mode ($n=0$) in the quasi-geostrophic range, $R_0 = U/fL \ll 1$. Instability to the internal modes does not set in until $R_0 > 1-\pi^{-1} = 0.682$, well-outside the quasi-geostrophic range. Characteristics of the instability are presented and discussed.

1. Introduction

The barotropic stability of a zonal flow $\bar{u}^*(y^*)$ to small-amplitude motions on a β-plane, has been analyzed under the assumption that the disturbances are either two-dimensionally nondivergent (Kuo, 1949) or that the divergence which is permitted in the model is that implied by first-order quasi-geostrophic theory (e.g., Lipps 1963; Jacobs & Wiin-Nielsen, 1966). In these studies it is possible to establish the necessary condition for instability, in analogy with the Rayleigh-Fjørtoft theorem (e.g., Drazin & Howard, 1966), because in each case the stability equation has the same form as the Rayleigh stability equation. Consequently, the physical mechanism of barotropic instability may also be interpreted in analogy with the Rayleigh problem.

The stability of a barotropic current to nongeostrophic disturbances apparently had not been considered until the investigation of Sela & Jacobs (1968). A numerical approach was used to determine stability characteristics of a cosine jet for relatively small values of the Rossby number, just outside the quasi-geostrophic range. This model, a shallow-water model with 'reduced' gravity and constant Coriolis parameter, leads to a complicated stability equation which is not easily analyzed for stability criteria of more general basic flows.

In a recent investigation, Blumen (1971) examined the stability of a barotropic flow of a Boussinesq fluid of depth H to three-dimensional non-divergent and nonhydrostatic disturbances. The Coriolis parameter f and Brunt-Väisälä frequency N were assumed constant. Under these conditions it was shown that, in a statically stable atmosphere ($N^2>0$) the sufficient condition for stability to a normal mode disturbance, characterized by vertical and zonal wave numbers $n\pi/H$ and α^*, is the Rayleigh-Fjørtoft criterion

$$\bar{u}^*/\bar{u}^*_{yy} > 0 \qquad (1)$$

together with

$$1-(\bar{u}^*/c_g)^2 > 0 \qquad (2)$$

Here c_g is the internal gravity-wave speed

$$c_g = \pm N\,[(n\pi/H)^2 + \alpha^{*2}]^{-\frac{1}{2}} \qquad (3)$$

and the basic flow is scaled so that $\bar{u}^*$ vanishes at its inflection point. Conditions (1) and (2) are not in general independent, since the form of the basic profile affects the allowable modes of oscillation. However in the limiting case, $N^2 = \infty$, the flow becomes two-dimensional and only condition (1) is applicable. In this latter case $\bar{u}^*/\bar{u}^*_{yy} < 0$ is a necessary condition for the existence of a neutral disturbance (e.g., Drazin & Howard, 1966). A fluid in stable stratification, but not rotating, is an analogue of the compressible flow problem studied by Lees & Lin (1946). By analogy, $\bar{u}^*/\bar{u}^*_{yy} < 0$ is a necessary condition for the existence of a neutral subcritical disturbance, i.e., a mode which satisfies $1 - (\bar{u}^*/c_g)^2 > 0$. We shall show, by example, that this latter result carries over when the fluid is in uniform rotation if $1 - (\bar{u}^*/c_{gi})^2 > 0$, where c_{gi} denotes the speed of a gravity-inertia wave. It is also interesting to note that although $\bar{u}^*/\bar{u}^*_{yy} > 0$ implies stability in the Rayleigh problem, instability of profiles of this type is not precluded when the fluid is stably stratified.

This investigation is concerned with the instability of a shear layer, represented by the hyperbolic-tangent velocity profile, for which $\bar{u}^*/\bar{u}^*_{yy} < 0$. A neutral solution, whose locus of eigenvalues in parameter space forms a stability boundary, is first displayed. Then the instability characteristics of the barotropic mode $(n = 0)$ and the lowest baroclinic mode $(n = 1)$ are computed and some comparisons made with the geostrophic model studied by Jacobs & Wiin-Nielsen.

2. Boussinesq model

We consider a rotating stratified atmosphere in hydrostatic balance. A basic zonal flow $\bar{u}^*(y^*)$ of amplitude U is directed along the x^*-axis, positive to the east, with transverse variation along the y^*-axis, positive to the north. There is no variation of this flow in the vertical z^*-direction, which is parallel to the rotation vector $f\mathbf{k}/2$ and antiparallel to gravity g.

Introduction of the Boussinesq approximation implies that the range of parameters is restricted by

$$\left.\begin{aligned} &(a) \qquad fLU/gH < 1 \\ &(b) \qquad H/H_s < 1 \end{aligned}\right\} \tag{4}$$

where L is a length scale associated with the transverse variation of the basic current and H_s denotes scale height. The neglect of the y^*-variation of the basic thermodynamic field, due to rotation, is implied by (4 a) (Greenspan, 1968); while the attendant approximations associated with (4 b) have been discussed elsewhere (e.g. Ogura & Phillips, 1962). For later reference, (4 a) may be written

$$fLU/gH = (f^2 L^2/gH)\, R_0 < 1 \tag{5}$$

where
$$R_0 = U/fL \tag{6}$$

denotes the Rossby number, whose magnitude will remain unspecified for the present.

We shall adopt the formalism of Ogura & Phillips (1962), assuming small deviations from an adiabatic atmosphere whose pressure p_a^* density ϱ_a^* and temperature T_a^* at the earth's surface are denoted by $(p_{ao}^*, \varrho_{ao}^*, T_{ao}^*)$. The basic temperature structure of the atmosphere $\bar{T}^* = -\Gamma z^*$ is characterized by a constant value of the Brunt-Väisälä frequency N, defined within the present model by

$$N^2 = g\,(\Gamma_a - \Gamma)/T_{ao}^*, \tag{7}$$

where Γ_a is the adiabatic lapse rate.

In order to cast the equations of this model into dimensionless form, we introduce the characteristic values:

$$x^*,\ y^* \sim L,\ z^* \sim H,\ t^* \sim L/U,\ u^*,\ v^* \sim U$$
$$w^* \sim UH/L,\ p^*/\varrho_{ao}^* \equiv \pi^* \sim U^2,\ T^*/T_{oa}^*$$
$$\equiv \theta^* \sim U^2/gH \tag{8}$$

where (u^*, v^*, w^*) are the (x^*, y^*, z^*) velocity components and t^* is time. Then the linearized set of nondimensional Boussinesq equations takes the form

$$\mathfrak{L}u + v(\bar{u}_y - R_0^{-1}) = -\pi_x \tag{9}$$

$$\mathfrak{L}v + R_0^{-1}u = -\pi_y \tag{10}$$

$$0 = -\pi_z + \theta \tag{11}$$

$$u_x + v_y + w_z = 0 \tag{12}$$

$$F^2\mathfrak{L}\theta + w = 0 \tag{13}$$

where $\mathfrak{L} = \partial/\partial t + \bar{u}\partial/\partial x$ and

$$F^2 = (U/NH)^2 \tag{14}$$

The fluid is assumed bounded from above and below by rigid horizontal surfaces. Consequently the vertical velocity satisfies

$$w(x, y, t) = 0, \quad z = 0, 1 \tag{15}$$

Each wave disturbance will be represented as a normal mode. In particular, the solution satisfying (15) may expressed as

$$w = \hat{w}(y) \sin n\pi z \exp [i\alpha(x - ct)] \tag{16}$$

where α denotes the real x-wave number and the complex phase speed is $c = c_r + i\, c_i$. Instability corresponds to $\alpha c_i > 0$. At the boundaries (y_1, y_2) the normal velocity vanishes. However, here we shall consider an unbounded fluid and require

$$\text{all perturbation amplitudes} = 0, \quad y = \pm\infty. \tag{17}$$

The perturbation energy equation of this system (9–13) is given by

$$\frac{\partial}{\partial t}\int_{\infty -}^{\infty} \mathcal{E}\, dy = \int_{-\infty}^{\infty} \varrho_{a0} \tau \bar{u}_y\, dy \tag{18}$$

where the perturbation kinetic plus available potential energy, averaged over one wave length $\lambda = 2\pi/\alpha$ and fluid depth, is denoted by

$$\mathcal{E} = \tfrac{1}{2}\int_0^1 \frac{1}{\lambda}\int_0^\lambda \varrho_{a0}(u^2 + v^2 + F^2\theta^2)\, dx\, dz \tag{19}$$

and

$$\tau = -\int_0^1 \frac{1}{\lambda}\int_0^\lambda uv\, dx\, dz \tag{20}$$

is the average of the Reynolds stress per unit mass. When instability occurs, (18) shows that the transfer of energy from the mean flow to the unstable disturbances is carried out via the horizontal Reynolds stress.

3. Neutral solution

In order to investigate the stability problem, posed by this atmospheric model, we shall first seek neutral solutions ($c_i = 0$) corresponding to the zonal flow

$$\bar{u} = \tanh y, \; (1 - \infty \leqslant y \leqslant \infty) \tag{21}$$

For this purpose, we eliminate u from (10) using (9), and eliminate θ between (11) and (13) to obtain

$$\{\mathcal{L}^2 - R_0^{-1}(\bar{u}_y - R_0^{-1})\}\, v = R_0^{-1}\pi_x - \mathcal{L}\pi_y \tag{22}$$

$$w = -F^2\mathcal{L}\pi_z. \tag{23}$$

Next w is eliminated from (23) by means of (12) and (9) is again used to eliminate u, yielding

$$\{\mathcal{L}v_y - (\bar{u}_y - R_0^{-1})v_x\} = \{F^2\, \mathcal{L}^2\pi_{zz} + \pi_{xx}\} \tag{24}$$

Guided by a previous analysis of the hyperbolic-tangent profile (Blumen, 1970), we set $c_r = c_i = 0$ and introduce into (22) and (24)

$$\left.\begin{aligned} \pi &= \Pi A \operatorname{sech}^\gamma y \cos n\pi z \exp(i\alpha x) \\ v &= A \operatorname{sech}^\gamma y \cos n\pi z \exp(i\alpha x) \end{aligned}\right\} \tag{25}$$

where γ, A and Π are constants. The resulting pair of equations contain only terms proportional to unity and $\tanh^2 y$. The following expressions result from setting the coefficients of unity and $\tanh^2 y$ separately equal to zero:

$$\Pi = i\alpha^{-1}(1 - R_0^{-1}) \tag{26}$$

$$\gamma = 1 - (n\pi)^2 F^2(1 - R_0^{-1}) \tag{27}$$

$$\alpha^2 + (n\pi)^2 F^2(1 - R_0^{-1})^2 = 1 \tag{28}$$

The locus of neutral eigenvalues in parameter space (α, n, F, R_0) is given by (28). However, we note from (25) that the amplitude of v, say,

$$v \to \exp(-\gamma|y|) \text{ as } |y| \to \infty \tag{29}$$

Boundary condition (17) is satisfied if $\gamma > 0$, which further constrains F and the allowable modes n if $R_0^{-1} < 1$. In order to elucidate this condition, we write

$$\gamma^2 = \{1 - (n\pi)^2 F^2(1 - R_0^{-1})^2\}\{1 - (n\pi)^2F^2\} + (n\pi)^2 F^2 R_0^{-2} \tag{30}$$

and then introduce (28) to obtain

$$\gamma^2 = \alpha^2 (n\pi)^2 F^2\{(n\pi)^{-2}F^{-2} + (R_0\alpha)^{-2} - 1\} \tag{31}$$

This expression (31) is positive if

$$1 - (U/c_{gi})^2 > 0 \tag{32}$$

where $$c_{gi} = \pm \left[\frac{N^2}{(n\pi/H)^2} + \frac{f^2}{\alpha^{*2}}\right]^{\frac{1}{2}} \tag{33}$$

denotes the phase speed of a hydrostatic gravity-inertia wave.

Thus we have shown that, within the scope of the present model, a neutral solution exists for modes satisfying (32) when $\bar{u}/\bar{u}_{yy} < 0$. Modes which satisfy a radiation condition at infinity are associated with imaginary γ. Hence they are not admissible solutions here, because (27) is real.

When $R_0^{-1} = 0$, (26), (27) and (28) reduce to

$$\left.\begin{aligned} \Pi &= i\alpha^{-1} \\ \gamma &= 1 - (n\pi)^2 F^2 = \alpha^2 \\ \alpha^2 + (n\pi)^2 F^2 &= 1 \end{aligned}\right\} \tag{34}$$

This case is the analogue of the compressible flow problem investigated by Blumen (1970). Correspondence is accomplished by replacing $n\pi F$ in (34) by the Mach number M.

Hereafter, we shall focus attention on the regime of large-scale mid-latitude flow in which $0 < R_0 \lesssim 1$. Under these conditions it is convenient to let

$$F^2 = \varepsilon R_0^2 \tag{35}$$

where $$\varepsilon = (fL/NH)^2 = O(1). \tag{36}$$

Typical values under consideration are:

$f \sim 10^{-4}\ \text{sec}^{-1}, \quad L \lesssim 5 \times 10^5$ m

$N \lesssim 2 \times 10^{-2}\ \text{sec}^{-1}, \quad H < 2 \times 10^3$ m

The restriction imposed by (5) is satisfied under these conditions, since $R_0 < 1$.

Introduction of (35) into (28) yields

$$\alpha^2 + (n\pi)^2 \varepsilon (1 - R_0)^2 = 1 \tag{37}$$

In the quasi-geotrophic regime of flow ($R_0 < 1$), (37) reduces to

$$\alpha^2 + (n\pi)^2 \varepsilon = 1 \tag{38}$$

which cannot be satisfied by any of the baroclinic disturbance modes ($n \neq 0$) for the range of parameters considered. Thus neutral modes of this type do not exist in the present model. However, (37) may be satisfied if $n = 0$, the barotropic disturbance mode. It follows from (9–13) and (16), that the vertical velocity and temperature perturbation vanish identically while the remaining perturbation quantities are independent of height and of the *fluid stratification*. The remaining equations combine to yield the Rayleigh stability equation

$$(\bar{u} - c)(\hat{v}_{yy} - \alpha^2 \hat{v}) - \bar{u}_{yy}\hat{v} = 0 \tag{39}$$

and the neutral eigenvalue is given by

$$\alpha = \gamma = 1 \tag{40}$$

This result (40) was obtained by Michalke (1964), who also computed the unstable eigenvalues and growth rates. The presence of $R_0^{-1} \neq 0$ does not affect these calculations, although the amplitude of the eigenfunction π is modified by (26).

4. Instability characteristics

Eigenvalues

It is convenient to consider ε (36) and the mode of oscillation n given and view (37) as the equation for an ellipse in the α, $1 - R_0$ plane. For $\varepsilon = 1$, say, there will be a neutral curve for each value of n chosen. The unstable eigenvalues were determined by the numerical method used previously by Blumen (1970) and described therein. We note from (37) and reference to Fig. 1, that instability for $n \neq 0$ does not set in until

$$R_0 > 1 - \varepsilon^{-\frac{1}{2}}\pi^{-1} = 0.682, \ (\varepsilon = 1) \tag{41}$$

well-outside the quasi-geotrophic range. The neutral curve for $n = 1$ is shown in Fig. 1 together with the curve of maximum growth rate αc_i. Table 1 displays some instability characteristics for modes $n = 0$, 1. Unstable modes ($c_r = 0$, $c_i > 0$), contiguous with the neutral solution, only occur for α less than the neutral α given by (37). This result has previously been established for barotropic modes which satisfy the Rayleigh equation (39), e.g., Drazin & Howard (1966). Moreover, numerical integrations, carried out as indicated above, have failed to reveal the existence of unstable baroclinic modes when α exceeds the neutral value.

Table 1. *The maximum growth rate αc_i, corresponding wave length λ and e-folding time τ_e are displayed as a function of the Rossby number*

	αc_i		λ (km)		τ_e (days)	
R_0	$n=0$	$n=1$	$n=0$	$n=1$	$n=0$	$n=1$
1.00	0.190	0.129	5 650	4 740	0.6	0.9
0.95		0.105		4 790		1.2
0.90		0.082		4 900		1.6
0.85		0.059		5 240	0.7	2.3
0.80		0.036		5 710		4.0
0.75		0.017		6 980	0.8	9.1
0.70		0.002		12 570	0.9	83.0
0.10		—		—	6.1	—

The computations were made with $\varepsilon = 1$, $L = 400$ km and $f = 10^{-4}$ sec^{-1}.

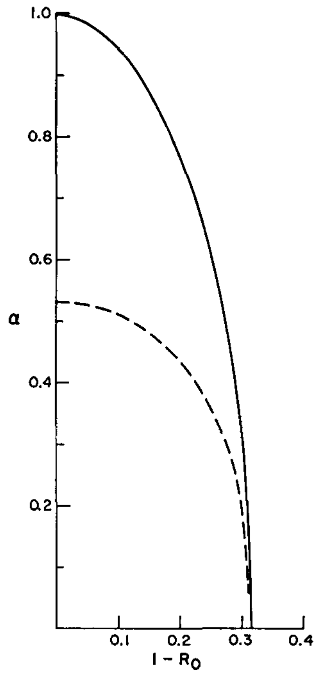

Fig. 1. Neutral curve (——) for $n = 1$, $\varepsilon = 1$. The maximum growth rate αc_i, for a given value of R_0, occurs along the dashed line.

Eigenfunctions

The amplitude A in (25) is arbitrary. We may, however, insure that the eigenfunctions are continuous at the intersection of the neutral curve with the α-axis by requiring $A \to 0$ as $\alpha \to 0$. A value which satisfies this requirement is

$$A = \alpha^2 \tag{42}$$

The eigenfunction $\hat{v}$ and the initial value of the Reynolds stress τ, associated with the most unstable baroclinic wave ($n = 1$), are displayed in Figs. 2 and 3 respectively. Since $\bar{u}_y = \operatorname{sech}^2 y > 0$ the total energy conversion is directed from the mean flow to the unstable disturbances as required by (18), although $\tau \bar{u}_y$ becomes slightly negative away from the origin.

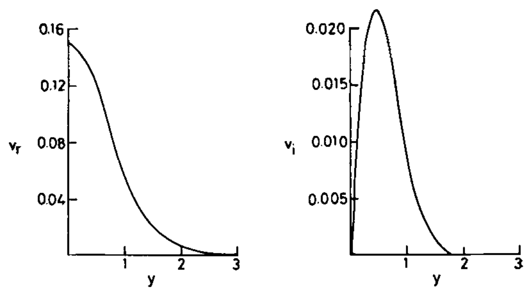

Fig. 2. The real v_r (symmetric) and imaginary v_i (antisymmetric) parts of $\hat{v}$ for $n = 1$, $\alpha = 0.53$, $R_0 = 1$.

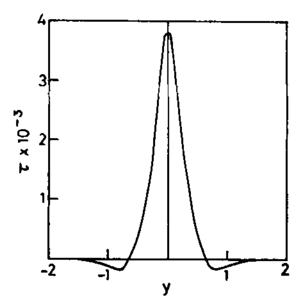

Fig. 3. The Reynolds stress τ associated with the eigenvalues given in Fig. 2.

5. Remarks

The present model was chosen to demonstrate some properties of barotropic instability, because the availability of a neutral solution led to a simple relationship between the various characteristic parameters (28 or 37) and thereby the numerical labor involved in delineating the unstable eigenvalues was considerably reduced. Yet this model is not unrealistic, in the sense that the results do provide some insight into the physical mechanism of barotropic instability in a stratified fluid.

Physically, the instability examined here is a manifestation of inertial instability associated with the inflection point in the basic velocity profile. Moreover instability may be realized by both two- and three-dimensional disturbances which correspond to $n=0$ and $n\neq 0$ respectively. However, when the motion is three-dimensional, some of the available energy from the basic flow must be used to do work against buoyancy before any becomes available to initiate instability. Hence, the maximum growth rate, for a given R_0, is associated with the two-dimensional barotropic solution. The fact that the *neutral* baroclinic disturbances reduce to two-dimensional inertial modes when $R_0=1$ does not alter the basic feature of the instability discussed above, because the *unstable* baroclinic modes associated with $R_0=1$ are *not* two-dimensional.

A detailed comparison with the results obtained by Jacobs & Wiin-Nielsen (1966) is not warranted, since they studied the stability of a barotropic jet flow of a stratified fluid to *geostrophic* disturbances on a β-plane. However, it is interesting to note that the growth rates and corresponding wave lengths of the most unstable geostrophic modes are of comparable magnitude with the unstable nongeostrophic modes, whose characteristics are displayed in Table 1. Moreover, their observation that the unstable internal modes should be expected to influence the flow pattern, since their growth rates are quite close to that of the barotropic mode, may be restated here under the provision that $R_0 \sim 1$.

The importance of barotropic instability in the more commonly occurring regime of small Rossby numbers, within and just outside the quasi-geostrophic range, has not been clearly assessed. For instance, the absence of quasi-geostrophic instability is a characteristic of the model which seems to be overly restrictive, since this phenomenon was found in the model used by Jacobs & Wiin-Nielsen. However, the ingredients of the two models are quite different and it can only be conjectured whether a different velocity profile, a variable static stability and/or the use of a β-plane, for example, would yield unstable modes as a continuous function of the Rossby number of whether instability would occur in discrete regions of the R_0, α plane.

Acknowledgments

This investigation was supported by the Atmospheric Science Section of the National Science Foundation, under Grant GA-1483. Acknowledgement is also made to the National Center for Atmospheric Research, which is sponsored by the National Science Foundation, for use of the Control Data 6600 computer. I wish to thank Patrick J. Kennedy for his assistance with the numerical computations and Dr Michael E. McIntyre for his critical appraisal of the manuscript.

REFERENCES

Blumen, W. 1970. Shear layer instability of an inviscid compressible fluid. *J. Fluid Mech. 40*, 769–781.

Blumen, W. 1971. On the stability of plane flow with horizontal shear to three-dimensional nondivergent disturbances. *Geophys. Fluid Dyn. 2*, 189–200.

Drazin, P. G. & Howard, L. N. 1966. Hydrodynamic stability of parallel flow of inviscid fluid. *Advanc. Appl. Mech. 9*. Academic Press, New York.

Greenspan, H. P. 1968. *The theory of rotating fluids.* University Press, Cambridge.

Jacobs, S. J. & Wiin-Nielsen, A. 1966. On the stability of a barotropic basic flow in a stratified atmosphere. *J. Amos. Sci. 23*, 682–687.

Kuo, H. L. 1949. Dynamic instability of two-dimensional nondivergent flow in a barotropic atmosphere. *J. Meteor. 6*, 105–122.

Lees, L. & Lin, C. C. 1946. Investigation of the stability of the laminar boundary layer in a compressible fluid. NACA TN no. 1115.

Lipps, F. B. 1963. Stability of jets in a divergent barotropic fluid. *J. Atmos. Sci. 20*, 120–129.

Michalke, A. 1964. On the inviscid instability of the hyperbolic-tangent velocity profile. *J. Fluid Mech. 19*, 543–556.

Ogura, Y. & Phillips, N. A. 1962. Scale analysis of deep and shallow convection in the atmosphere. *J. Atmos. Sci. 19*, 173–179.

Sela, J. & Jacobs, S. J. 1968. On baroclinic and barotropic ageostrophic stability. Tech. Rpt. 07344-2-T, Dept. Meteor. and Ocean., University of Michigan, Ann-Arbor, 98 pp.

ОБ УСТОЙЧИВОСТИ БАРОТРОПНОГО ПОТОКА С ПОПЕРЕЧНЫМ СДВИГОМ СКОРОСТИ К НЕГЕОСТРОФИЧЕСКИМ ВОЗМУЩЕНИЯМ

С помощью линейного анализа изучается устойчивость к трехмерным возмущениям баротропного потока устойчиво стратифицированной жидкости глубины H с профилем скорости $\bar{u}_* = U\text{th}(y^*/L)$. Как для основного, так и для возмущенного потоков предполагается гидростатичность, а частота Брента–Вяисяля и параметр Кориолиса считаются постоянными. При условии, что $1-(U/C_{gi})^2 > 0$, где C_{gi}-фазовая скорость гравитационно-инерционной волны с зональным и вертикальным волновыми числами α^* и $n\pi/H$, найдено нейтральное решение. С помощью этого решения численными методами определены некоторые характеристики неустойчивых волн. Затем показано, что для величин параметров, характерных для крупномасштабных движений средних широт, основной поток устойчив по отношению к внутренним бароклинным модам ($n \neq 0$), но неустойчив к баротропной моде ($n = 0$) в квазигеострофическом интервале чисел $\text{Ro} = U/fL \ll 1$. Неустойчивость к внутренним модам не возникает до тех пор, пока $\text{Ro} > 1 - \pi^{-1} = 0{,}682$, т. е. задолго после области квазигеострофичности. Приводятся и обсуждаются характеристики этой неустойчивости.