The stability of continuous baroclinic models with planetary vorticity gradient

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ABSTRACT

The linearized continuous baroclinic model of Eady with planetary vorticity gradient is studied by considering a normal mode disturbance. The problem is reduced to a confluent hypergeometric equation with proper boundary conditions. The stability of this problem is expressed in terms of the behavior of an analytical function. A proof is given to show the existence of neutral waves. Fjörtoft's advective model with planetary vorticity gradient is obtained by letting the static stability vanish in Eady's model with planetary vorticity gradient. A comparison of the two models shows that they are very similar. Static stability leads to a lower level of non-divergence than in the pure advective case. An expansion technique is used to study the deviation near the advective model. Although this method is useful only for medium-scale unstable waves, it gives a simple formula for the phase speed and growth rate.

1. Introduction

The problem of baroclinic instability has been studied extensively for more than two decades. The pioneer works of Charney (1947), Eady (1949) and Fjörtoft (1950) showed different aspects of the same problem. Charney's model includes the effect of the planetary vorticity gradient (β-effect) but without the upper boundary. Eady's model has both upper and lower rigid boundaries but no β-effect. These two authors considered quasi-geostrophic normal mode disturbances. Fjörtoft considered the bounded, non-geostrophic, advective model.

In order to study both the β-effect and the effects of the rigid top-and-bottom boundaries, Green (1960) solved Eady's model (with the β-effect) numerically by finite difference approximations. He found that unstable waves existed in the long wave spectrum where Charney found only neutral waves. Burger (1962) showed analytically that the unstable waves existed in Charney's model except for an infinite number of discrete wave lengths for a given zonal wind shear. Later Miles (1964*a*, *b*, *c*, *d*) obtained similar results analytically. Hirota (1968) solved Green's problem numerically again by finite difference approximations in order to study the wave structures and the energetic processes. The most recently published work of Garcia & Norscini (1970) solved Green's problem by numerically truncating the series solution, and confirmed and extended the results of Green & Hirota.

While there are extensive studies of the statically stable models, relatively few published studies have been made with regard to the statically neutral model (i.e. Fjörtoft's advective model) e.g. Holmboe (1959), Wiin-Nielsen (1967). The physical mechanism of the waves was clearly exhibited in Holmboe's study where he introduced the method of symmetric waves.

In the present study we discuss (1) the stability of the statically stable model (Eady's model, $\beta \neq 0$) and (2) the stability of the advective model as the limiting case when the static stability vanishes in Eady's model ($\beta \neq 0$). A bridge is established between the two models by analytical methods whenever possible and by qualitative physical arguments when the analytical methods no longer work. Hereafter $\beta \neq 0$ is implied in all cases.

2. Eady's model

The model consists of a baroclinic layer bounded above and below by rigid horizontal planes separated by a distance $2h$. The entropy

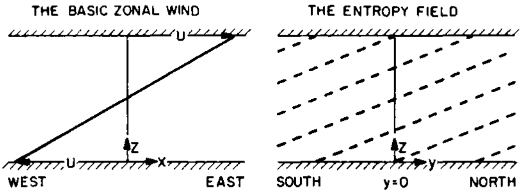

Fig. 1. The statically stable model (Eady's model).

field increases linearly upward and southward with no variation in the zonal direction in the undisturbed state (Fig. 1). The entropy with the factor C_p (specific heat at constant pressure) omitted can be written as

$$\ln \theta = \ln \theta_0 - \tau y + \sigma z \tag{2.1}$$

where θ is potential temperature, y is distance northward, z is distance upward and τ and σ are positive constants. Within the Boussinesq approximation the temperature field is balanced by the zonal wind with constant shear, namely,

$$fU_z = g\tau \tag{2.2}$$

where f is the Coriolis parameter at a constant latitude, g is the acceleration of gravity, and the subscript z denotes differentiation with respect to z. In a frame of reference which moves with the air at the middle level, the air moves to the east with the speed U at the upper boundary and the same speed to the west at the lower boundary.

Let us superimpose on this basic state a small amplitude wave disturbance which has a sinusoidal variation in the zonal direction and no variation in the meridional direction. The entropy is then given by

$$\ln \theta = \ln \theta_0 - \tau(y - Y) + \sigma z \tag{2.3}$$

where Y is the meridional displacement of the isentropic surface from its equilibrium latitude.

With the assumption that the motion is adiabatic, the isentropic surfaces are material surfaces which move with the air. The adiabatic equation is obtained by taking the individual change of (2.3) and letting it vanish, namely:

$$\sigma w = \tau\left(v - \frac{DY}{Dt}\right) \tag{2.4}$$

Vertical differentiation of the geostrophic equation with the hydrostatic and Boussinesq approximations finally gives the thermal wind formula,

$$Y_x = v_z / U_z \tag{2.5}$$

This equation states that the wind shear is parallel to the isentropic contour lines.

The vertical component of the vorticity equation with the Boussinesq approximation is given by

$$\frac{D}{Dt} v_x + \beta v = f w_z \tag{2.6}$$

From (2.2), (2.4), (2.5) and (2.6) we obtain the so-called linearized quasi-geostrophic potential vorticity equation,

$$\left(\frac{\partial}{\partial t} + U\frac{z}{h}\frac{\partial}{\partial x}\right)\left[v_{xx} + \left(\frac{f}{N}\right)^2 v_{zz}\right] + \beta v_x = 0 \tag{2.7}$$

where subscript x denotes differentiation with respect to x and N is the buoyancy frequency which is defined as $N^2 = \sigma g$.

Let us consider the disturbance to be a normal mode such that

$$v = \text{Real part of } \{V(z) \exp [ik (x + Ct)]\} \tag{2.8}$$

With the definitions,

$$\varkappa = khN/f \tag{2.9 a}$$

$$\zeta = \varkappa\left(\frac{C}{U} + \frac{z}{h}\right) \tag{2.9 b}$$

$$\alpha = \varkappa U_R / U \quad (U_R = \beta / k^2) \tag{2.9 c}$$

$$\phi = Ve^{\zeta}/\zeta \tag{2.9 d}$$

substitution of (2.8) in (2.7) gives a confluent hypergeometric equation,

$$\zeta\ddot{\phi} + (2 - 2\zeta)\dot{\phi} + (\alpha - 2)\phi = 0 \tag{2.10}$$

where the super-dot means differentiation with respect to ζ. The boundary conditions are that the vertical velocity vanishes at the level rigid boundaries. Let us put $w = 0$ in (2.4). Then with the aid of (2.5), (2.8) and (2.9a, b, d), we obtain

$$\phi = \dot{\phi} \quad \text{at} \quad \zeta = \zeta^{\pm} = \varkappa\left(\frac{C}{U} \pm 1\right) \tag{2.11}$$

(2.10) and (2.11) form the boundary value problem which we are going to solve. (2.10) is a second-order differential equation which has two independent solutions φ and ψ such that

$$\phi = A\varphi + B\psi \tag{2.12}$$

where A and B are arbitrary constants whose relation to each other has to be determined from the boundary conditions (2.11). The series solutions of φ and ψ can be obtained by the method of Frobenius, the results are

$$\varphi = \sum_0^\infty a_r \zeta^r \quad \left(a_0 = 1, a_r = Ra_{r-1}, R = \frac{2r-\alpha}{r(r+1)}\right) \tag{2.13}$$

$$\psi = \varphi \ln \zeta - \frac{1}{\alpha}\left(\frac{1}{\zeta} + \sum_0^\infty b_r \zeta^r\right) \quad (b_0 = 2+\alpha) \tag{2.14}$$

$$b_r = Rb_{r-1} + \frac{\alpha}{r(r+1)}\left(a_r + 2\frac{r-1-\alpha}{r+1}a_{r-1}\right)$$

With the definition,

$$F(\zeta) = -\frac{\dot\psi - \psi}{\dot\varphi - \varphi} = -\frac{P(\dot\varphi - \varphi) + \dot P \varphi}{\dot\varphi - \varphi} \quad (\psi = P\varphi) \tag{2.15}$$

substitution of (2.12) in (2.11) yields

$$\frac{A}{B} = F(\zeta^-) = F(\zeta^+) \tag{2.16}$$

Let us differentiate F with respect to ζ, and substitute φ and $P\varphi$ separately in (2.10). The results are

$$\dot F = \dot P(1 - \alpha/\zeta)\left(\frac{\varphi}{\dot\varphi - \varphi}\right)^2 \tag{2.17}$$

where

$$\dot P = \lambda[e^\zeta/(\zeta\varphi)]^2 \tag{2.18}$$

The constant λ is determined by substituting the series solutions (2.13) and (2.14) in (2.18) and taking the limit as $\zeta \to 0$. The result is

$$\lambda = \lim_{\zeta \to 0}[\dot P(\zeta\varphi)^2] = \frac{1}{\alpha} \tag{2.19}$$

Substituting (2.19) in (2.18) and (2.18) in (2.17), we obtain

$$\dot F = \left(\frac{1}{\alpha} - \frac{1}{\zeta}\right)\left[\frac{e^\zeta}{\zeta(\dot\varphi - \varphi)}\right]^2 \tag{2.20}$$

Let us now consider a neutrally stable mode. Since ζ is real in this mode, φ in (2.13) and its derivative are real. Therefore, $\dot F$ in (2.20) is real and, except for an unknown integration constant, $F(\zeta)$ is a real function of ζ. But this integration constant does not affect the boundary conditions in (2.16), so without loss of generality we may take $F(\zeta)$ to be real. For any given value of α we see that $\dot F = \infty$ at $\zeta = 0$ and $\dot\varphi = \varphi$ from (2.20), so the F-curve has vertical asymptotes at $\zeta = 0$ and $\dot\varphi = \varphi$ as shown in Fig. 2*a*. It is continuous between the asymptotes and has a minimum point at $\zeta = \alpha$. Any horizontal line above the minimum point intersects the F-curve in a pair of points whose abscissas are $\zeta = \zeta^\pm$. From the definitions of $\zeta^\pm$ in (2.11) we can determine $\varkappa$ and C as follows

$$\varkappa = (\zeta^+ - \zeta^-)/2 \tag{2.21}$$

$$c = \frac{\zeta^+ + \zeta^-}{\zeta^+ - \zeta^-} U \geqslant U \tag{2.22}$$

The minimum point of the F-curve in Fig. 2*a* gives the very long modes,

$$\varkappa \to 0: \zeta^\pm \to \varkappa C/U \to \alpha = \varkappa U_R/U \text{ or } C = U_R$$

This result has already been obtained earlier by Holmboe (1959, 1968) for the very long modes in a general quasi-geostrophic model. For the

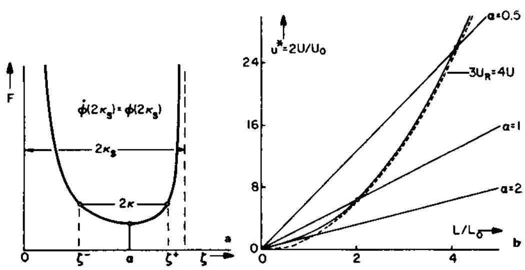

Fig. 2. (*a*) Schematic graph of the $[F(\zeta) = -(\dot\psi - \psi)/(\dot\varphi - \varphi)]$-curve for the neutrally stable modes. (*b*) The wave length of the shortest neutrally stable mode as a function of the wind shear of the baroclinic layer (*solid curve*).

shortest neutrally stable mode, we have from the distance between the asymptotes,

$$\varkappa \to \tfrac{1}{2}\zeta^+ \to \varkappa_s\colon\ \zeta^- \to 0;\quad \dot{\varphi}(2\varkappa_s) = \varphi(2\varkappa_s)$$

Substituting (2.22) in ζ^-, and (2.13) in φ and $\dot{\varphi}$ above, we obtain

$$\varkappa \to \varkappa_s\colon\quad C = U \qquad \sum_0^\infty \frac{r-\alpha}{r+2} a_r (2\varkappa_s)^r = 0 \tag{2.23}$$

The series in (2.23) represents $\varkappa_s$ as a function of $\alpha = \varkappa_s U_R/U$. To show this curve in a wind shear-wave length diagram, let us first select a velocity parameter U_0 such that

$$\beta = k_0^2 U_0 \tag{2.24}$$

where

$$k_0 = f/C_L \tag{2.25}$$

and C_L is the phase speed of the long gravity wave with inertia frequency f. Next let us define $\varkappa_0 = k_0 Nh/f$. Then $\varkappa_0 = \pi/2$ (Holmboe, 1968) and α becomes

$$\alpha = \varkappa U_R/U = \pi \frac{L}{L_0} \Big/ u^* = \pi^2/(2u^*\varkappa) \tag{2.26}$$

where

$$L = 2\pi/k \text{ and } u^* = 2U/U_0 \tag{2.27}$$

With (2.26) substituted in the series in (2.23), we obtain numerically the boundary of the neutrally stable waves in the wind shear-wavelength diagram (solid curve in Fig. 2*b*). To see the analytic properties of the boundary of the neutrally stable waves, we rewrite the series in (2.23) as follows:

$$\varkappa \to \varkappa_s\colon\ \alpha/\varkappa_s = U_R/U = 2(2-\alpha)\left[\frac{1}{3}(1-\alpha) + \sum_2^\infty \frac{r-\alpha}{r+2}\frac{a_r}{a_1}(2\varkappa_s)^{r-1}\right] \tag{2.28}$$

For very weak static stability ($N \to 0$), we have that $\varkappa \to \alpha \to 0$, so for such cases the boundary of the neutral waves is very near the asymptotic curve, namely:

$$N \to 0\colon\ 3U_R = 4U \text{ or } (k_0/k_s)^2 = (L_s/L_0)^2 = \tfrac{2}{3}u^* \tag{2.29}$$

which is shown as the dashed-parabola in Fig. 2*b*. For very short waves, near the origin in Fig. 2*b*, the boundary of the neutral waves is tangent to the ($\alpha = 2$)-line, namely:

$$\varkappa_s \to \infty\colon\ \alpha = 2 \text{ or } u^* = \tfrac{1}{2}\pi k_0/k_s = \tfrac{1}{2}\pi L_s/L_0 \tag{2.30}$$

Next, let us consider the shortest neutrally stable mode. We note that this mode moves westward with the speed $C = U$ relative to the air at the middle level, so it is stationary relative to the air at the lower boundary.

The function $F(\zeta^\pm)$ in (2.16) is infinite at the boundary of the neutral waves, so B in (2.12) is zero. Thus, the boundary conditions for this mode are satisfied by the φ-series in (2.13), and from (2.9d) we have the vorticity amplitude of the mode,

$$\varkappa = \varkappa_s\colon\ V = V_0 \zeta \varphi e^{-\zeta};\ \zeta = \varkappa_s(1 + z/h) \tag{2.31}$$

V is zero at the lower boundary ($z = -h$).

Given the vorticity amplitude, the slope of the isentropes is given by the thermal wind equation (2.5)

$$UY_x = hv_z = \varkappa v_\zeta = \varkappa \dot{V} \cos kx$$

The amplitude of the isentropic contour ($Y = A \sin kx$) is then given by

$$UkA = \varkappa V = \varkappa V_0[1 + \zeta(\dot{\varphi}/\varphi - 1)]\varphi e^{-\zeta}$$

Let $A_0 = \varkappa V_0/Uk$ denote the amplitude of the isentropic contour at the lower boundary ($z = -h$). At other levels inside the layer, the amplitude of the isentropic contour is given by

$$\varkappa = \varkappa_s\colon\ A = A_0[1 + \zeta(\dot{\varphi}/\varphi - 1)]\varphi e^{-\zeta} \tag{2.32}$$

The vorticity field (2.31) has the amplitude

$$\varkappa = \varkappa_s\colon\ V = A_0 k_s U(1 + z/h)\varphi e^{-\zeta} \tag{2.33}$$

Next we wish to find the divergence amplitude of this mode. Before doing this let us first find the relation between the divergence and the vorticity at an arbitrary level in the layer. Substitution of the continuity equation

$$w_z = -u_x \tag{2.34}$$

in the vorticity equation (2.6) gives

$$\frac{D}{Dt} v_x + \beta v = -f u_x \tag{2.35}$$

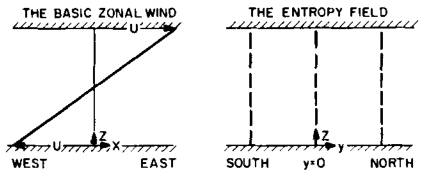

Fig. 3. The statically neutral model (Fjörtoft's advective model).

Note that u is the zonal component of the perturbation velocity. On an arbitrary level let us select the origin of x such that

$$v = V \sin (kx + \gamma),\ u = -D \cos kx \qquad (2.36)$$

where γ is the distance between the trough of the vorticity wave and the nearest place of zero divergence. In a frame which moves with the air on this level, (2.35) becomes

$$v_t - U_R v_x = -fu \quad \left[(\ )_t = \frac{\partial}{\partial t}(\ )\right] \qquad (2.37)$$

Substitution of (2.36) in (2.37) and separation of the sine wave and the cosine wave yield

$$\begin{aligned} (\ln V)_t &= f(D/V) \sin \gamma \\ \gamma_t &= f(D/V) \cos \gamma + kU_R \end{aligned} \qquad (2.38)$$

γ_t is the intrinsic upwind phase speed of the vorticity wave through the air.

Let us apply (2.38) to the shortest neutrally stable mode. Since it is neutral, we have that $(\ln V)_t = 0$, so from (2.38) $\gamma = 0$ and

$$f(D/V) = \gamma_t - k_s U_R$$

Relative to the air at the middle level

$$\gamma_t = k_s U(1 + z/h) \quad (C = U)$$

Therefore the divergence amplitude of the mode is given by

$$\varkappa = \varkappa_s\!: fD = k_s U V(1 + z/h - U_R/U)$$

Substitution of (2.33) in the expression immediately above gives

$$\begin{aligned} \varkappa = \varkappa_s\!: fD = A_0(k_s U)^2\ (1 + z/h) \\ \times (1 + z/h - U_R/U)\varphi e^{-\zeta} \end{aligned} \qquad (2.39)$$

For Fjörtoft's advective model (Fig. 3), $N = 0$, so $\varkappa_s = 0$. Hence, we have $\varphi e^{-\zeta} = 1$. For weak static stability, $N \to 0$, $\varkappa_s \to 0$, $\varphi e^{-\zeta} \to 1$, and $3U_R \to 4U$ [from (2.29)] and we see that it has almost the same kinematic structure as the $(3U_R = 4U)$-mode in Fjörtoft's advective model. The variation of $\varphi e^{-\zeta}$ for $\alpha = 0$, 0.5, 1, and the curves of V and fD for $\alpha = 0$ are shown in Fig. 4. The static stability reduces both the vorticity amplitude and the divergence amplitude by the factor $\varphi e^{-\zeta}$, and lowers the level of zero divergence so that the total divergence in the air column vanishes. We see that in Fig. 4 the static stability reduces the amplitude more in the upper part of the layer than in the lower part of the layer, the effect is to shift the energy of the mode to the lower levels. A similar effect was found in the two-isentropic layer model (Holmboe 1966, 1970).

We have considered the shortest neutrally stable mode. For the long neutrally stable $(\varkappa < \varkappa_s)$-mode we find the corresponding C and $\varkappa$ as follows. For a given α, we substitute φ in (2.13) and ψ in (2.14) into (2.15) to compute F numerically by truncating the series solutions φ and ψ. F^{-1}-curves for $\alpha = 1$, 2/3, 1/2 are computed and shown in Fig. 5. The values of $\varkappa$ and C are obtained graphically from the formulas (2.21) and (2.22) respectively for a given value of α.

3. Advective model

In the last section we have touched upon Fjörtoft's advective model ($\beta \neq 0$) for the $(3U_R = 4U)$-mode. In this section we shall

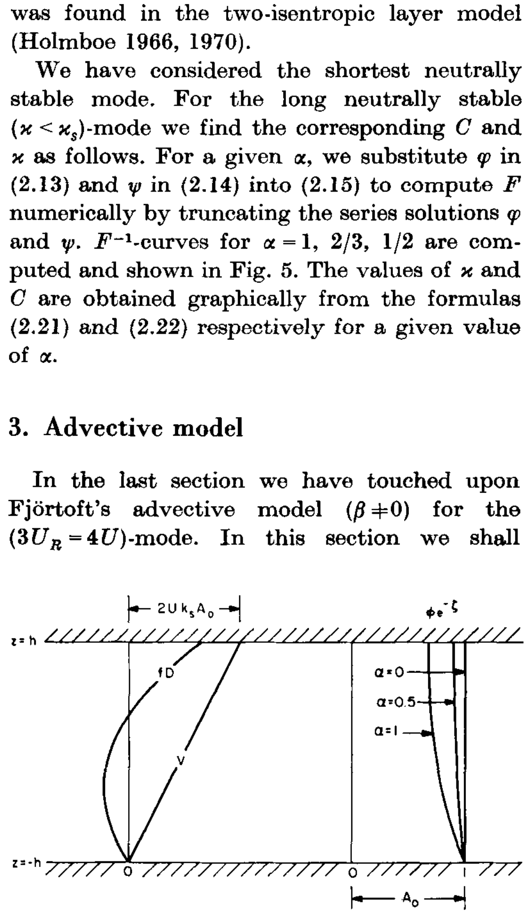

Fig. 4. To the left: divergence amplitude (D) and vorticity amplitude (V) in the shortest neutrally stable $(3U_R = 4U)$-mode in a layer with no static stability ($\alpha = 0$). *To the right:* the height variation of the $(\varphi e^{-\zeta})$-reduction factor of the amplitudes of the shortest neutrally stable mode in layers with weak static stability ($\alpha = 0.5$) and in layers with strong static stability ($\alpha = 1$).

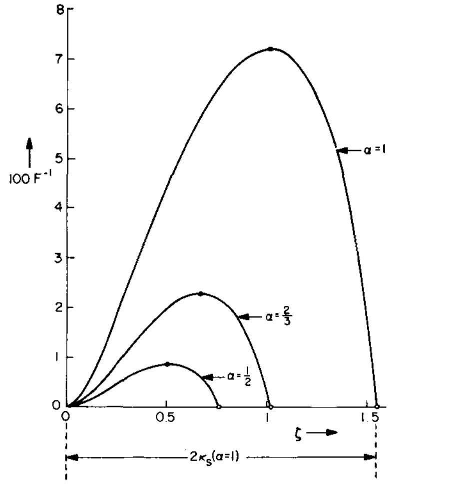

Fig. 5. Three F^{-1}-curves which determine the phase velocity, $C=(1+\zeta^-/\varkappa)U$, of the long neutrally stable ($\varkappa<\varkappa_s$)-modes.

derive the advective model as a special case of Eady's model ($\beta \neq 0$). For the cases of weak static stability ($N<f/kh$) we recall from (2.9a) that $\varkappa<1$. Using (2.13) and (2.14) for such cases, we have

$$N<f/kh:\ \dot{\varphi}-\varphi = -(\tfrac{1}{2}\alpha-\tfrac{1}{3}\zeta)(1+\zeta)+O(\varkappa^3), \quad (3.1)$$
$$\alpha\zeta^2(\dot{\psi}-\psi) = 1+\zeta+\alpha\zeta+O(\varkappa^3)$$

The function $F(\zeta)$ in (2.15) then becomes

$$N<f/kh:\ F^{-1}(\zeta) = \alpha\zeta^2(\tfrac{1}{2}\alpha-\tfrac{1}{3}\zeta)+O(\varkappa^6) \quad (3.2)$$

Substituting (3.2) in the inverse of (2.16), $F^{-1}(\zeta^-)=F^{-1}(\zeta^+)$, and dividing it by the factor $1/2\ \alpha\varkappa^3$, we have, in the limit, the equation for the advective model (statically neutral model),

$$N=0:\left(\frac{\zeta^-}{\varkappa}\right)^2\left(\frac{\alpha}{\varkappa}-\frac{2}{3}\frac{\zeta^-}{\varkappa}\right)=\left(\frac{\zeta^-}{\varkappa}+2\right)^2\left[\frac{\alpha}{\varkappa}-\frac{2}{3}\left(\frac{\zeta^-}{\alpha}+2\right)\right] \quad (3.3)$$

With $\alpha/\varkappa=U_R/U$ substituted from (2.9c) and $\zeta^-/\varkappa=C/U-1$ substituted from (2.11), (3.3) becomes

$$N=0: C^2-U_RC+\frac{1}{3}U^2=0 \quad \left(C=\frac{U_R}{2}\pm\sqrt{\left(\frac{U_R}{2}\right)^2-\frac{U^2}{3}}\right) \quad (3.4)$$

We see that C is complex for $U_R<2U/\sqrt{3}$, and real for $U_R\geqslant 2U/\sqrt{3}$. Hence, for the advective model the stability boundary is at $U_R=2U/\sqrt{3}$. For a given basic zonal wind shear, waves with wave lengths longer than this wave length are neutrally stable; waves with wave lengths shorter than this wave length are unstable.

Let us examine the long neutral modes. From the pair of roots in (3.4) we see that

(i) $4U/3>U_R>2U/\sqrt{3}$: both modes $C<U$;
(ii) $4U/3=U_R$: one mode $C=U$, other mode $C=U/3$;
(iii) $4U/3<U_R$: one mode $C>U$, other mode $C<U/3$.

In case (i) both modes have a critical level (the level at which the phase speed equals the wind speed) inside the layer. In case (ii) one mode has a critical level at the lower boundary (similar to the kinematic structure in the statically stable case in Section 2), while the other mode has a critical level inside the layer. In case (iii) one mode does not have a critical level, but the other has a critical level.

Let us try to explain qualitatively, for the statically stable case, why the mode with a critical level cannot be neutral. Substitution of the adiabatic equation (2.4) in the vorticity equation (2.6) and setting $\sigma=0$ we obtain, together with continuity equation (2.34), the following equation when the normal mode solution is substituted,

$$\left(C+U\frac{z}{h}-U_R\right)\left(C+U\frac{z}{h}\right)Y_x=-\frac{f}{k^2}w_z=\frac{f}{k^2}u_x\Bigg]$$
$$\left[v=\left(C+U\frac{z}{h}\right)Y_x\right] \quad (3.5)$$

Differentiation of v in the adiabatic equation with $\sigma=0$ and substitution of the thermal wind equation (2.5) yield that Y is independent of height. Suppose there is a critical level. Then from (3.5) there are no horizontal motions on this level, but there is maximum vertical velocity. In the advective model with vertical isentropic surfaces, the vertical motion at the level with no horizontal motion does not cause interference. But in a baroclinic layer with the entropy increasing with height, the mode with zero horizontal motion inside the layer cannot be neutral, because in this case a buoyancy force is created and an additional temperature wave is excited, i.e., a gravity wave at the critical level is excited. Therefore, we expect qualitatively that for the statically stable case, the modes in case (i) cannot be neutral; only one mode in case (ii) can be neutral ($C = U$, $4U/3 \approx U_R$); in case (iii) one mode is neutrally stable ($C > U$), but the other cannot be neutrally stable. In fact the numerical results of Green (1960), Hirota (1968) and Garcia & Norscini (1970) showed that the unstable waves in case (iii) exist.

4. The neutrally stable mode in baroclinic layers with weak static stability

The phase velocities of the neutrally stable ($3U_R > 4U$)-modes for given values of $\alpha = 1$, $2/3$, $1/2$ were given in Fig. 5. In view of the steep slope of F^{-1} at $\dot{\varphi}(2\varkappa_s) = \varphi(2\varkappa_s)$ if $\varkappa_s - \varkappa < \frac{1}{2}\alpha$, that is in the relatively short neutral ($4U < 3U_R < 12U$)-modes, a good approximation of the phase velocity can be obtained by replacing the right-hand part of the F^{-1}-curve by its limiting tangent at $\zeta = 2\varkappa_s$. Substituting (2.20) in the inverse of (2.16), we have

$$\varkappa_s - \varkappa < \alpha: F^{-1}(\zeta^-) = F^{-1}(\zeta^+) \approx (2\varkappa_s - \zeta^- - 2\varkappa) \times \left(\frac{1}{\alpha} - \frac{1}{2\varkappa_s}\right) \left[\frac{e^\zeta}{\zeta(\dot{\psi} - \psi)}\right]^2_{\zeta = 2\varkappa_s} \tag{4.1}$$

Considering the case with very weak static stability ($\varkappa < 1$), we have from (2.14),

$$\varkappa < 1: \left[\frac{e^\zeta}{\zeta(\dot{\psi} - \psi)}\right]^2_{\zeta = 2\varkappa_s} = (2\alpha\varkappa_s)^2 + 0(\varkappa^6)$$

With (3.2) substituted in F^{-1} (ζ^-) in (4.1), equation (4.1) with $O(\varkappa^6)$ ignored becomes

$$\varkappa_s - \alpha < \varkappa < 1: \alpha(\zeta^-)^2 \left(\frac{1}{2}\alpha - \frac{1}{3}\zeta^-\right) \approx \left(1 - \frac{1}{2}\frac{\zeta^-}{\varkappa_s} - \frac{\varkappa}{\varkappa_s}\right) \left(\frac{2\varkappa_s}{\alpha} - 1\right)(2\alpha\varkappa_s)^2$$

With this divided by $2\,(\alpha\varkappa_s)^2$, and substituting $\alpha = \varkappa\, U_R/U = 4\,\varkappa_s/3$, we obtain in the limit for the statically neutral model,

$$\mathrm{N} = 0,\ U_R < 4U: (\tfrac{1}{2}\zeta^-/\varkappa_s + 1)\ (\tfrac{1}{2}\zeta^-/\varkappa_s - 1)^2 \approx 4U/3U_R \tag{4.2}$$

where $\frac{1}{2}\zeta^-/\varkappa_s = \frac{3}{4}(C - U)/U_R$

Table 1. ($N = 0$), U_R/U *tabulated*

$(C-U)/U_R$	Exact (3.4)	Approx. (4.2)
0	1.333	1.333
1/4	1.648	1.646
1/3	1.812	1.803
1/2	2.309	2.250

For the ($3U_R = 4U$)-mode (4.2) is satisfied because $C = U$. For the longer modes ($4U < 3U_R < 12U$), (4.2) is a cubic equation which has a real root almost the same as the larger root of the exact quadratic equation (3.4). The difference between the exact and the approximate value of the wave lenth for $(C - U)/U_R = 0$, $1/4$, $1/3$, $1/2$ are shown in Table 1. We see that the difference is very small.

Table 2 shows the values of U_R/U obtained from (4.1) for $\alpha = 1$ and $1/2$, and the same selected values of $(C - U_R)/U_R$ as in Table 1. The values of U_R/U in Table 2 for the selected values of $(C - U)/U_R$ do not differ much from the corresponding values in Table 1 in the statically neutral case. With the same wind shear the neutrally stable modes propagate westward relative to the air at the middle level a little faster in the statically stable layer than in the statically neutral layer. However, the difference is small. The results shown in Tables 1 and 2 are shown in Fig. 6. The exact asymptotic values in Table 1 are shown by the dashed parabolas in Fig. 6 with the formula:

$$u^* = 2U/U_0 = 2(U/U_R)\ (\mathrm{L}/\mathrm{L}_0)^2 \tag{4.3}$$

5. Expansion near the advective model

The boundary value problem given by (2.10) and (2.11) can also be written non-dimensionally

$$(z' + c')\,(V'_{z'z'} - \varkappa^2 V') + U'_R \varkappa^2 V' = 0 \tag{5.1}$$

$$(z' + c')\, V'_{z'} - V' = 0 \quad \text{at } z' = \pm 1 \tag{5.2}$$

where $z' = z/h$, $c' = C/U$, $U'_R = U_R/U$, V' is non-dimensional and all these have the order of unity respectively. $\varkappa$ is a small quantity, say order of 10^{-1}. Hereafter for convenience let us drop the primes.

Let us expand V and c in terms of $\varkappa$,

$$V = V_0 + \varkappa V_1 + \varkappa^2 V_2 + \varkappa^3 V_3 + \varkappa^4 V_4 + \ldots \tag{5.3}$$

$$c = c_0 + \varkappa c_1 + \varkappa^2 c_2 + \varkappa^3 c_3 + \varkappa^4 c_4 + \ldots \tag{5.4}$$

Table 2. U_R/U *tabulated*

$\frac{C-U}{U_R}$	$\alpha=1$ Fig. 5	$\alpha=1$ Eq. (4.1)	$\alpha=1/2$ Fig. 5	$\alpha=1/2$ Eq. (4.1)
0	1.291	1.291	1.311	1.311
1/4	1.610	1.600	1.641	1.614
1/3	1.778	1.750	1.806	1.765
1/2	2.281	2.152	2.304	2.181

where $V_0, V_1, V_2, V_3, V_4, \ldots, c_0, c_1, c_2, c_3, c_4, \ldots$ are all of the order of unity.

Substituting (5.3) and (5.4) in (5.1) and (5.2) and separating each order of $\varkappa$, we have boundary-value problem for each order of $\varkappa$. Solving each boundary value problem, we obtain

$$O(\varkappa^0):\ V_0 = z + c_0 \quad (z + c_0 \neq 0) \tag{5.5}$$

$$O(\varkappa^1):\ V_1 = a_1(z + c_0) + c_1 \tag{5.6}$$

$$O(\varkappa^2):\ c_0 = \frac{U_R}{2} \pm \sqrt{\frac{U_R^2}{4} - \frac{1}{3}} \tag{5.7}$$

$$V_2 = \frac{1}{6}z^3 + \frac{1}{2}(c_0 - U_R)z^2 + $$

$$+ a_2(z + c_0) + c_0 + c_1 a_1 + c_2 - \frac{U_R}{2}$$

$$O(\varkappa^3):\ \text{for } 2c_0 \neq U_R, \quad c_1 = 0 \tag{5.8}$$

$$\text{and } V_3 = a_1\left[\frac{1}{6}z^3 + \frac{1}{2}(c_0 - U_R)z^2\right] + a_3(z + c_0)$$

$$+ (c_0 + c_2)a_1 + c_3 - \frac{U_R}{2}$$

$$O(\varkappa^4): c_2 = \frac{\frac{16}{45} + \frac{2}{3}U_R - U_R^2 - \frac{1}{4}U_R^3 + \frac{1}{4}U_R^4 \pm \left(1 - \frac{1}{3}U_R - \frac{1}{2}U_R^2 + \frac{1}{2}U_R^3\right)\sqrt{\left(\frac{U_R}{2}\right)^2 - \frac{1}{3}}}{\pm 2\sqrt{\left(\frac{U_R}{2}\right)^2 - \frac{1}{3}}} \tag{5.9}$$

Since we are only interested in c, the expression for V_4 is not written down. c in (5.4) is then

$$c = c_0 + \varkappa^2 c_2 + O(\varkappa^3) \tag{5.10}$$

where c_0 is given by (5.7) and c_2 by (5.9). Note that (5.7) is exactly (3.4).

First let us take an example to see the deviation from the advective model. Select $U_R = 1$ in (5.7) and (5.9). (5.10) becomes

$$c = \frac{1}{2} + \frac{\varkappa^2}{3} \pm (0.288675 - 0.038490\varkappa^2)\,i + O(\varkappa^3)$$

The percentage deviation for this case is

$$\varepsilon_{Cr} = \frac{2}{3}\varkappa^2$$

$$\varepsilon_{Ci} = \frac{1}{7.2}\varkappa^2$$

This example shows that medium-scale unstable waves (wave length shorter than that at $U_R = 2/\sqrt{3}$, but longer than that of maximum growth) with static stability, has a faster westward phase speed relative to the air at the middle level than those with vanishing static stability (advective case), and the growth rate is smaller.

Next, as an example, let us compare a value of c computed from (5.10), (5.7) and (5.9) with that obtained by substituting (2.13) and (2.14) in (2.16) (Tang, 1970) and solving numerically. The parameters of this example are $\alpha = 1/2$, $L/L_0 = \pi/2\varkappa = 3.003$, $\varkappa = 0.52308$. Strictly speaking, this value of $\varkappa$ is not small, but to the order of $\varkappa^3$ as in (5.10), $O(\varkappa^3)$ is small enough. The results are

this method: $c = 0.56840 \pm 0.29456\ i + O(\varkappa^3)$
numerical method: $c = 0.54022 \pm 0.30105\ i$

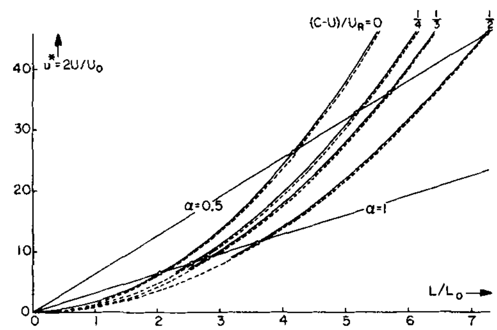

Fig. 6. Solid curves: the wave length of the neutrally stable ($\varkappa < \varkappa_s$)-mode for selected values of the phase velocity in Table 2. *Broken curves:* the corresponding asymptotic values from Table 1 and eq. (4.3).

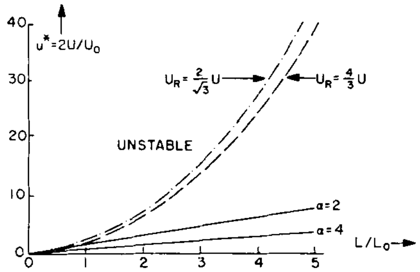

Fig. 7. Stability diagram.

Ignoring $O(\varkappa^3)$, we find the error in c_r is 5% and the error in c_i is 2%.

6. Concluding remarks

Let us consider Fig. 7. For waves to the left of those at $U_R = 2U/\sqrt{3}$, we find from (5.10) that the stronger the basic zonal wind shear the more the instability. The static stability is a stabilizing effect in this case. The waves with $4U/3 > U_R > 2U/\sqrt{3}$ have the same property as those with $U_R < 2U/\sqrt{3}$ in the sense that the stronger the basic zonal wind shear the more the instability. For waves to the right of those at $3U_R = 4U$, starting from the top of the diagram, the static stability destabilizes one of the long modes in the advective model (the long mode with critical level inside the layer). But the properties of instability on both sides of the curve $3U_R = 4U$ are different. To the left of the curve $3U_R = 4U$, the more the basic zonal wind shear the more the instability. To the right of the curve $3U_R = 4U$, it is different. We can show from the adiabatic equation (2.4), the thermal wind equation for the basic state (2.2), and the vorticity equation (2.6), that with no basic zonal wind shear the normal mode disturbances are Rossby waves $C = U_R$ which are neutrally stable. This is the case that the insentropic surfaces are horizontal surfaces ($\tau = 0$, $w = 0$, but $\sigma \neq 0$, the waves on the abscissa of Fig. 7). Therefore, if unstable waves exist at all to the right of the curve $3U_R = 4U$, there exists at least one maximum instability between infinite basic zonal wind shear and zero basic zonal wind shear. It turns out numerically that there are several maxima (Green, 1960; Hirota, 1968; Garcia & Norscini, 1970). Burger (1962) showed that in an upper unbounded model (Charney's model) there were an infinite number of maximum instabilities separated by $\alpha = 2, 4, 6, 8, 10, \ldots$ However, the recent numerical calculation of Garcia & Norscini (1970) in Eady's model ($\beta \neq 0$) shows that the neutral lines represented by the even number of α (positive) in the stability diagram (Fig. 7) were the tangents at the origin of the stability diagram.

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УСТОЙЧИВОСТЬ НЕПРЕРЫВНЫХ БАРОКЛИННЫХ МОДЕЛЕЙ С ГРАДИЕНТОМ ПЛАНЕТАРНОЙ ЗАВИХРЕННОСТИ

Проводится исследование на собственные значения линеаризованной непрерывной модели Иди с градиентом планетарной завихренности. Задача сводится к изучению вырожденного гипергеометрического уравнения с соответствующими граничными условиями. Проблема устойчивости в этой задаче связывается с поведением некоторой аналитической функции. Дано доказательство существования нейтральных волн. Путем стремления к нулю параметра статической устойчивости в модели Иди с градиентом планетарной завихренности получена соответствующая адвективная модель Фьортофта. Сравнение двух моделей показывает, что они очень похожи. Статическая устойчивость приводит к более низкому уровню бездивергентности, чем в чисто адвективном случае. Для изучения отклонений от адвективной модели используется техника разложений. Хотя этот метод полезен только для неустойчивых волн средних масштабов, он дает простые формулы для фазовой скорости волн и скорости роста их амплитуды.