On baroclinic instability of ultra-long waves[1]

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ABSTRACT

Baroclinic instability of ultra-long waves (wave numbers 1, 2, and 3) is studied with a 40-layer, geostrophic, hydrostatic, adiabatic model which is later modified to include Newtonian heating. The assumption of a meridionally-varying zonal current linear in pressure upon which are superimposed localized wavelike perturbations traveling in the west-east direction and static stability dependent only upon the inverse of pressure allows the reduction of the basic equations to a single second-order homogeneous ordinary differential equation in vertical pressure velocity having latitude as an implicit parameter. The "shooting method", a numerical search procedure to determine the eigenvalue, perturbation phase velocity, is employed under the simplifying boundary conditions that vertical pressure velocity is zero at the top and bottom of the atmosphere. This method requires one exact solution, and a procedure is given to determine it. The adiabatic and Newtonian models are each solved for a zonal wind profile based upon the annual average, 1963 500 mb zonal wind from 25° N to 80° N. It is found that perturbation instability, which is decreased by Newtonian heating, is maximum at 35° N which, in the adiabatic model, corresponds to an e-folding time for the third harmonic of six days. Here, the phase speed, equivalent for adiabatic and Newtonian flow and independent of wave number, is 6 m sec^{-1}. Near equivalence between the models is observed in the vertical structure, the waves tilting westward with decreasing pressure. Instantaneous energetics of the adiabatic and Newtonian models is studied by calculation of the vertical variation and total in a latitudinal strip 10° wide centered on the latitude of maximum instability of the normalized conversion from zonal to eddy available potential energy, $C(A_z, A_E)$, and normalized conversion from eddy available potential energy to eddy kinetic energy, $C(A_E, K_E)$. Here it is found, after summing over all pressures, that $C(A_z, A_E) > 0$ and $C(A_E, K_E) > 0$ for both models which agrees with the time-averaged energetics as observed in the atmosphere.

1. Introduction

On hemispherical weather charts atmospheric motion can be described as zonal (west-east) flow on which are superimposed disturbances. These disturbances represent the superposition of components having horizontal scales in a very wide range. On hemispheric 500 mb maps of mid-latitudinal westerly flow one clearly observes the existence of very long waves having wave lengths from 10 000–20 000 km. These waves are either stationary or slow-moving. Eliasen (1958) describes them as slowly-moving systems whose amplitude doubles in a few days, oscillating around certain preferred geographical positions. For the purpose of mathematically modeling the ultra-long waves one may consider a superposition of two distinct wave types—the stationary or quasi-stationary waves and the transient, traveling, or free waves. The former necessarily result from constant large-scale external forcing such as terrain or heat sources and sinks, while the latter may be subdivided into barotropic and baroclinic modes, the barotropic describing the vertically-averaged component and the baroclinic the deviation from the vertical average.

Study of the effects of large-scale external forces on the existence of quasi-stationary long waves was pioneered by Charney & Eliassen (1949), who showed the importance of large-

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scale mountains and friction, and Smagorinsky (1953), who studied the ultra-long waves created by large-scale heat sources and sinks and modified by friction. Later, long wave studies by Döös (1962), who included diabatic heating; Saltzman (1965), who incorporated mountains and heating; and Derome (1968), who investigated the effect of lower boundary topography, distribution of heat sources and sinks, and friction on the maintenance of the time-averaged standing eddies, augmented these works.

Recent studies of transient ultra-long waves and their representation by a barotropic model include those of Deland (1964, 1965) and Eliasen & Machenhauer (1965, 1969). In Deland's 1965 study, an amplification of his 1964 investigation, he makes a comparison between wave speeds calculated from daily surface-spherical harmonic expansions of 500 mb height in the Northern Hemisphere and theoretical nondivergent Rossby-Haurwitz wave speeds. It is observed that there exists a systematic difference between these wave speeds, i.e., all of the observed waves show greater eastward, or less rapid westward, motion than the Rossby-Haurwitz predicted wave speeds. A similar study over the Northern Hemisphere by Eliasen & Machenhauer (1965) uses the spherical harmonic expansions of 500- and 1 000-mb height to obtain a representation of the nondivergent part of the large-scale horizontal motion in terms of the spherical-harmonic components of the stream function. Mean values of the velocity of propagation obtained from the stream field for the different harmonic components are then compared with the velocities determined by the Rossby effect modified by weak divergence and found nearly in accordance. In Eliasen & Machenhauer (1969) the large-scale waves are represented by spherical harmonic components of the height field for the whole earth as well as for each of the two hemispheres and no attempt is made to compute stream function components. Velocities of propagation and amplitudes of fluctuations of the very long waves are then, as in their earlier work, compared with the Rossby effect combined with a divergence effect and essential agreement realized. These studies of Deland, Eliasen, and Machenhauer thus conclude that the transient part of the ultra-long waves to some extent can be described by a divergent barotropic model.

In attempting to explain characteristics of the very long transient waves in the baroclinic mode through the use of a mathematical model, both the analytical and numerical approaches have been applied. Burger (1958) derived through the vehicle of scale analysis a quasi-stationary vorticity equation in the very long wave regime, an expression derivable from the geostrophic relations. This effectively threw doubt on the simplified vorticity equation from which most earlier baroclinic theories began. Phillips (1963) later amplified Burger's conclusions in a review of the general problem of baroclinic instability within the framework of geostrophic motion. Welander (1961) found analytically explicit solutions for only neutral traveling waves of very long wave length in a model in which both wind speed and density were linear in pressure. His solution, worked out for the beta-plane approximation, consisted of slowly-moving waves, independent of wave number.

Another analytical attack on the internal dynamics of ultra-long baroclinic transient waves was launched by Wiin-Nielsen (1961) which might be considered an extension of Welander's (1961) study containing more simplifications but no restriction to neutral waves. Utilizing Burger's (1958) quasi-stationary vorticity equation, he investigated the stability and structure of the waves. In the two-parameter model he showed that the stationarity of the vorticity equation (neglect of $\partial\zeta/\partial t$) acts as a filter to eliminate Rossby waves from the solution of the linearized equations, while the remaining waves move with a speed independent of wave number. Expanding to the three-parameter model, the smallest resolution allowing for the possibility of unstable solutions, he found only slowly-moving waves possible and the appearance of unstable waves for sufficiently large vertical wind shears.

The studies by Miles (1964, 1965) are both analytical. In the earlier study, for adiabatic, nonviscous motion of a perfect gas with small Rossby number (Ro) and large Richardson number (Ri), the equations are reduced to a singular Sturm-Liouville problem and approximations deduced for small and large wave numbers. The latitude (φ) appears as an implicit parameter. Much attention is devoted to the singular Sturm-Liouville problem, but no physical interpretations of unstable waves such as their structure or energy conversion process are offered. In his later investigation, Miles formulated the eigenvalue problem governing baroclinic disturbances relative to zonal flow of waves having length comparable to the circumference of the earth. Several theorems governing the existence of such disturbances were deduced, but calculations of the complex phase velocity were made in only two limiting cases: $\beta \to 0$, ∞ where $\beta = \mathrm{Ro\ Ri} \cot \varphi$.

Most recently Hirota (1968), using the quasi-geostrophic equations, investigated numerically the stability of a zonal current and characteristics of wave perturbations by a finite difference approximation of the linearized perturbation equations applied to a multi-layer model. He found three types of baroclinic transient wave solution: (1) a pair of amplifying and decaying waves, (2) neutral waves without steering levels (levels where the basic zonal current and wave speed are equivalent) in the basic current, and (3) neutral waves with a steering level. Thorough discussion is given to the nature of the unstable waves—their vertical structure and energy conversion process—and to the comparison between the characteristics from theory and those of the ultra-long waves observed in the atmosphere.

This study most nearly resembles Hirota's (1968). Like him, a numerical solution to the linearized perturbation equations for a wave perturbation on a zonal current in a multi-layer model is obtained and much effort given to the vertical structure and energy conversion process of the unstable, baroclinic transient waves. Unlike him, however, we assume geostrophy rather than quasigeostrophy and, besides investigating adiabatic flow, obtain complete solutions for the unstable waves modified by the addition of Newtonian heating. Detailed comparison between the two models is made with the intent of determining the effect of Newtonian heating on the wave stability, structure, and energetics.

2. The adiabatic model

We neglect mountains and friction, not that these are of no importance, but partly for convenience in solving the equations and partly because we desire to isolate the system's internal dynamical factors. For the ultra-long waves investigated, we consider zonal wave number n = 1, 2, 3, giving a horizontal scale of motion, $L \simeq 10^7$ m, the order of magnitude of the radius of the earth, $a = (2/\pi) \times 10^7$ m. The vertical scale of motion, H, approximates 10^4 m, while we assume the characteristic horizontal and vertical speeds, from observation, to be $V \simeq 15$ m sec^{-1} and $W \leqslant 0.1\ V$, respectively. For these scale values the equation for the vertical component of velocity can then be approximated by the hydrostatic relation, in which case it is convenient to use pressure instead of height as vertical coordinate.

For $\mathrm{Ro} = V/fL < 1$, where the Coriolis parameter $f = 2\ \Omega \sin \varphi \simeq 10^{-4}$ sec^{-1} for mid-latitudes and $\Omega = 7.3 \times 10^{-5}$ sec^{-1} is the earth's angular velocity, the horizontal pressure and Coriolis forces in the horizontal equations of motion nearly balance each other so that the geostrophic approximation is well satisfied as shown by Burger's (1958) and Miles' (1965) scale analyses. Further, the type of motion we consider is an example of geostrophic motion of type 2 (see Phillips, 1963) and, as shown by Phillips, use of the β-plane is invalid. In spherical coordinates the basic equations take the form

$$u = -\frac{1}{2\Omega a \sin \varphi} \frac{\partial \Phi}{\partial \varphi} \tag{2.1}$$

$$v = \frac{1}{2\Omega a \sin \varphi \cos \varphi} \frac{\partial \Phi}{\partial \lambda} \tag{2.2}$$

$$\frac{\partial \Phi}{\partial p} = -\alpha \tag{2.3}$$

$$\frac{\partial \omega}{\partial p} = -\frac{1}{a \cos \varphi}\left(\frac{\partial u}{\partial \lambda} + \frac{\partial v \cos \varphi}{\partial \varphi}\right) \tag{2.4}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial p}\right) + \frac{u}{a \cos \varphi} \frac{\partial}{\partial \lambda}\left(\frac{\partial \Phi}{\partial p}\right) + \frac{v}{a} \frac{\partial}{\partial \varphi}\left(\frac{\partial \Phi}{\partial p}\right) + \sigma \omega = 0 \tag{2.5}$$

where we assume the equation of state is the ideal gas law,

$$p\alpha = RT \tag{2.6}$$

Equations (2.1) and (2.2) express the geostrophic relations, (2.3) the hydrostatic equation, (2.4) the continuity equation and (2.5) the thermodynamic energy equation for adiabatic flow. The independent variables are time (t), latitude

(φ), longitude (λ), and pressure (p). The dependent variables are the velocity components (u, v, ω), where u increases eastward, v northward, and $\omega = dp/dt$ is the "vertical velocity"; geopotential $\Phi = gz$, where g is the acceleration of gravity and z is the height above mean sea level of isobaric surface p; and specific volume α. T represents the temperature and R is the gas constant for air. In eq. (2.5) σ, a measure of the static stability, is expressed as

$$\sigma = -\alpha \frac{\partial \ln \theta}{\partial p} \tag{2.7}$$

where θ is potential temperature. From Poisson's equation

$$\theta = T\left(\frac{p}{p_0}\right)^{-R/c_p} \tag{2.8}$$

where c_p is the specific heat capacity at constant pressure for air. Substituting (2.3), (2.6), and (2.8) into (2.7) we obtain

$$\sigma = \frac{\partial^2 \Phi}{\partial p^2} + \frac{c_v}{c_p}\frac{1}{p}\frac{\partial \Phi}{\partial p} \tag{2.9}$$

where $c_v = c_p - R$ is the specific heat capacity at constant volume for air.

Inserting (2.1) and (2.2) into (2.4) we find

$$\frac{\partial \omega}{\partial p} = \frac{1}{2\Omega a^2 \sin^2 \varphi}\frac{\partial \Phi}{\partial \lambda} \tag{2.10}$$

In solving (2.1), (2.2), (2.5), (2.9), and (2.10) we desire to linearize the equations by perturbation theory. Consider a steady, axially symmetric basic state having only zonal wind with perturbations superimposed upon it,

$$u = U(\varphi, p) + u'(t, \lambda, \varphi, p) \tag{2.11 a}$$

$$v = v'(t, \lambda, \varphi, p) \tag{2.11 b}$$

$$\omega = \omega'(t, \lambda, \varphi, p) \tag{2.11 c}$$

$$\Phi = \Phi_z(\varphi, p) + \Phi'(t, \lambda, \varphi, p) \tag{2.11 d}$$

$$\sigma = \sigma_z(\varphi, p) + \sigma'(t, \lambda, \varphi, p) \tag{2.11 e}$$

where the primed terms are perturbations. Because we work with spherical coordinates, it is convenient to define the basic state angular velocity,

$$\Lambda(\varphi, p) = \frac{U(\varphi, p)}{a \cos \varphi} \tag{2.12}$$

In determining the basic state solution we assume $U(\varphi, p)$ is given and express $\Phi_z(\varphi, p)$ and $\sigma_z(\varphi, p)$ in terms of it. From (2.11) into (2.1) and (2.9), we find

$$\frac{\partial \Phi_z}{\partial \varphi} = -2\Omega a \sin \varphi U = -2\Omega a^2 \sin \varphi \cos \varphi \Lambda \tag{2.13}$$

$$\sigma_z = \frac{\partial^2 \Phi_z}{\partial p^2} + \frac{c_v}{c_p}\frac{1}{p}\frac{\partial \Phi_z}{\partial p} \tag{2,14}$$

Differentiating (2.13) with respect to pressure gives

$$\frac{\partial}{\partial \varphi}\left(\frac{\partial \Phi_z}{\partial p}\right) = -2\Omega a^2 \sin \varphi \cos \varphi \frac{\partial \Lambda}{\partial p} \tag{2.15}$$

while by integrating (2.13) with respect to latitude we get

$$\Phi_z(\varphi, p) = \Phi_{zL}(p) - 2\Omega a \int_{\varphi_L}^{\varphi} U(\varphi', p) \sin \varphi' d\varphi' \tag{2.16}$$

where φ_L is the lower latitude limit, 25°N, φ' the integration variable, and $\Phi_{zL} = \Phi_z$ at φ_L. Inserting (2.16) into (2.14) completes the basic state solution,

$$\sigma_z = \sigma_{zL} - 2\Omega a \int_{\varphi_L}^{\varphi} \sin \varphi' \left(\frac{\partial^2 U}{\partial p^2} + \frac{c_v}{c_p}\frac{1}{p}\frac{\partial U}{\partial p}\right) d\varphi' \tag{2.17}$$

where

$$\sigma_{zL} = \sigma_z \text{ at } \varphi_L$$

To obtain perturbation equations we substitute (2.11) and (2.12) into (2.5) and (2.10), giving

$$\frac{\partial}{\partial t}\left(\frac{\partial \Phi'}{\partial p}\right) + \Lambda \frac{\partial}{\partial \lambda}\left(\frac{\partial \Phi'}{\partial p}\right) + \frac{v'}{a}\frac{\partial}{\partial \varphi}\left(\frac{\partial \Phi_z}{\partial p}\right) + \sigma_z \omega' = 0 \tag{2.18}$$

$$\frac{\partial \omega'}{\partial p} = \frac{1}{2\Omega a^2 \sin^2 \varphi}\frac{\partial \Phi'}{\partial \lambda} \tag{2.19}$$

After re-expressing the third term of (2.18) using (2.2) and (2.15), (2.18) becomes,

$$\frac{\partial}{\partial t}\left(\frac{\partial \Phi'}{\partial p}\right) + \Lambda \frac{\partial}{\partial \lambda}\left(\frac{\partial \Phi'}{\partial p}\right) - \frac{\partial \Lambda}{\partial p}\frac{\partial \Phi'}{\partial \lambda} + \sigma_z \omega' = 0 \tag{2.20}$$

In solving (2.19) and (2.20) we consider the solutions as local, a solution at one latitude being uncoupled with respect to a solution at another latitude. Using the method of the separation of variables in this localized sense, we introduce perturbations of the form

$$(\quad)' = (\hat{\ }) e^{in(\lambda - ct)} \tag{2.21}$$

where $(\hat{\ })$ is assumed to be a function of p only and c, the phase velocity of the perturbations, is in general complex, $c = c_r + i\, c_i$. We thus assume that the perturbations are wavelike disturbances, periodic in the east-west direction and time, and that the flow can be represented by a single component of a Fourier series in λ. For $c_i > 0$ the rate of growth is $n\, c_i$, the motion being unstable through the growth factor, $e^{nc_i t}$, multiplying all the dependent variables. For $c_i = 0$, we term the flow neutral and for $c_i < 0$, stable.

Equation (2.21) into (2.19) and (2.20) gives

$$\frac{d\hat{\omega}}{dp} = \frac{in}{2\Omega a^2 \sin^2 \varphi} \hat{\Phi} \tag{2.22}$$

$$in\left[(\Lambda - c)\frac{d\hat{\Phi}}{dp} - \frac{d\Lambda}{dp}\hat{\Phi}\right] + \sigma_z \hat{\omega} = 0 \tag{2.23}$$

which, after substituting (2.22) into (2.23) reduces to the single equation,

$$(\Lambda - c)\frac{d^2\hat{\omega}}{dp^2} - \frac{d\Lambda}{dp}\frac{d\hat{\omega}}{dp} + \frac{\sigma_z}{2\Omega a^2 \sin^2\varphi}\hat{\omega} = 0 \tag{2.24}$$

We specify the boundary conditions for $\hat{\omega}$ to be $\hat{\omega} = 0$ at $p = 0$ and $p = p_0 = 1\,000$ mb. The lower condition is obtained by expanding ω in the z-coordinate system, removing the horizontal pressure advection through the geostrophic approximation, and assuming a negligible pressure tendency to arrive at

$$\omega \simeq w\frac{\partial p}{\partial z}$$

Assumption of level terrain at pressure p_0 implies $w(p = p_0) = 0$ from which we get the desired boundary condition.

In seeking a numerical solution we enhance the accuracy of our calculations by eliminating in (2.24) the first derivative, recasting the eigenvalue problem incanonical form. This is achieved by letting $\hat{\omega} = \sqrt{M}\, W$ for $M = \Lambda - c$ and substituting in (2.24) to get, after dividing by $M^{3/2}$

$$\frac{d^2W}{dp^2} + \left[\frac{1}{2}\frac{1}{M}\frac{d^2M}{dp^2} - \frac{3}{4}\frac{1}{M^2}\left(\frac{dM}{dp}\right)^2 + \frac{\sigma_z}{2\Omega a^2 \sin^2\varphi}\frac{1}{M}\right] W = 0 \tag{2.25}$$

It is convenient to nondimensionalize pressure and phase velocity by

$$p_* = \frac{p}{p_0},\; c_* = \frac{c}{|\Lambda_0|} \tag{2.26}$$

where $|\Lambda_0| = s\Lambda_0$, implying that $s = +1$ for $\Lambda_0 > 0$ and $s = -1$ for $\Lambda_0 < 0$.

We assume a linear variation of U with pressure at each latitude,

$$U(\varphi, p_*) = U_0(\varphi)\,(1 - p_*) \text{ or}$$
$$\Lambda(\varphi, p_*) = \Lambda_0(\varphi)\,(1 - p_*) \tag{2.27}$$

the validity of which is shown by Welander (1961).

From observations σ_z is proportional to p_*^k where k varies between -1 and -2. For mathematical expediency we choose the former,

$$\sigma_{zL} = \frac{\sigma_0}{p_*} \tag{2.28}$$

We use throughout this study $\sigma_0 = 1$ in the MTS system, this being based upon results obtained by Peixoto (1960).

Now substituting (2.17), (2.26), (2.27), and (2.28) into (2.25) gives us, for $M = \Lambda_0(1 - p_* - sc_*)$

$$\frac{d^2W}{dp_*^2} - \left[\frac{1}{p_*[p_* - (1 - sc_*)]}\left(\frac{\sigma_0 p_0^2}{2\Omega\Lambda_0 a^2 \sin^2\varphi} + \frac{\varepsilon\cos\varphi}{U_0 \sin^2\varphi}\int_{\varphi_L}^{\varphi} U_0 \sin\varphi'\, d\varphi'\right) + \frac{3}{4}\frac{1}{[p_* - (1 - sc_*)]^2}\right] W = 0 \tag{2.29}$$

where $\varepsilon = c_v/c_p$.

The boundary conditions are $W = 0$ at $p_* = 0$, 1. We avoid regularity difficulties at the equator by confining the range of latitudes to 25–80° N. Further, we examine only cases having $U_0 \neq 0$, precluding a similar type of singularity. Finally, investigation of (2.29) reveals the existence of singularities at $p_* = 0$ and $p_* = 1 - sc_*$. The former is avoided by virtue of the fact that it coincides with the upper boundary where (2.29) is not applied. The latter will only occur when c_* is real. For the latitudes considered only one yielded real c_*;

Latitude	U_0	c_*
25°	14 m sec^{-1}	$(-.3\,521\,928,0)$

Here, $s = +1$. The eigenvalue $c_* < 0$ implies that singularity can only occur for $p_* > 1$. Thus, all singularities are avoided in the numerical solution of (2.29). For given φ and $U_0(\varphi')$, $\varphi_L \leqslant \varphi' \leqslant \varphi$, we solve for c_* using the "shooting method". This is a numerical search procedure in the complex plane which requires an exact solution c_{*e} for some U_{0e} at latitude φ_e to start it and provide a test condition for the search procedure. We now desire to find this exact solution.

Inserting (2.17), (2.26), (2.27), and (2.28) into (2.24) we obtain

$$[p_* - (1 - sc_*)]\frac{d^2\hat{\omega}}{dp_*^2} - \frac{d\hat{\omega}}{dp_*} - \frac{1}{p_*}\left[\frac{\sigma_0 p_0^2}{2\Omega\Lambda_0 a^2 \sin^2\varphi} + \frac{\varepsilon\cos\varphi}{U_0\sin^2\varphi}\int_{\varphi_L}^{\varphi} U_0 \sin\varphi'\,d\varphi'\right]\hat{\omega} = 0 \tag{2.30}$$

Transforming to a new independent variable, $\xi = p_*/1 - sc_*$, (2.30) becomes

$$\xi(\xi-1)\frac{d^2\hat{\omega}}{d\xi^2} - \xi\frac{d\hat{\omega}}{d\xi} - \left[\frac{\sigma_0 p_0^2}{2\Omega\Lambda_0 a^2\sin^2\varphi} + \frac{\varepsilon\cos\varphi}{U_0\sin^2\varphi}\int_{\varphi_L}^{\varphi} U_0\sin\varphi'\,d\varphi'\right]\hat{\omega} = 0 \tag{2.31}$$

under the boundary conditions $\hat{\omega} = 0$ at $\xi = 0$, $1/1 - sc_*$. Equation (2.31) is Gauss's differential equation, known also as the hypergeometric equation, a special case of the generalized hypergeometric equation. The standard form of Gauss's equation is

$$\xi(\xi-1)\frac{d^2\hat{\omega}}{d\xi^2} + [(\alpha+\beta+1)\xi + \gamma]\frac{d\hat{\omega}}{d\xi} + \alpha\beta\hat{\omega} = 0 \tag{2.32}$$

Equating (2.31) and (2.32), and setting $\varphi = \varphi_e = \varphi_L$,

$$\gamma = 0, \quad \alpha + \beta + 1 = -1 \tag{2.33}$$

$$\alpha\beta = -q \tag{2.34}$$

where

$$q = \frac{\sigma_0 p_0^2}{2\Omega\Lambda_{0e} a^2 \sin^2\varphi_e} \tag{2.35}$$

Combining (2.33) and (2.34),

$$\alpha = -1 + r, \quad \beta = -1 - r \tag{2.36}$$

where

$$r = \sqrt{1+q} \tag{2.37}$$

From Kamke (1961), page 467, for $\gamma = 0$ and α and β as determined in (2.36), the solution to (2.32) is

$$\hat{\omega} = C_1\hat{\omega}_1 + C_2\hat{\omega}_2 \tag{2.38}$$

where

$$\hat{\omega}_1 = \xi\,{}_2F_1(r, -r; 2; \xi) \tag{2.39a}$$

$$\hat{\omega}_2 = (1 - r^2)\xi \ln \xi\,{}_2F_1(r, -r; 2; \xi) + \text{power series in } \xi \text{ starting with "1"} \tag{2.39b}$$

Here, Gauss's function is

$${}_2F_1(a, b; d; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n x^n}{(d)_n n!} \tag{2.40}$$

where $(a)_n = a(a+1)\,(a+2)\ldots(a+n-1)$ and a, b, d, and x can be real or complex, but d cannot be a negative integer.

From (2.39a) and (2.39b), for the upper boundary condition at $\xi = 0$, $\hat{\omega}_1 = 0$ and $\hat{\omega}_2 = 1$. Since $\hat{\omega} = 0$ here, C_2 must be 0. To satisfy the lower boundary condition we seek $\xi = \xi_0$ such that

$${}_2F_1(r, -r; 2; \xi_0) = 0 \tag{2.41}$$

where c_* is determined from $\xi_0 = 1/1 - sc_*$. Thus,

$$c_* = \frac{\xi_0 - 1}{s\xi_0} \tag{2.42}$$

From Kamke (1961), for integer n (not zonal wave number),

$${}_2F_1(n+a, -n;\ c; \xi) = \frac{\xi^{1-c}(1-\xi)^{c-a}}{c(c+1)\ldots(c+n-1)}\frac{d^n}{d\xi^n}[\xi^{c+n-1}(1-\xi)^{a+n-c}] \tag{2.43}$$

Combining (2.41) and (2.43),

$$n = r, \quad a = 0, \quad c = 2 \tag{2.44}$$

and calling upon (2.37), (2.35), and (2.12), we have

$$U_{0e} = \frac{\sigma_0 p_0^2 \cos \varphi_e}{2\Omega a \sin^2 \varphi_e (n^2 - 1)} \tag{2.45}$$

Given $\varphi_e = \varphi_L$, this varies only with arbitrary integer $n > 1$. Combining (2.39a), (2.43), and (2.44) we now get

$$\hat{\omega} = C_1 \frac{(1-\xi)^2}{(n+1)!} \frac{d^n}{d\xi^n} [\xi^{n+1}(1-\xi)^{n-2}] \tag{2.46}$$

from which we can easily determine c_{*e}. Let us consider several special cases.

(1) $n = 2$

Equation (2.46) simplifies to $\hat{\omega} \propto \xi(1-\xi)^2$. At the lower boundary condition, $\hat{\omega} = 0$ obtains from $\xi_0 = 0$, 1 which substituted into (2.42) and discarding the meaningless solution at $\xi_0 = 0$ gives $c_{*e} = 0$.

(2) $n = 3$

Equation (2.46) simplifies to

$$\hat{\omega} \propto \xi(1-\xi)^2 (1 - \tfrac{5}{2}\xi).$$

To satisfy the lower boundary condition,

$\xi_0 = 0$, 1, 2/5 which gives $c_{*e} = 0$, -1.5.

(3) $n = 4$

Equation (2.46) simplifies to $\hat{\omega} \propto \xi(1-\xi)^2 (7\xi^2 - 6\xi + 1)$. The lower boundary condition is satisfied when $\xi_0 = 0$, 1, $3/7 \pm \sqrt{2}/7$ which gives $c_{*e} = 0$, $-2 \pm \sqrt{2}$.

Either of these is a satisfactory exact solution for $\varphi = \varphi_L$.

3. The Newtonian heating model

Here we retain all of the assumptions made previously except the adiabatic assumption. Thus, the thermodynamic energy equation, (2.5), now becomes

$$\frac{\partial}{\partial t}\left(\frac{\partial \Phi}{\partial p}\right) + \frac{u}{a \cos \varphi} \frac{\partial}{\partial \lambda}\left(\frac{\partial \Phi}{\partial p}\right) + \frac{v}{a} \frac{\partial}{\partial \varphi}\left(\frac{\partial \Phi}{\partial p}\right) + \sigma\omega = -\frac{R}{c_p}\frac{1}{p} Q \tag{3.1}$$

where Q is the diabatic heating per unit time and unit mass. We consider the diabatic heating to be Newtonian in form so that

$$Q = c_p q (T_E - T) \tag{3.2}$$

where q is the heating or cooling coefficient and $T_E(\varphi, p)$ is the equilibrium temperature toward which the diabatic processes are driving the atmosphere. This is the hypothetical temperature which would be established in the absence of large-scale motion but in the presence of radiation and small-scale convection. A value of 0.4×10^{-6} sec^{-1} for q was obtained by Wiin-Nielsen, Vernekar & Yang (1967) using the calculated value of the generation of eddy available potential energy from observations. This constant value for q will be used throughout this study. The Newtonian heating assumption, eq. (3.2), says that the atmosphere is heated in regions where the temperature is below the equilibrium temperature, while it is cooled where the temperature exceeds the equilibrium temperature. It certainly represents an oversimplification of the heating in the atmosphere, especially so because of the role of condensation in large-scale flow patterns as studied by Fjørtoft (1959). However, eq. (3.2) does agree with the average behavior of the atmosphere, and it is on this basis that we justify its use.

Expressing temperature as a steady, axially symmetric basic state upon which is superimposed a perturbation, we have

$$T = T_z(\varphi, p) + T'(t, \lambda, \varphi, p) \tag{3.3}$$

Inserting (2.11a)–(2.11e), (3.2), and (3.3) into (3.1) we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial \Phi_z}{\partial p} + \frac{\partial \Phi'}{\partial p}\right) + \frac{(U+u')}{a \cos \varphi} \frac{\partial}{\partial \lambda}\left(\frac{\partial \Phi_z}{\partial p} + \frac{\partial \Phi'}{\partial p}\right) + \frac{v'}{a} \frac{\partial}{\partial \varphi}\left(\frac{\partial \Phi_z}{\partial p} + \frac{\partial \Phi'}{\partial p}\right) + (\sigma_z + \sigma')\omega'$$

$$= \frac{-Rq}{p}(T_E - T_z - T') \tag{3.4}$$

which gives the basic state expression

$$\frac{-Rq}{p}(T_E - T_z) = 0 \tag{3.5}$$

In seeking a revised perturbation energy equation we first express T' as a function of $\partial\Phi'/\partial p$ through the ideal gas law and hydrostatic equation, and then substitute into (3.4), to get

$$\frac{\partial}{\partial t}\left(\frac{\partial\Phi'}{\partial p}\right) + \Lambda\frac{\partial}{\partial\lambda}\left(\frac{\partial\Phi'}{\partial p}\right) - \frac{\partial\Lambda}{\partial p}\frac{\partial\Phi'}{\partial\lambda} + \sigma_z\omega' = -q\frac{\partial\Phi'}{\partial p} \tag{3.6}$$

Introducing perturbations of the form (2.21) into (3.6) we arrive at

$$in\left[(\Lambda - c)\frac{d\hat{\Phi}}{dp} - \frac{d\Lambda}{dp}\hat{\Phi}\right] + \sigma_z\hat{\omega} = -q\frac{d\hat{\Phi}}{dp}$$

Eliminating $\hat{\Phi}$ by (2.22), it follows that

$$\left[(\Lambda - c) + \frac{q}{in}\right]\frac{d^2\hat{\omega}}{dp^2} - \frac{d\Lambda}{dp}\frac{d\hat{\omega}}{dp} + \frac{\sigma_z}{2\Omega a^2\sin^2\varphi}\hat{\omega} = 0$$

which is easily transformed, after applying (2.17), (2.26), (2.27), and (2.28) to

$$\left[p_* - (1 - sc_{*N}) + \frac{q}{n\Lambda_0}i\right]\frac{d^2\hat{\omega}}{dp_*^2} - \frac{d\hat{\omega}}{dp_*} - \frac{1}{p_*}\left[\frac{\sigma_0 p_0^2}{2\Omega\Lambda_0 a^2\sin\varphi} + \frac{\varepsilon\cos\varphi}{U_0\sin^2\varphi}\int_{\varphi_L}^{\varphi}U_0\sin\varphi'\,d\varphi'\right]\hat{\omega} = 0 \tag{3.7}$$

where c_{*N} is the nondimensional phase velocity for the Newtonian heating model. In seeking to determine c_{*N} it is very useful to compare (3.7) with its adiabatic equivalent

$$[p_* - (1 - sc_{*a})]\frac{d^2\hat{\omega}}{dp_*^2} - \frac{d\hat{\omega}}{dp_*} - \frac{1}{p_*}\left[\frac{\sigma_0 p_0^2}{2\Omega\Lambda_0 a^2\sin^2\varphi} + \frac{\varepsilon\cos\varphi}{U_0\sin^2\varphi}\int_{\varphi_L}^{\varphi}U_0\sin\varphi'\,d\varphi'\right]\hat{\omega} = 0 \tag{2.30}$$

where c_{*a} is the nondimensional phase velocity for the adiabatic model. We note the equivalence of the two equations in all terms but the coefficient of $d^2\hat{\omega}/dp_*^2$ which, for the Newtonian heating model, may be expressed as $\{p_* - [1 - s(c_{*N} + (q/sn\,\Lambda_0)i)]\}$ as compared with $[p_* - (1 - sc_{*a})]$ for the adiabatic model. It is therefore clear that, for given φ, n, and Λ_0, a solution c_{*a} will suffice to calculate c_{*N} from the equality,

$$c_{*N} + \frac{q}{sn\Lambda_0}i = c_{*a}$$

or applying (2.12),

$$c_{*N} = c_{*a} - \frac{qa\cos\varphi}{snU_0}i \tag{3.8}$$

This precludes the necessity to apply the "shooting method" for the Newtonian heating model once the adiabatic phase velocities have been obtained. We note that, whereas c_{*a} is independent of zonal wave number n, c_{*N} is dependent upon it in the sense that increasing n tends to cause perturbations in both westerlies and easterlies to become more unstable, as s $U_0 > 0$ in either case. In contrasting the two models, one can show that complex values of phase velocity necessarily occur in complex conjugate pairs for the adiabatic model but not for the Newtonian heating model.

4. Method of solution

For given φ and $U_0(\varphi')$, $\varphi_L \leqslant \varphi' \leqslant \varphi$, we desire to solve eq. (2.29) under the boundary conditions, $W = 0$ at $p_* = 0, 1$, for c_* by a numerical search procedure in the complex plane. This search procedure is referred to as both the "shooting method" and the "Biblical method", the latter deriving from the Scriptural text, Matthew 7:7, "Seek and ye shall find", for the process is based upon an iterative algorithm whereby a crude eigenvalue is continuously changed until the boundary conditions are "satisfied". After the finite difference equivalent of (2.29) is obtained and the atmosphere divided into N levels—$N = 40$ was chosen for all eigenvalue calculations—$W = 0$ is specified at the upper boundary and an arbitrary value at the first level below the upper boundary. Successive values of W are then calculated from the finite difference equation at lower levels until W at $p_* = 1$ is determined. This absolute value is then normalized by dividing by the maximum absolute value of W. We notate this normalized W at the lower boundary by G. In the absence of computer round-off and truncation errors, $G = 0$ for each eigenvalue. This not being the situation, however, a small number, δ, must be calculated to serve as an upper limit for the lower boundary condition in eigenvalue calculations. It is in the determination of δ that an exact solution, c_{*e} for some U_{0e} at latitude φ_e, to (2.29) is required. To eliminate the integral in (2.29) φ_e is chosen as φ_L, while U_{0e} from (2.45) and c_{*e} are determined by selection of integer $n > 1$. After some numerical tests it was found that, for $n = 4$ and $c_{*e} = -2 + \sqrt{2}$, $\delta = 0.560 \times 10^{-3}$ gave the optimum convergence criterion.

5. Results

Having assumed a linear variation of basic zonal velocity with pressure as shown in (2.27), we obtain the meridional distribution of U_0 by doubling values of the zonal average of U at 500 mb from data taken during 1963. We consider only the annual average as shown in Fig. 1.

Fig. 2 displays instability as measured by c_i as a function of latitude for adiabatic and Newtonian flow. Eq. (2.29) reveals the former to be independent of wave number, while eq. (3.8) shows this not to be the case in the Newtonian model. Maximum instability is observed to occur at 35° N with a monotonic decrease to the north and south. In the Newtonian model, for $q > 0$ and $s\, U_0 > 0$, it is clear from (3.8) that $(c_{*N})_i < (c_{*a})_i$, indicating that heating tends to stabilize the flow. The degree of stabilization is directly proportional to q, the heating or cooling coefficient, and $\cos \varphi$, while inversely proportional to wave number and amplitude of basic zonal velocity. The maximum stabilizing effect of heating takes place when the degree of instability is maximum, while the difference between

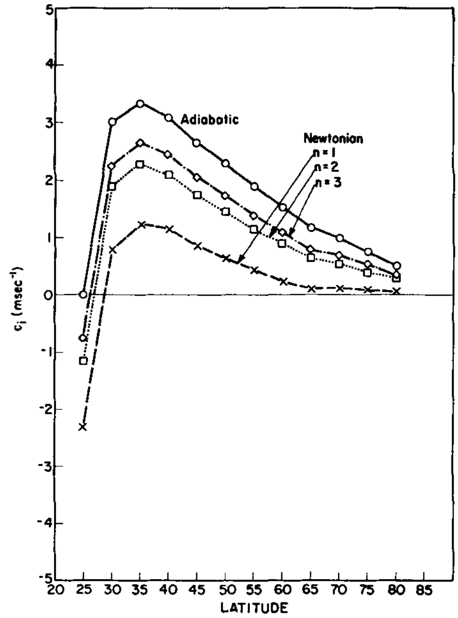

Fig. 2. Imaginary component of phase velocity as a function of latitude for adiabatic model; and Newtonian model at $n = 1$, 2, and 3.

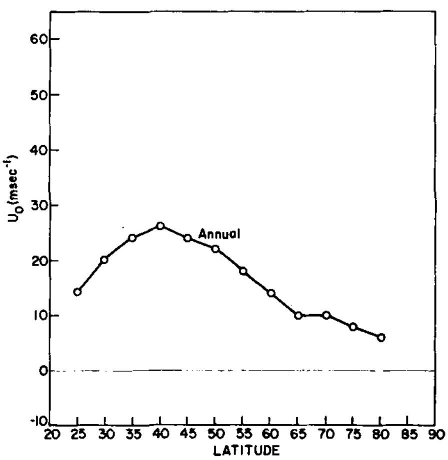

Fig. 1. Zonal wind at $p = 0$ as a function of latitude as calculated for a linear profile by doubling 500 mb data of the annual average, 1963.

adiabatic and Newtonian instability diminishes with increasing latitude. The similar shapes of the waves indicate that variation of instability with latitude in the adiabatic and Newtonian models is very much alike. As wave number increases, the Newtonian curves approach the limiting adiabatic curve, crowding closer together. Further, whereas the minimum c_i in adiabatic flow was non-negative, for Newtonian heating c_i can be negative, indicating damping or stable flow at particular latitudes and wave lengths. Maximum damping occurs, of course, for $n = 1$.

Fig. 3 displays another measure of instability, e-folding time (T_e)—the time it takes for the amplitude of the perturbations to increase by a factor of e—as a function of latitude for the adiabatic model. From (2.21) it follows that $T_e = 1/n\, c_i$, and thus that, for either model, there must be wave length dependence. Fig. 3 reveals a minimum e-folding time, or maximum instability, for the third harmonic of about six days in adiabatic flow. Larger e-folding times would necessarily follow for Newtonian flow.

Fig. 4 displays the relationship of phase speed, defined as the real part of the perturbation phase velocity, to latitude. The curve repre-

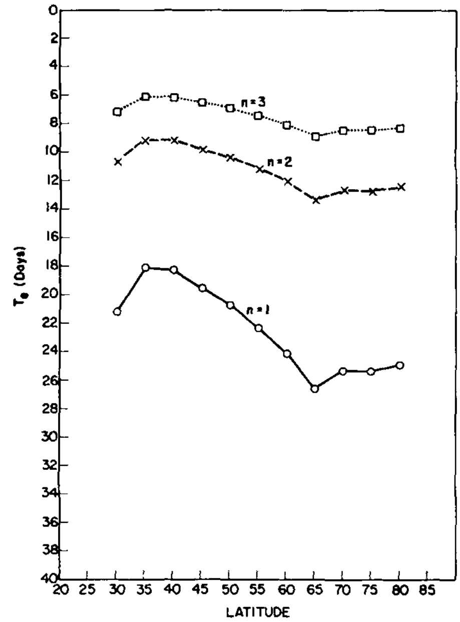

Fig. 3. e-Folding time as a function of latitude for adiabatic model at $n = 1$, 2, and 3.

sents all wave lengths of the ultra-long waves, both for adiabatic flow and Newtonian heating. This is apparent when we recall that c_* for adiabatic flow is independent of wave number and, from (3.8), that the real parts of c_{*N} and c_{*a} are identical. Only their imaginary parts differ by a factor dependent upon wave number. Comparing Figs. 4 and 1 we note a striking resemblance between U_0 vs. φ and c_r vs. φ. The maximum value of phase speed is 8.2 m sec^{-1} at 40° N, while, for the most unstable waves, the phase speed is 6.3 m sec^{-1} at 35° N. These values are in good agreement with those obtained in the empirical study by Bradley & Wiin-Nielsen (1968), where the phase speeds ranged from 5–15 m sec^{-1}. As can be seen, all the waves move eastward relative to the earth with the single exception of the retrogressive wave at 25° N.

If our model is at all an approximation to reality, the unstable waves which are solutions to the linearized equations should have a vertical structure which is in a gross sense comparable to the structure of ultra-long waves in the atmosphere. In particular we investigate the vertical structure of the most unstable waves at 35° N. The real component of any perturbation parameter as in (2.21) normalized at a given φ with respect to the maximum absolute value over all pressures may be expressed, at time $t = 0$,

$$\mathrm{Re}\,[(\ )'_N] = A \cos (n\lambda - \delta)$$

where subscript "N" indicates normalization,

$$A = \sqrt{a^2 + b^2}$$

and

$$\delta = \tan^{-1}\left(\frac{-b}{a}\right) \text{ for } a = \mathrm{Re}\,[(\ \hat{}\ )_N] \text{ and}$$

$$b = \mathrm{Im}\,[(\ \hat{}\ )_N]$$

One of the striking results of this study is the pronounced similarity between the adiabatic and Newtonian models in the structure of the most unstable waves, especially in regard to the vertical variation of the normalized amplitude, A, and approximately for the vertical variation of phase angle, δ.

Figs. 5–10 portray the vertical variation of the phase angle and amplitude of normalized geopotential, vertical velocity, and temperature, respectively, for the most unstable waves in the adiabatic and Newtonian models. Variation with

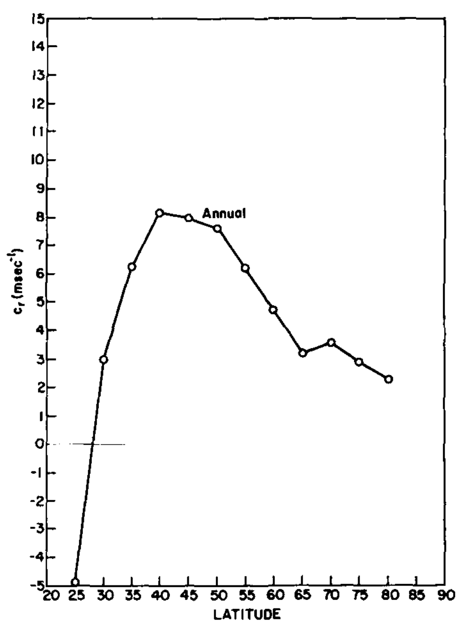

Fig. 4. Phase speed as a function of latitude for adiabatic and Newtonian models.

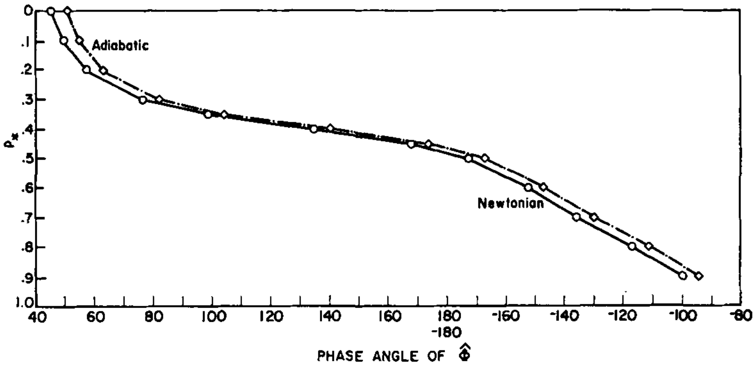

Fig. 5. Phase angle of perturbation geopotential of most unstable wave as a function of pressure for adiabatic and Newtonian models.

wave number is negligible. Further, amplitude deviations between the models are effectively infinitesinal, leaving us with just one curve on the amplitude figures and two on the phase angle graphs, where the differences between the adiabatic and Newtonian models are large enough to be significant.

In Fig. 5, depicting phase angle of geopotential, the most striking feature in the east-west tilt with decreasing pressure, which indicates northward sensible heat transport, is the maximum rate of change for $0.3 < p_* < 0.5$, with less rapid change occurring in the lower and upper atmosphere. The range of δ is approximately 220°. This compares very favorably with an empirical study on the transient part of the ultra-long waves undertaken by Bradley & Wiin-Nielsen (1968) in which the tilt of surface harmonics of both forced and free modes of geopotential was calculated to be westward from 850 to 500 mb and approximately vertical from 500 to 200 mb. Extrapolating their values in the 850 to 500 mb regime for the free mode to the rest of the atmosphere, we find the range of δ to be 233° for $n = 1$ and 2 and 156° for $n = 3$, in good agreement with our computed value of 220°. Phase angle for Newtonian heating is about 5° west of that for adiabatic flow. Fig. 6 depicts the vertical variation of normalized geopotential amplitude. Here we note a maximum in the wave's intensity at the top of the atmosphere, a secondary maximum at the ground, and a minimum around $p_* = 0.4$. We thus see in comparing Figs. 5 and 6, that the wave exhibits amplitude maxima where the tilt is least, and an amplitude minimum where the phase angle change in the vertical is greatest.

In Fig. 7 we plot phase angle of vertical velocity, $\hat{\omega}$, vs. pressure. Here as with $\hat{\Phi}$, both the

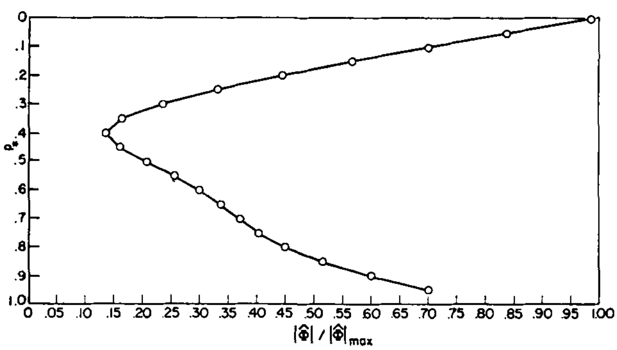

Fig. 6. Normalized amplitude of perturbation geopotential of most unstable wave as a function of pressure for adiabatic and Newtonian models.

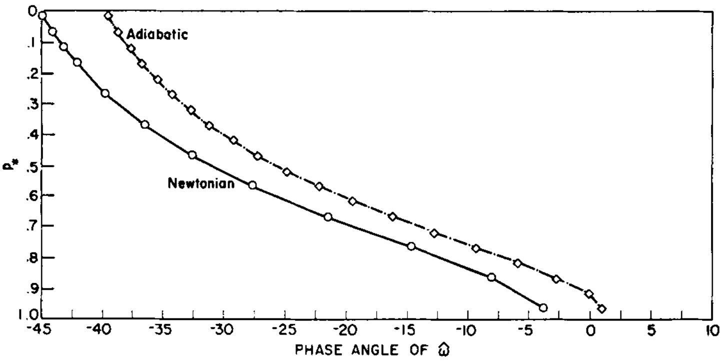

Fig. 7. Phase angle of perturbation vertical (pressure) velocity of most unstable wave as a function of pressure for adiabatic and Newtonian models.

tilt is greatest in mid-troposphere and the phase angle for Newtonian heating is about 5° west of that for adiabatic flow, but the range of δ is very much less, approximately 40°, indicating a slight east-west slope of $\hat{\omega}$ compared with $\hat{\Phi}$. Turning to normalized amplitude of $\hat{\omega}$ in Fig. 8, we observe a variation very nearly opposite to that of $\hat{\Phi}$. Here, minima occur at the bottom and top of the atmosphere to satisfy the lower and upper boundary conditions, respectively, while the maximum appears at $p_* = 0.4$.

Like the phase angle variation of $\hat{\Phi}$ and $\hat{\omega}$, we observe in Fig. 9 a maximum slope of T in mid-troposphere, with Newtonian heating phase angle 5° west of that for adiabatic flow. The range of δ is 85–90°. In contrast with $\hat{\Phi}$, $\hat{T}$ has a maximum tilt for $0.5 < p_* < 0.7$, below the layer of maximum slope for $\hat{\Phi}$. The variation of the normalized amplitude of $\hat{T}$ is seen in Fig. 10, where there is a maximum at the bottom boundary and minimum at the top of the atmosphere This is reasonable due to the fact that $\hat{T} \propto p_*$ from the ideal gas law. Further, we observe a secondary maximum around $p_* = 0.4$, which, from Fig. 6, is the level of minimum in $|\Phi|$. Combining the perfect gas and hydrostatic equations, we obtain $T \propto \partial\Phi/\partial p$ or, alternatively, $T \propto \partial\Phi/\partial \ln p$ which supports this behavior.

In the study of unstable waves it is of paramount importance to consider the energetics or energy transfer processes involved, in particular how the amplifying perturbation obtains its energy. The foundation of most current energy studies was laid by Lorenz (1955), included in which he considered energy conversion processes of unstable long waves associated with maintenance of the general circulation. He showed that, for an atmosphere in baroclinic instability in its normal state, $C(A_z, A_E) > 0$ and $C(A_E, K_E) > 0$, where, $C(A_z, A_E)$ is the conversion from zonal

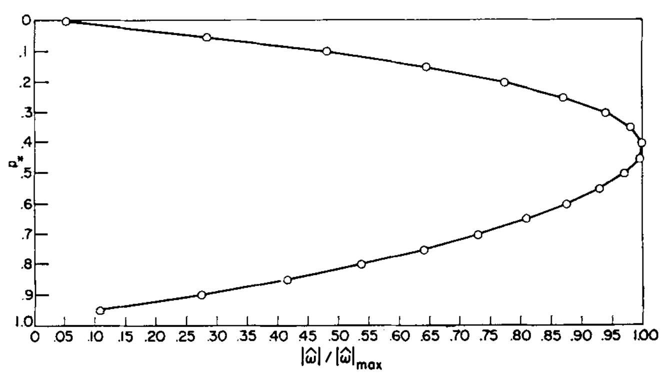

Fig. 8. Normalized amplitude of perturbation vertical (pressure) velocity of most unstable wave as a function of pressure for adiabatic and Newtonian models.

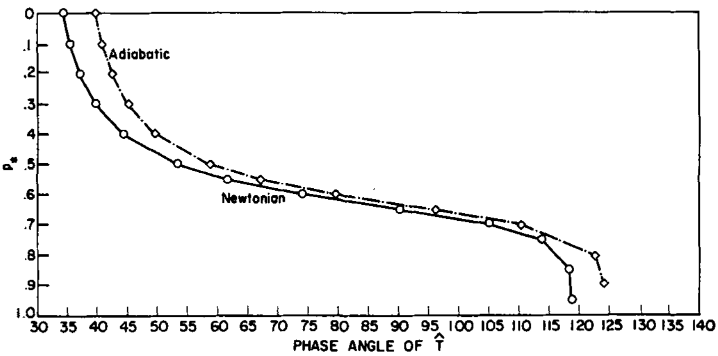

Fig. 9. Phase angle of perturbation temperature of most unstable wave as a function of pressure for adiabatic and Newtonian models.

to eddy available potential energy (APE) and $C(A_E, K_E)$ is the conversion from eddy APE to eddy kinetic energy (KE). When the instantaneous normalized energy conversions are calculated in a thin latitudinal strip of width 5° on each side of the latitude of maximum instability (35° N), as shown in Tables 1 and 2, we find for both adiabatic and Newtonian flow complete agreement with Lorenz (1955) at each wave number. A_z is converted into K_E via A_E.

From Table 1 it is seen for each wave number that the addition of Newtonian heating reduces the total normalized $C(A_z, A_E)$ in the strip, denoted $C_{NO}(A_z, A_E)$, relative to its adiabatic counterpart by an order of magnitude, while Table 2 presents a reduction in normalized $C(A_E, K_E)$, denoted $C_{NO}(A_E, K_E)$, of two to three orders of magnitude due to Newtonian heating. At each wave number comparison of Tables 1 and 2 reveals, in the adiabatic model, values of

Table 1. *Strip normalized $C(A_z, A_E)$ as a function of wave number in adiabatic and Newtonian models*

n	Adiabatic	Newtonian
1	1.976	0.184
2	0.988	0.092
3	0.659	0.061

$C_{NO}(A_E, K_E)$ two orders of magnitude smaller than $C_{NO}(A_z, A_E)$, while in the Newtonian model the values are three to four orders of magnitude smaller. These results agree qualitatively with Murakami's (1962) scale analysis that only the interaction between zonal and eddy available potential energy is predominant for ultra-long waves, exceeding all other energy conversions by two orders of magnitude. From Table 1 it is

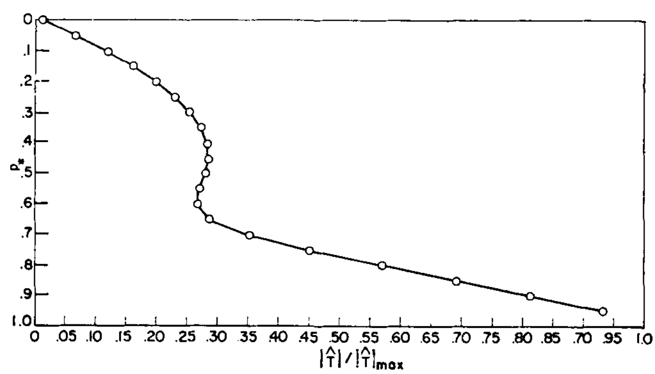

Fig. 10. Normalized amplitude of perturbation temperature of most unstable wave as a function of pressure for adiabatic and Newtonian models.

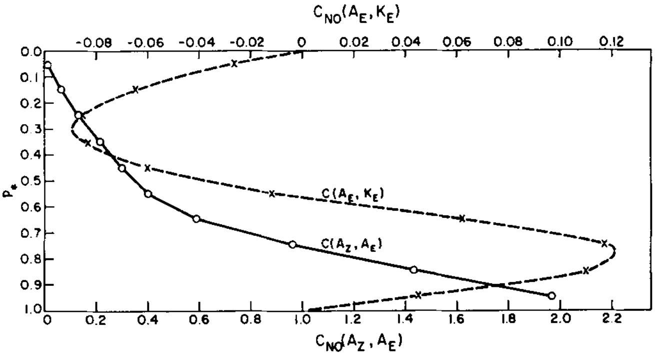

Fig. 11. Normalized conversion from zonal to eddy APE and from eddy KE as a function of pressure in latitudinal strip around latitude of maximum instability for adiabatic model at $n = 1$.

Table 2. *Strip normalized $C(A_E, K_E)$ as a function of wave number in adiabatic and Newtonian models*

n	Adiabatic	Newtonian
1	0.250×10^{-2}	0.198×10^{-4}
2	0.125×10^{-2}	0.200×10^{-5}
3	0.083×10^{-2}	0.823×10^{-5}

further seen that $C_{NO}(A_z, A_E) \propto n^{-1}$ in both models, while Table 2 presents the result that $C_{NO}(A_E, K_E) \propto n^{-1}$ in only the adiabatic model. Investigation of the general expressions for these conversions and their particular form in the adiabatic and Newtonian models support these correlations. Fig. 11 depicts the vertical variation of $C_{NO}(A_z, A_E)$ and $C_{NO}(A_E, K_E)$ for the adiabatic model at $n = 1$, showing that conversion proceeds from A_z to A_E at every level, increasing as pressure increases, while K_E is converted into A_E for $p_* < 0.56$, while the reverse is the case below. The near balance between these regions explains why total $C_{NO}(A_E, K_E)$ in the strip is negligible compared with total $C_{NO}(A_z, A_E)$. Similar vertical variation of $C_{NO}(A_z, A_E)$ and $C_{NO}(A_E, K_E)$ is obtained for the Newtonian model at $n = 1$ and for both models at $n = 2$ and 3.

6. Concluding remarks

We have examined the internal dynamics with regards to stability, structure, and energetics of ultra-long, transient, baroclinic wave perturbations superimposed on a meridionally-varying zonal current in adiabatic, frictionless, geostrophic flow and how the addition of Newtonian heating modifies the results. The zonal wind is assumed linear in pressure and is based upon the annual average, 1963, 500 mb wind between 25° N and 80° N. Applying the perturbation theory, a single second-order ordinary differential equation for adiabatic flow is derived and numerically solved by the "shooting method" in a 40-layer model for the normally complex perturbation phase velocity, c. From this the corresponding quantity for Newtonian heating is easily determined. Knowledge of the eigenvalue, phase velocity, suffices for complete analysis of the structure and energetics of the disturbance in the two models.

Investigation of instability as measured by the imaginary component of phase velocity, c_i, reveals no dependence upon wave number, n, for adiabatic flow, while instability increases as wave number increases for Newtonian heating, which itself exerts a stabilizing effect. This effect is maximum at maximum instability which occurs at 35° N. Here, in the adiabatic model, e-folding time for the third harmonic is approximately six days. The phase speed, defined as the real part of the phase velocity of the perturbation wave, is equivalent for both adiabatic and Newtonian flow and is independent of wave number. Its variation with latitude closely resembles the input distribution of zonal wind with latitude. The maximum value of phase speed is 8.2 m sec^{-1} at 40° N, while the most unstable waves travel at 6.3 m sec^{-1}. The vertical wave structure of only the most unstable waves is investigated and is found strikingly similar for the adiabatic and Newtonian models at $n = 1$, 2, and 3. Wave number is negligible as well as amplitude variation between the models. Only phase angle, δ, which tilts westward with decreasing pressure for every parameter, reveals any distinction, those for Newtonian heating being approximately 5° west of those for adiabatic flow at each level. Instantaneous energetics of the adiabatic and Newtonian models is studied by calculation of vertical variation and total over all pressures in a latitudinal strip 10° wide centered on the latitude of maximum instability of the normalized conversion from zonal to eddy available potential energy, $C_{NO}(A_z, A_E)$, and normalized conversion from eddy available potential energy to eddy kinetic energy, $C_{NO}(A_E, K_E)$. We conclude that the instantaneous energetics in both models in the region of maximum instability is in agreement with the time-averaged energetics as observed in the atmosphere and predicted by Lorenz (1955). It is found for each wave number that $C_{NO}(A_z, A_E)$ is positive at every level and increases monotonically with increasing pressure from zero at the top of the atmosphere, while $C_{NO}(A_E, K_E)$ is negative for $p_* < 0.55$ and positive below, the total being several orders of magnitude smaller than $C_{NO}(A_z, A_E)$ in both models. Addition of Newtonian heating reduces the total adiabatic strip $C_{NO}(A_z, A_E)$ by an order of magnitude and the total adiabatic $C_{NO}(A_E, K_E)$ by two to three orders of magnitude. Variation of the energy conversions with wave number—$C_{NO}(A_z, A_E) \propto n^{-1}$ for both models and $C_{NO}(A_E, K_E) \propto n_*^{-1}$ in only the adiabatic model—obtain from the numerical results, but may also be shown theoretically.

This study pertains solely to the transient baroclinic mode of the ultra-long waves. The existence, in addition, of a stationary mode and a transient barotropic mode suggests that an empirical investigation be conducted in the ultra-long wave regime to determine the division between the three modes.

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О БАРОКЛИННОЙ НЕУСТОЙЧИВОСТИ УЛЬТРАДЛИННЫХ ВОЛН

Бароклинная неустойчивость ультрадлинных волн (волновые номера, 1, 2 и 3) изучается с помощью 40-уровенной, геострофической, гидростатической, адиабатической модели, которая в дальнейшем модифицируется с целью включения ньютоновского нагревания. Малые возмущения наложены на зональный поток, изменяющийся по меридиану, и линейный по давлению. Статическая устойчивость предполагается обратно пропорциональной давлению. Принятые допущения позволяют свести исходные уравнения к одному однородному обыкновенному дифференциальному уравнению второго порядка для вертикальной скорости в p-системе координат с параметрической зависимостью от широты. Для численного определения собственных значений (фазовых скоростей возмущений) используется метод проб при упрощенных краевых условиях — равенстве нулю вертикальной скорости в p-системе координат на верхней и нижней границах атмосферы. Такой метод требует одного точного решения, и поэтому приводится процедура получения последнего. В обеих моделях — адиабатической и ньютоновской — профиль зонального ветра основан на среднегодовых данных за 1963 г. для поверхности 500 *мб* в полосе от 25° с. ш. до 80° с. ш.

Найдено, что неустойчивость по отношению к малым колебаниям, уменьшающаяся при наличии ньютоновского нагревания, достигает максимума при 35° с. ш., что в адиабатической модели соответствует увеличению амплитуды третьей гармоники в e раз за шесть дней. Фазовая скорость, одинаковая для обеих моделей, не зависит от волнового числа и составляет 6 *м*/сек. Приближенная эквивалентность моделей проявляется в вертикальной структуре — в сдвиге волн с высотой к западу.

Изучается мгновенная энергетика адиабатической и ньютоновской моделей путем вычисления энергетических характеристик в полосе шириной 10° с центром на широте максимальной неустойчивости. Вычисляются нормализованные превращения зональной в доступную вихревую потенциальную энергию $C(A_Z, A_E)$ и вихревой доступной потенциальной энергии в вихревую кинетическую энергию $C(A_E, K_E)$. Определяется зависимость этих величин от высоты и их суммарные значения. Найдено, что после суммирования по всем уровням давления $C(A_Z, A_E) > 0$, и $C(A_E, K_E) > 0$ для обеих моделей, что согласуется с наблюдаемой средней по времени энергетикой атмосферы.