Energetics of low-order spectral systems

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ABSTRACT

The potential vorticity equation and the barotropic vorticity equation represented in spectral form in terms of solid spherical harmonics have been truncated to allow interaction between only one planetary wave and an arbitrary zonal flow. The analytic solutions in time which ensue from these truncated systems have been investigated in terms of the allowed nonlinear energy transfers. An energy flow diagram including potential and kinetic energy has been prepared and shows the direction and magnitude of exchange as it depends on the initial conditions. Maximum energy transfers and limitations of exchange amongst wave components are discussed. The linear solutions are considered and evaluated in terms of the exact solutions. Phase propagation and momentum transport in these systems are also investigated.

1. Introduction

The exchange and conversion of energy amongst different scales of motion in atmospheric flow are phenomena whose detailed understanding is handicapped by our limited knowledge of nonlinear processes. Although many theoretical, linear studies have given valuable insight into the scale of maximum instability of the energy conversion process—a scale range which has been substantiated by observation—the scale transfer of energy subsequent to conversion is a purely nonlinear event which is generally not accessible to analytic analysis. There exist, fortunately, highly truncated systems of equations which represent barotropic and baroclinic flows, probably realizable only in "dishpan" type experiments, which have analytic solutions in time, thereby allowing a detailed discussion of energy exchange and/or conversion. While we emphasize that the flows described by the truncated systems may not be physically realizable in the atmosphere, which always has many active scales, the insight gained from the exact solutions should prove valuable in discussing less truncated and hence more realistic flows.

The systems for which energy exchanges will be considered are highly truncated forms of the spectral barotropic and baroclinic models, the barotropic model having been discussed by Platzman (1960) and the baroclinic model by Baer (1970). With the spectral expansion given by solid spherical harmonics, both models allow for an arbitrary zonal flow with two active wave components each having the same planetary wave number. The exact solutions show periodic amplitude fluctuations in time given by elliptic functions. The character of the solution is therefore determined by the initial energy distribution and the choice of truncation (zonal configuration and wave components).

Although highly truncated, the models show characteristics which are somewhat comparable to observation. They furthermore yield relationships amongst energy components (wave and zonal, kinetic and potential) in the wave domain which are obscured by more general models. One may see clearly, for example, how the amplitude of energy exchange between the waves and the zonal flow varies during one cycle depending on the initial distribution. Furthermore, the models are compared to their linear counterparts and the deviations therefrom discussed.

Properties of the systems not directly related to the energy exchange—but influenced by it—are also considered in terms of the initial configuration. These include momentum transport, behavior of trough and ridge lines, in fluence of phase angles, and the tilt of waves with height (baroclinic) or latitude (barotropic).

2. The low-order equations

The physical systems for which exact solutions may be determined by suitable truncation include the non-forced barotropic, quasi-geostrophic model, and a two level baroclinic model with fixed static stability as described by Lorenz (1960). The prediction equations describing those systems are

$$\frac{\partial}{\partial t}\nabla^2\psi = J(\nabla^2\psi + f, \psi) + J(\nabla^2\tau, \tau)$$

$$\frac{\partial}{\partial t}(\nabla^2 - r^2)\tau = J[(\nabla^2 - r^2)\tau, \psi] + J(\nabla^2\psi + f, \tau) \quad (1)$$

where ψ represents the mean stream field, τ is the shear stream field, $r^2 = f_0^2/\sigma$ with σ a constant stability parameter, and the other symbols have their usual notation. Clearly for the barotropic case the shear field vanishes; i.e., $\tau = 0$, leaving only the first of (1). The stream fields which depend on longitude (λ), sin of latitude (μ) and time (t) may be represented by a zonal field (independent of λ) and a wave field:

$$\psi = \bar{\psi}(u, t) + \psi'(\mu, \lambda, t)$$
$$\tau = \bar{\tau}(\mu, t) + \tau'(\mu, \lambda, t). \quad (2)$$

Assuming an expansion in terms of solid spherical harmonics $Y_\alpha(\lambda, \mu)$—see for example Hobson (1955)—which satisfy the differential equation

$$\nabla^2 Y_\alpha = -c_\alpha Y_\alpha$$
$$c_\alpha = n_\alpha(n_\alpha + 1) \quad (3)$$

in a spherical surface, the zonal flow may be represented by an arbitrary series of zonal polynomials with time-dependent coefficients. The wave component is described by two harmonics in the same planetary wave, $\alpha = n_\alpha + il$ and $\beta = n_\beta + il$, where n denotes the ordinal wave number and l is the planetary number. For the barotropic model both α and β describe components of the wave field, whereas in the baroclinic model the α component represents the mean wave field and the β component corresponds to the shear wave field. Thus we have,

$$\bar{\psi}_{BT}(\mu, t) = \sum_\gamma \psi_\gamma(t)\, Y_\gamma(\mu)$$

$$\psi'_{BT}(\lambda, \mu, t) = \psi_\alpha(t) Y_\alpha(\lambda, \mu) + \psi_\beta(t)\, Y_\beta(\lambda, \mu)$$

$$\bar{\tau}_{BC}(\mu, t) = \sum_\gamma \psi_\gamma(t)\, Y_\gamma(\mu)$$

$$\psi'_{BC}(\lambda, \mu, t) = \psi_\alpha(t)\, Y_\alpha; \quad \tau'_{BC}(\lambda, \mu, t) = \psi_\beta(t)\, Y_\beta$$

$$\bar{\psi}_{BC}(\mu, t) = \sum_\gamma \Psi_\gamma Y_\gamma, \Psi_\gamma \text{ indep. of time} \quad (4)$$

where BT and BC represent barotropic and baroclinic respectively, and $\gamma = n_\gamma$ for all desired values. The time dependent coefficients are in general complex (although the zonals are real). Substitution of (4) into (1) results in the following equation for determining the rate of change of the zonal components:

$$\dot{\psi}_\gamma = 2a_\gamma \operatorname{Im} \psi_\alpha \psi_\beta^* \quad (5)$$

where a_γ is a constant depending on the γ index, the dot represents time differentiation and the asterisk denotes conjugation. Since (5) implies a linear relationship between all ψ_γ, we need concern ourselves with only one zonal component, say ψ_γ for $\gamma = n$. In terms of this component we may write the complete system for the unknowns (ψ_n, ψ_α, ψ_β) as

$$\dot{\psi}_n = 2a_n \operatorname{Im} \psi_\alpha \psi_\beta^* \quad (6)$$
$$\dot{\psi}_\alpha = -i\varrho_\alpha\psi_\alpha + ih_{\alpha\beta}\psi_\beta + ig_{\alpha\alpha}\psi_n\psi_\alpha + ig_{\alpha\beta}\psi_n\psi_\beta$$
$$\dot{\psi}_\beta = -i\varrho_\beta\psi_\beta + ih_{\beta\alpha}\psi_\alpha + ig_{\beta\beta}\psi_n\psi_\beta + ig_{\beta\alpha}\psi_n\psi_\alpha$$

The details for generating spectral equations has been presented by Platzman (1960). The coefficients in (6) depend entirely on the spectral truncation (choice of γ, α, β). If we now represent the complex wave coefficients in terms of amplitude and phase such that,

$$\psi_\gamma = B_\gamma; \quad \psi_\alpha = (2)^{-\frac{1}{2}} B_\alpha e^{i\theta_\alpha}, \quad \text{etc.},$$

Eq. (6) reduces to

$$(\dot{B}_n)^2 = \sum_{i=0}^{4} b_i B_n^i \quad (7)$$

which has solutions in terms of elliptic functions. The details of the reduction and also the solution of (7) and the phase angles have been presented by the writer (Baer, 1970, to be referred to in the sequel as LOAS) and will not be repeated here. It is sufficient to note from (7) that the solution to the entire system (1) may be given in terms of B_n and that n may be chosen with complete generality as the smallest index for which an active zonal component exists.

3. Energy considerations

Since we have seen from Section 2 that the entire solution is dependent only on the time variation of the lowest active zonal component, B_n, it should be possible to describe the energy (both kinetic and potential) as well as the energy changes and the transfer processes in terms of this variable also, and consequently in terms of time.

The kinetic energy per unit mass for the models considered herein and described by (1) may be written as,

$$K = -\int \psi \nabla^2 \psi dA - \int \tau \nabla^2 \tau dA \tag{8}$$

where dA represents an element of area on the unit sphere. Although τ, which describes the shear stream field for the baroclinic problem, is undefined for the barotropic problem, we may redefine the barotropic variables so that (8) will be applicable to both cases. With reference to (4), let ψ in (8) represent the α-wave and the inactive zonal components (those components for which $a_\gamma = 0$). If we now let τ represent the β-wave *and* the remaining (active) zonal components, (8) will describe also the energy in the barotropic case except for $n_\alpha = n_\beta$, a disallowed barotropic truncation (see LOAS).

The available potential energy necessary to satisfy the energy conservation conditions applicable to (1) is given as

$$P = r^2 \int \tau^2 dA \tag{9}$$

where r^2 has been defined in Section 2. Since there is no available potential energy in the barotropic case, we define r^2 (barotropic) $= 0$.

Not only is the energy distributed between potential and kinetic, but in each of these categories some of the energy is in the zonal flow and some in the planetary wave. The distribution of the energy in all categories may be determined by introducing the expansions for the stream field, given by (4) into (8) and (9) and performing the integration. The results are as follows:

$$K = \bar{K} + K_z + K_\alpha + K_\beta$$

$$P = P_z + P_\beta$$

$$K + P = \text{const.} \tag{10}$$

In (10), $\bar{K}$ represents the kinetic energy in the zonal flow which does not change with time (the mean zonal flow for the baroclinic case), K_z represents the KE of the changing zonal flow (the zonal shear flow for the baroclinic case), K_α and K_β represent the KE in the α- and β-waves respectively, P_z represents the PE in the zonal shear flow and P_β is the PE in the β-wave (shear flow). Clearly the last two quantities do not exist for the barotropic case. Applying the relations for amplitude dependence on B_n from LOAS, we have,

$$\bar{K} = \sum_\gamma c_\gamma \Psi_\gamma^2 \quad (\gamma\text{'s for which } a_\gamma = 0, \text{ the barotropic case})$$

$$K_z = \sum_\gamma c_\gamma B_\gamma^2 = \bar{c}_\gamma(t) \sum B_\gamma^2 = \bar{c}_\gamma (k_{2z} B_n^2 + k_{1z} B_n + k_{0z})$$

$$P_z = r^2 \sum B_\gamma^2 = \frac{r^2}{\bar{c}_\gamma} K_z = r^2 (k_{2z} B_n^2 + k_{1z} B_n + k_{0z})$$

$$K_\beta = c_\beta B_\beta^2 = c_\beta (k_{2\beta} B_n^2 + k_{1\beta} B_n + k_{0\beta})$$

$$P_\beta = r^2 B_\beta = \frac{r^2}{c_\beta} K_\beta$$

$$K_\alpha = c_\alpha B_\alpha^2 = c_\alpha (k_{2\alpha} B_n^2 + k_{1\alpha} B_n + k_{0\alpha}) \tag{11}$$

The coefficients used in (11) depend on the spectral truncation and initial conditions.

We note, immediately, that the potential and kinetic energies in the β-wave (wave energy in the shear flow) are related to each other by a fixed constant, $r^2 c_\beta^{-1}$, for all time. Similarly, by defining a mean eigenvalue for the zonal flow, $\bar{c}_\gamma(t)$, we may show that there is a relationship between the zonal kinetic and potential energies. Although this relationship is time dependent, we may establish limits on the ratio. From the definition of the eigenvalues c_γ (3), we note that they are positive and monotonically increasing for increasing n_γ. Thus $\bar{c}_\gamma$ is bounded by the values

$$c_\gamma\,(\min) \leqslant \bar{c}_\gamma(t) \leqslant c_\gamma\,(\max)$$

which is readily verified from (11). Thus the potential energy in the zonal shear flow is con-

Table 1. *Extremes of energy components*

Component	B_n for Extreme
K_z	$-a_n \sum c_\gamma s_\gamma a_\gamma / \sum c_\gamma a_\gamma^2$
P_z	$-k_{1z}/2k_{2z}$
K_β, P_β	$-k_{1\beta}/2k_{2\beta}$
K_α	$-k_{1\alpha}/2k_{2\alpha}$

fined by the kinetic energy in the shear flow. We shall discuss the significance of this result later in this section.

If we are interested in the combined energy in different components (zonal or wave) it is simply necessary to replace the eigenvalue c_γ by $d_\gamma = c_\gamma + r^2$, an eigenvalue appropriate to the baroclinic problem. Thus, for example,

$$K_z + P_z = \sum_\gamma d_\gamma B_\gamma^2 = \bar{d}_\gamma \sum_\gamma B_\gamma^2$$

The formulas for the energy components (11) all depend on the dependent variable B_n which in turn has been described in LOAS as a periodic function of time. It can be seen that the energy components have their maxima and minima at the same times as the function B_n. They have, moreover, other possible extremes which are listed in Table 1; the quantities s_γ depend on initial conditions.

For initial conditions appropriate to the earth's atmosphere, the extremes listed in Table 1 do not seem to be attainable. Therefore, in the following discussion, we shall concentrate our attention on the energy extremes which correspond to the extremes of B_n. Since we shall suggest in the following section that the initial phase angle between the α- and β-wave may be chosen as $\theta_0 = (\theta_\alpha - \theta_\beta)_0 = 0$ without loss of generality, it can be shown that B_n must have an extreme at $t = 0$ and, since it is periodic with period T, it will also have an extreme at $t = T/2$ (see LOAS for a detailed discussion).

The unique property of the nonlinear solution available here is the time variation of the amplitudes of the individual components (or, equivalently, energy components), a property not available from the linearized equations. The maximum changes of these amplitudes may be found by calculating their difference between their maximum and minimum times. With reference to energy components, such a calculation will specify the amount of energy which any component can exchange with the other active components. For precision we shall define energy range of any component as the value at its first extreme minus the value at its second extreme. Since we have already seen that the energy components have their extremes at the same times as B_n, and furthermore that B_n has extremes at $t = 0$ and $t = T/2$, we have for any energy component E,

$$\Delta E = E(B_n(0)) - E(B_n(T/2)) \tag{12}$$

Let us now define the mean and difference of the variable B_n at its extremes, as well as similar quantities for the time-dependent eigenvalue $\bar{c}_\gamma$ (t) as

$$\begin{aligned} \bar{B} &\equiv B_n(0) + B_n(T/2) \\ \Delta B &\equiv B_n(0) - B_n(T/2) \\ \bar{c} &\equiv \bar{c}_\gamma(0) + \bar{c}_\gamma(T/2) \\ \Delta c &\equiv \bar{c}_\gamma(0) - \bar{c}_\gamma(T/2) \end{aligned} \tag{13}$$

The latter two definitions are necessary to describe the range of the zonal kinetic energy. If we now apply the range operator (12) to the different energy components (11), we may express the maximum energy exchange in each component in terms of the mean difference of the basic variable B_n (13) as follows:

$$\begin{aligned} \Delta K_z &= \frac{\bar{c}\Delta B}{2}(k_{2z}\bar{B} + k_{1z}) + \frac{\Delta c}{2} \\ &\quad \times \left(\frac{k_{2z}}{2}[(\Delta B)^2 + \bar{B}^2] + k_{1z}\bar{B} + k_{0z}\right) \\ \Delta P_z &= r^2 \Delta B(k_{2z}\bar{B} + k_{1z}) \\ \Delta K_\beta &= c_\beta \Delta B(k_{2\beta}\bar{B} + k_{1\beta}) = \frac{c_\beta}{r^2}\Delta P_\beta \\ \Delta K_\alpha &= c_\alpha \Delta B(k_{2\alpha}\bar{B} + k_{1\alpha}) \end{aligned} \tag{14}$$

Since the maximum exchange of energy described by (14) depends on the coefficients $k_{\delta\varepsilon}$, we see that the initial values and the system truncation are of primary importance in determining what these exchanges will actually be. Furthermore, one may normalize the exchange values listed above by the initial value of the time-dependent energy, thereby determining the percent of activity of each component. Because of the relatively different magnitudes of the kinetic and potential energies, it is preferable to normalize the kinetic energies by the total initial KE and the potential energies by the total initial PE. The interchange between the different components of energy and its magnitude may then be readily studied as a function of the initial values and the spectral truncation.

As an example, as ΔB approaches zero, we see that the energy ranges approach zero (except for the zonal kinetic) and thus the system under which this condition prevails ($\Delta B \to 0$) may be described by linear processes. The nonvanishing of the zonal kinetic energy implies a nonlinear transfer amongst the individual zonal components. If, however, only one zonal component is allowed to exist, Δc will vanish and ΔK_z will be proportional to ΔB. Furthermore, under this last assumption, the kinetic and potential energies of the zonal flow will be proportional by a time-independent constant since $\bar{c}_\gamma(t) = c_\gamma =$ constant.

By using standard atmosphere temperature values, one finds that the nondimensional constant $r^2 \simeq 200$. To measure the relationship of kinetic to potential energy, we must compare r^2 with c_β for the wave energy and with $\bar{c}_\gamma$ for the zonal energy. Recalling that the eigenvalues (c_γ) are given approximately by the square of the latitudinal index of the solid harmonics (n_γ), we see that the two forms of energy are roughly equivalent for indices $n_r \simeq$ 13–14. Characteristic zonal profiles of the atmosphere including jets are well represented by coefficients with eigenvalues less than $c_r = n_r(n_r+1)$ and hence we may conclude that since $\bar{c}_\gamma < c_r$, the zonal potential energy will always have more energy than the zonal kinetic. More specifically, it is rarely necessary to include zonal indices $n_\gamma > 9$. Thus we may say that in the models considered in this paper,

$$K_z < \tfrac{1}{2} P_z$$

which is also observed in the atmosphere (Wiin-Nielsen, 1967) and in less truncated general circulation models (Phillips, 1956; Smagorinsky et al., 1965).

For the planetary waves, we see that the potential energy of the longest waves will be considerably greater than the kinetic, whereas for shorter waves, $l > 12$, the kinetic energy will exceed the potential. For very short waves, the kinetic energy will dominate, but the approximations made in deriving this system, (1), may not be appropriate to such small scales.

The average properties of the energy exchange are described by (14). However, since we know the solution for all time, it is possible to describe the instantaneous exchange at any time. For convenience of notation, we shall define a time-dependent variable R_γ such that

$$R_\gamma \equiv \frac{-2\dot{B}_n}{d_\beta - c_\alpha}\left(\frac{d_\gamma a_\gamma^2}{a_n^2} B_n + \frac{d_\gamma a_\gamma s_\gamma}{a_n}\right). \tag{15}$$

On summation over all γ we have that

$$A(t) \equiv \sum_\gamma R_\gamma = \frac{-2\dot{B}_n}{d_\beta - c_\alpha} \sum_\gamma \frac{d_\gamma a_\gamma^2}{a_n^2}(B_n - B_{zex}) \tag{16}$$

where B_{zex} represents the value of B_n at the time when the total zonal energy (PE plus KE) would have an extreme value, not including the times $\dot{B}_n = 0$. This extreme value,

$$B_{zex} \equiv \frac{a_n \sum_\gamma d_\gamma a_\gamma s_\gamma}{\sum_\gamma d_\gamma a_\gamma^2}$$

has already been discussed (see Table 1) in terms of energy extremes.

The function $A(t)$ is clearly a periodic function of time with period T and changes sign where $\dot{B}_n = 0$. From limited computations which show that the energy does not reach extremes other than at extremes of B_n, we may assume for the subsequent discussion that B_n does not generally take on the value B_{zex} and hence that $A(t)$ changes sign only once during a period (provided that $\dot{B}_n(0) = 0$). Let us further define the sum,

$$\sum_\gamma c_\gamma R_\gamma \equiv c_\gamma^*(t)\, A(t) \tag{17}$$

where $c_\gamma^*(t)$ is an averaged value of the c_γ weighted by the R_γ and is not necessarily bounded. We shall, however, concern ourselves primarily with times at which

$$c_\gamma\,(\min) \leqslant c_\gamma^*(t) \leqslant c_\gamma\,(\max) \tag{18}$$

To discuss the instantaneous energy exchange, we now write the time rate of change of the energy components listed in (11):

$$\dot{K}_z = c_\alpha A - c_\beta A - r^2 \sum_\gamma \frac{c_\gamma + c_\alpha - c_\beta}{d_\gamma} R_\gamma$$

$$\dot{P}_z = -r^2 A + r^2 \sum_\gamma \frac{c_\gamma + c_\alpha - c_\beta}{d_\gamma} R_\gamma$$

$$\dot{K}_\beta = -c_\alpha A + c_\gamma^* A + r^2 \frac{(c_\beta - c_\alpha - c_\gamma^*)}{d_\beta} A$$

$$\dot{P}_\beta = r^2 A - r^2 \frac{(c_\beta - c_\alpha - c_\gamma^*)}{d_\beta} A$$

$$\dot{K}_\alpha = c_\beta A - c_\gamma^* A \tag{19}$$

These equations are derived by differentiation of (11) and substitution of the amplitude form of (6). The barotropic exchanges may be seen clearly simply by setting $r = 0$.

The contributions from divergence listed as the last terms in (19) and describing the exchange between potential and kinetic energies in the zonal and wave separately, may be calculated from the vorticity equation and thermodynamic equation for the baroclinic model before they are combined to form the second of Eqs. (1) (see LOAS for details). To examine the nature of this exchange in more detail, let us assume that

$$r^2 \sum_\gamma \frac{(c_\gamma + c_\alpha - c_\beta)}{d_\gamma} R_\gamma \simeq (c_\gamma^* + c_\alpha - c_\beta) A \tag{20}$$

based on our earlier observation that we may truncate γ with reasonable fidelity to atmospheric zonal wind profiles such that $c_\gamma(\max) < r^2/2$. Recalling the constraint on active γ-components given by Baer & Platzman (1961),

$$|n_\alpha - n_\beta| \leqslant n_\gamma \leqslant n_\alpha + n_\beta$$

thereby establishing the maximum and minimum values of n_γ as

$$n_\gamma(\min) \simeq |n_\alpha - n_\beta|$$

$$n_\gamma(\max) \simeq n_\alpha + n_\beta$$

we may write

$$c_\alpha - c_\beta \simeq \pm [c_\gamma(\max)\, c_\gamma(\min)]^{\frac{1}{2}} \quad \begin{array}{l} + \text{for } n_\alpha > n_\beta \\ - \text{for } n_\alpha < n_\beta \end{array} \tag{21}$$

$$c_\alpha + c_\beta \simeq \tfrac{1}{2}[c_\gamma(\max) + c_\gamma(\min)]$$

Substitution of (21) and (20) into the divergence terms of (19) now allows us to establish the exchange of energy within the zonal and wave separately. Subject to the condition that $c_\gamma^* > 0$ (considerably weaker than (18)) and $A(t) > 0$ which occurs during half the time period, Table 2 describes the direction of this energy flow. Clearly for the other half period when $A(t) < 0$, these directions are reversed.

We note from this table that if the shear wave is longer than the mean wave ($n_\alpha > n_\beta$), the direction of flow does not depend on the magnitude of c_γ^*, but only on the sign of $A(t)$. If, however, the shear wave is shorter, we see that the direction of flow from PE to KE is the same for the zonal and wave exept for the restricted range where

$$\sqrt{c_\gamma(\min)\, c_\gamma(\max)} < c_\gamma^* < \tfrac{1}{2}(c_\gamma(\max) + c_\gamma(\min)).$$

The entire energy flow diagram, listing all five energy terms defined in (11) may be seen from Fig. 1. The direction of flow shown in the figure is based on the condition that $A(t) > 0$, $c_\gamma^* > 0$, and the choice of transfer within the zonal and wave can be established from Table 2. The barotropic exchange is described by simply ignoring the boxes for available potential energy.

Applying the following replacements to Fig. 1,

$$K_z \to K_z + P_z$$

$$K_\beta \to K_\beta + P_\beta$$

$$c_\alpha \to c_\alpha^* \equiv c_\alpha - r^2 \tag{22}$$

the lower three boxes will also describe the exchange of total energy without distinguishing between potential or kinetic.

When approximation (20) is substituted into (19) and one considers the cycle during which $A(t) > 0$, it is easily seen that the zonal kinetic energy decreases whereas the shear wave kinetic energy (K_β) increases. By combining the kinetic and potential energies, the following equations result:

Table 2. *Energy flow within zonal and wave for different values of $c_\gamma^* > 0$ and $A(t) > 0$*

Range	Zonal $n_\alpha > n_\beta$	Zonal $n_\beta > n_\alpha$	β-wave
$c_\gamma^* < \sqrt{c_\gamma(\text{min})\, c_\gamma(\text{max})}$	$K_z \to P_z$	$K_z \leftarrow P_z$	$K_\beta \leftarrow P_\beta$
$c_\gamma^* \gtrless \begin{matrix}\sqrt{c_\gamma(\text{max})\, c_\gamma(\text{min})} \\ \frac{1}{2}(c_\gamma(\text{max}) + c_\gamma(\text{min}))\end{matrix}$	$K_z \to P_z$	$K_z \to P_z$	$K_\beta \leftarrow P_\beta$
$c_\gamma^* > \frac{1}{2}(c_\gamma(\text{max}) + c_\gamma(\text{min}))$	$K_z \to P_z$	$K_z \to P_z$	$K_\beta \to P_\beta$

$$\dot{E}_z \equiv \dot{K}_z + \dot{P}_z = (c_\alpha^* - c_\beta) A \;:\; (c_\gamma^*)$$

$$\dot{E}_\beta \equiv \dot{K}_\beta + \dot{P}_\beta = (c_\gamma^* - c_\alpha^*) A \;:\; (c_\beta)$$

$$\dot{E}_\alpha \equiv \dot{K}_\alpha = (c_\beta - c_\gamma^*) A \;:\; (c_\alpha^*) \tag{23}$$

The quantities in parentheses after the colon in (23) may be considered as "modified eigenvalues" associated with the energy function which each follows. Equations (23) then state that *energy will flow into or out of the component whose "modified eigenvalue" is of intermediate value, at the expense of the other two components.* This result is identical to one derived by Fjortoft (1953) for the two dimensional barotropic model and compares here to the case where $r = 0$ and $c_\alpha^* = c_\alpha$. Certain observations may be drawn from this result.

(*a*) When the energy of the wave mean flow is in a long wave, i.e., $c_\alpha < r^2$, c_α^* will be negative and on the assumption that $c_\gamma^* > 0$, the mean wave energy will never be the sole recipient of energy from the other two (shear energy) sources, or vice versa.

(*b*) If the shear wave energy resides in a wave shorter than or equal to the mean wave, the "modified eigenvalue" of the shear wave will always be greater than that of the mean wave, $c_\beta > c_\alpha^*$.

(*c*) In application to typical atmospheric zonal wind profiles, if the energy in the shear wave is not in too long a wave, both the mean wave and the shear wave will feed the zonal energy and vice versa; i.e., $c_\alpha^* < c_\gamma^* < c_\beta$.

The above results, especially with regard to maximum energy transfer during a half period between extremes of $B_n(t)$, may also be derived by invoking the concept of conservation of potential vorticity appropriate to the baroclinic model discussed herein and following the technique utilized by Fjortoft (1953).

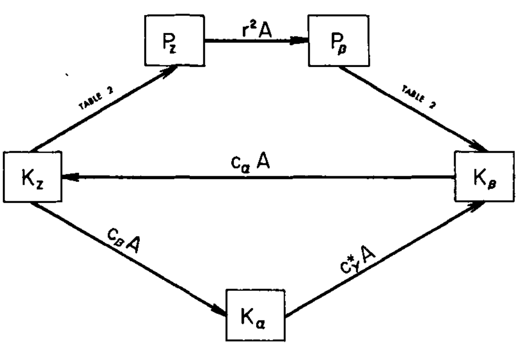

Fig. 1. Energy transfer diagram for all allowed energy forms based on the conditions that $A(t) > 0$ and $c_\gamma^* > 0$ (see text).

4. Linear Analysis

One means of linearizing system (6) which has real physical significance, and whereby some of the characteristics of the system may be simplified, is to assume that the energy in the wave components (α, β) is of perturbation amplitude in comparison with the zonal energy. Since we have shown in Section 3 that the potential energy is roughly proportional to the kinetic energy, no loss of generality will ensue if we confine our discussion to kinetic energy. In symbolic form, we may say that

$$\psi' < \bar{\psi}$$
$$\tau' < \bar{\tau} \tag{24}$$

where the barred and primed quantities have the same meaning as in (2). Following customary linearizing procedures, we shall neglect all second order or higher terms (products of primed quantities) on substitution of (24) into (6). It is immediately evident that the perturbation changes of the zonal coefficients (if they exist) cannot be calculated directly from the differential equation, since such change is of second order, but we shall see that the first order changes (implied) can be established from the energy relations. The wave perturbation equation may be written

$$\frac{d}{dt}\begin{pmatrix}\psi_\beta\\ \psi_\alpha\end{pmatrix} \equiv \dot{\psi} = i\begin{pmatrix}-n_\beta & \bar{G}_{\beta\alpha}\\ -n_\alpha & \bar{G}_{\alpha\beta}\end{pmatrix}\psi$$

$$\eta_\beta \equiv \varrho_\beta + h_{\beta\beta} - \bar{G}_{\beta\beta}, \quad \text{etc.}$$

$$\bar{G}_{\beta\beta} = g_{\beta\beta}\bar{B}_n + h_{\beta\beta}, \quad \text{etc.} \tag{25}$$

The barred quantities, $\bar{G}_{\beta\beta}$, etc. can be established by substituting the initial value of the zonal coefficient (B_n) and holding this term constant, recognizing of course that the initial values of the other zonal terms (B_γ) are incorporated in $g_{\beta\beta}$ and $h_{\beta\beta}$, etc.

System (25) is a linear homogeneous set having two frequencies ($\psi \propto e^{i\sigma t}$). These frequencies, the roots of the coefficient matrix, are

$$\sigma_{a,b} = \sigma_1 \pm \sigma_2$$

$$\sigma_1 \equiv -\frac{\eta_\alpha + \eta_\beta}{2}; \quad \sigma_2 \equiv \tfrac{1}{2}((\eta_\alpha - \eta_\beta)^2 + 4\bar{G}_{\alpha\beta}\bar{G}_{\beta\alpha})^{\frac{1}{2}} \tag{26}$$

We first observe that this system is capable of instability which depends on the initial zonal configuration *and* the wave number and wave profile. The condition in terms of the problem parameters is

$$4\bar{G}_{\alpha\beta}\bar{G}_{\beta\alpha} \gtrless -(\eta_\alpha - \eta_\beta)^2 \quad \begin{cases}\text{stable}\\ \text{unstable}\end{cases}$$

For the barotropic problem, this stability criteria should be contrasted to that given by Kuo (1949), which may be written in terms of spherical geometry as

$$\frac{d^2}{d\mu^2}((1-\mu^2)^{\frac{1}{2}}u) - 2 = 0$$

where u represents the initial zonal wind distribution. The disparity between the two conditions is apparent in that Kuo's condition depends *only* on the zonal configuration and not on the wave.

The condition for instability with regard to the baroclinic problem may be compared with that established by Phillips (1954) for a 2-layer model and written

$$V^2(\alpha\alpha^*)^2\left(4 - \left(\frac{\alpha\alpha^*}{r^2}\right)^2\right) > \left(\frac{\partial f}{\partial \phi}\right)^2$$

where V represents a zero-order value of the zonal wind shear, α is a characteristic wave vector as defined in Section 2 (Phillips deals with only one wave number in each of the two horizontal dimensions), r^2 is the previously employed stability parameter and the asterisk denotes conjugation. The stability properties of the low-order systems (both linear and nonlinear) are currently under investigation and will be discussed in a subsequent report.

We shall, for the remainder of this section consider only stable solutions with the implied condition that σ_2 is real. The complete solutions for the wave components ψ_α, ψ_β require a specification of their initial values and must depend on both frequencies; they may be written as

$$\psi = e^{i\sigma_1 t}\begin{pmatrix}q_1 A & q_2 B\\ A & B\end{pmatrix}\begin{pmatrix}e^{i\sigma_2 t}\\ e^{-i\sigma_2 t}\end{pmatrix} \tag{27}$$

Substituting this equation into (25) and evaluating at $t=0$ when $\psi_\alpha = \psi_{\alpha 0}$, $\psi_\beta = \psi_{\beta 0}$, we find that the coefficients q, dependent on the differential equations, and the amplitude factors A and B, dependent on the initial values, have the following form:

$$q_1 = \frac{\bar{G}_{\beta\alpha}}{\sigma_a + \eta_\beta} = \frac{\sigma_a + \eta_\alpha}{\bar{G}_{\alpha\beta}}; \quad q_2 = \frac{\bar{G}_{\beta\alpha}}{\sigma_b + \eta_\beta} = \frac{\sigma_b + \eta_\alpha}{\bar{G}_{\alpha\beta}}$$

$$A = \frac{q_2\psi_{\alpha 0} - \psi_{\beta 0}}{q_2 - q_1} \equiv |A|\, e^{i\theta_a};$$

$$B = \frac{q_1\psi_{\alpha 0} - \psi_{\beta 0}}{q_2 - q_1} \equiv |B|\, e^{i\theta_b} \tag{28}$$

The latter definitions for A and B imply that the initial phase angles need not vanish. More specifically,

$$\tan\theta_a = \frac{\mathrm{Im}(q_2\psi_{\alpha 0}-\psi_{\beta 0})}{\mathrm{Re}(q_2\psi_{\alpha 0}-\psi_{\beta 0})}$$

$$\tan\theta_b = \frac{\mathrm{Im}(q_1\psi_{\alpha 0}-\psi_{\beta 0})}{\mathrm{Re}(q_1\psi_{\alpha 0}-\psi_{\beta 0})}$$

Let us now use (27) to establish the energy variation with time for the linearized system. We have for the α-component,

$$K_\alpha = 2c_\alpha\psi_\alpha\psi_\alpha^* = 2c_\alpha(|A|^2+|B|^2+2|A||B|\cos(2\sigma_2 t+\theta_a-\theta_b)) \tag{29}$$

If we now create a new time variable which is linearly related to t by the equation

$$t' = t - T$$

$$T \equiv \frac{\theta_b-\theta_a}{2\sigma_2}$$

we find the equation for the energy in the α-wave,

$$K_\alpha = 2c_\alpha(|A|^2+|B|^2+2|A||B|\cos 2\sigma_2 t') \tag{30}$$

Since the frequency $(2\sigma_2)$ is the same in both (29) and (30), and since (29) has a form identical with (30) for zero initial phase angles, we must conclude that except for a shift in time of the maximum and minimum of the α-energy, the selection of non-zero phase angles does not contribute information to the solution of (25) different from the more simple condition of zero initial phase angles. We shall see subsequently that the other energy components (β, γ) have the same time dependence as (29) and thus need not be considered in detail in this discussion. Our selection of zero initial phase angles as initial conditions to the nonlinear problem and our confidence in the generality of those conditions is based on the above results from the linear solution. A complete proof would, of course, require direct analysis of the nonlinear equations.

Returning to the discussion of energy variation, we note from the definition of the β-wave energy in terms of ψ_β, Eqs. (11) and (27), that the time dependence is identical to (29) which describes the α-wave energy. For the zonal energy, multiplying by $2\psi_\gamma$, ignoring the variable part on the right-hand side $(\psi_\gamma \to \bar{\psi}_\gamma)$, and integrating, the variation may be given as

$$K_\gamma = 4c_\gamma a_\gamma \bar{\psi}_\gamma \int \mathrm{Im}\,\psi_\alpha\psi_\beta^* dt + \text{const.} \tag{31}$$

which is a second order variation ($\bar{\psi}_\gamma$ is the initial value of ψ_γ). Assuming now that A and B are real—based on our observation that zero initial phase angles are completely general—and substituting (27) and (28) into (31), we find

$$K_z = 4\sum_\gamma c_\gamma a_\gamma \bar{\psi}_\gamma \frac{AB}{\bar{G}_{\alpha\beta}} \cos 2\sigma_2 t + \text{const.} \tag{32}$$

With this presentation of the different energy components, we may investigate the range of energy variation (maximum energy exchange) as defined by (12). Since the extremes occur at $t=0$ and $t=\pi/2\sigma_2$, we have from (29), (32) and the β-energy relation,

$$\Delta K_z = -8\frac{\sum c_\gamma \bar{\psi}_\gamma a_\gamma}{\bar{G}_{\alpha\gamma\to\beta}} AB$$

$$\Delta K_\beta = 8c_\beta \frac{\bar{G}_{\beta\alpha}}{\bar{G}_{\alpha\beta}} AB$$

$$\Delta K_\alpha = -8c_\alpha AB \tag{33}$$

Consider now the total kinetic energy in the system to reside in the zonal flow such that it can be characterized by a mean zonal wind $\bar{u}_0$. Further, consider latitudinal profile parameters A_γ specified such that the initial stream coefficients may be written as,

$$\psi_\gamma = g_l A_\gamma \tag{34}$$

where γ represents α and β for $l \neq 0$, the wave components, and g_l is proportional to $\bar{u}_0$. Finally we define a parameter ϱ which measures the relative energy in the l-wave to the total energy in the system. The dependence of the exchange on the level of total energy ($\bar{u}_0$) and the relative energy in the wave (ϱ) may now be established from the product AB. The initial stream coefficients used in A and B may be represented in terms of the amplitude g_l and profile parameters A_α, A_β as seen from (34). Combining this form with the definition of ϱ and the dependence of K_l on g_l from (8) and (34), we find

Table 3. *Comparison between exact nonlinear solution and linear solution for several amplitude values*

Listed as a function of ϱ—the ratio of perturbation energy to total energy—are the exact energy range $\Delta K_\alpha/K_a$ and the percent error of the linear value to the exact

	$\bar{u}_0 = 0.075$		$\bar{u}_0 = 0.135$		$\bar{u}_0 = 0.275$	
ϱ	$\Delta K_\alpha/K_a$ exact	% of error	$\Delta K_\alpha/K_a$ exact	% of error	$\Delta K_\alpha/K_a$ exact	% of error
.02	−.01917	− 1.26	−.02249	− 1.23	−.02358	− 1.23
.05	−.04614	− 5.20	−.05419	− 5.01	−.05687	− 4.94
.10	−.08681	−11.8	−.1022	−11.3	−.1074	−11.1
.15	−.1228	−18.6	−.1451	−17.7	−.1527	−17.3
.20	−.1549	−25.4	−.1835	−24.1	−.1934	−23.5
.30	−.2087	−39.4	−.2493	−37.0	−.2637	−35.8
.40	−.2518	−54.2	−.3033	−50.1	−.3233	−48.1
Linear Slope $= (8c_\alpha S/K_a)$	−0.9708		−1.138		−1.194	

$$\varrho \equiv \frac{K_l}{K} = \frac{g_l^2}{8K}(c_\alpha A_\alpha^2 + c_\beta A_\beta^2)$$

such that

$$AB = -\varrho S$$

$$S \equiv \frac{\overline{G}_{\alpha\beta}^2 K}{8\sigma_2^2(c_\alpha A_\alpha^2 + c_\beta A_\beta^2)} \times (q_1 q_2 A_\alpha^2 + A_\beta^2 - (q_1+q_2) A_\alpha A_\beta) \quad (35)$$

In (35), K represents the zero order energy initially placed into the system and does not change for variation in ϱ. Therefore we see that the *maximum energy exchange among the components is linearly proportional to the relative amount of energy in the wave components.* In the limiting case of no initial energy in the wave—i.e., $\varrho = 0$—no motion would occur, a result which is evident also from (6).

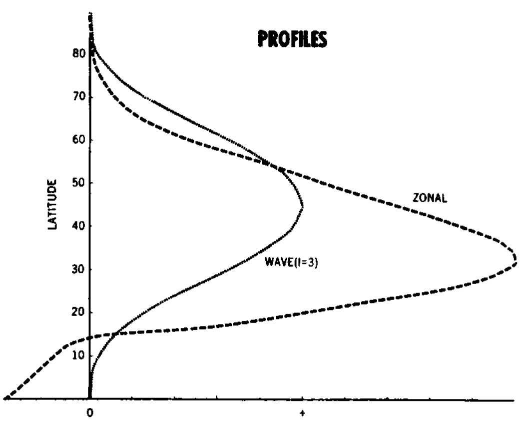

Fig. 2. Zonal wind profile and wave profile for $l = 3$ of meridional wind component plotted on a relative scale vs. latitude.

Whereas the energy range (ΔK) is linearly proportional to ϱ with zero intercept, the slope of the line is dependent on the amplitude $\bar{u}_0$. Noting from (34) that the initial zonal coefficients ($\bar{\psi}_\gamma$) are proportional to $\bar{u}_0$ and the profile parameters A_γ, and that $\overline{G}_{\alpha\beta}$, $\overline{G}_{\beta\alpha}$ are proportional to the $\bar{\psi}_\gamma$, we may observe from (33) that the variation of slope with $\bar{u}_0$ is incorporated entirely in the function S (35). The definitions of q_1 and q_2 from (28) yield required coefficients in the function S, (35), in terms of known quantities as follows:

$$q_1 + q_2 = \frac{s_1 + s_2 \bar{u}_0}{\bar{u}_0}$$

$$\sigma_2^2 = \frac{\overline{G}_{\alpha\beta}^2}{4\bar{u}_0^2}((s_1 + s_2\bar{u}_0)^2 - 4q_1 q_2 \bar{u}_0^1)$$

$$K = \frac{\bar{u}_0^2}{4}\sum c_\gamma A_\gamma^2 \quad (36)$$

where s_1 and s_2 depend only on the profile parameters. Substituting the results of (36) into (35), the $\bar{u}_0$ dependence of S is shown to be

$$S = \left(\frac{\sum c_\gamma A_\gamma^2}{2(c_\alpha A_\alpha^2 + c_\beta A_\beta^2)}\right)\bar{u}_0^3 \times \frac{(q_1 q_2 A_\alpha^2 + A_\beta^2 - s_2 A_\alpha A_\beta)\bar{u}_0 - s_1 A_\alpha A_\beta}{(s_1 + s_2\bar{u}_0)^2 - 4q_1 q_2 \bar{u}_0^2} \quad (37)$$

This is not a simple formula. We may, however, note that since we have chosen to consider only stable solution, σ_2^2 will not change sign and therefore the denominator will also be of one sign for all allowed values of $\bar{u}_0$. The slope will change sign, however, about the critical value of $\bar{u}_0$,

$$\bar{u}_0(c) \equiv \frac{s_1 A_\alpha A_\beta}{q_1 q_2 A_\alpha^2 + A_\beta^2 - s_2 A_\alpha A_\beta}.$$

When $\bar{u}_0 = \bar{u}_0(c)$, the linear system is inactive, independent of the value of ϱ. Such initial conditions have been observed in calculations of the nonlinear system and have been discussed elsewhere (Baer, 1968).

To describe the comparison between linear and nonlinear solutions, Table 3 has been prepared, based on calculations with a given initial state but variable ϱ. The initial configuration applies to the barotropic problem with the atmospheric zonal jet and profile for wave $l = 3$ described in Fig. 2. Three amplitude values ($\bar{u}_0$), nondimensionalized using the earth's rotation rate for time and the mean radius of the earth for space, have been chosen to show the variation of solutions both with ϱ *and* $\bar{u}_0$. We have chosen to describe the energy range in the α-component; the ranges of the other components are available (and computable from (33)) and show similar variations. The linear slope described in Table 3 is proportional to S and is given as

$$\text{Linear Slope} = \frac{8c_\alpha S}{K_a}$$

where K_a is the initial available kinetic energy, differing from the initial kinetic energy only by inactive zonal components. The error in the linear solution, described as the difference between the exact and linear values normalized by the exact value and represented as a percent, follows an expected pattern. For increasing values of ϱ, the linearization hypothesis becomes less and less valid. Although the activity of the α-wave increases for increased energy in the system—larger $\bar{u}_0$—the errors decrease slightly. We may conclude, therefore, that for $\varrho < 1$, linear calculations give a reasonable approximation to the nonlinear solutions.

5. Phase characteristics and other flow properties

The phase angles θ_α and θ_β may be calculated as functions of time, as described in LOAS. These solutions indicate that the angles need not be periodic with period T, although their difference ($\theta = \theta_\alpha - \theta_\beta$) must. If we assume that the phase angle periods differ from T (the nonlinear energy exchange period), then the spatial distribution of a wave component added to the zonal field need not show a repeating pattern until a modulation cycle between the energy (amplitude) period and the phase period is completed. This is especially true in the barotropic problem where the wave configuration may be more complicated.

For the baroclinic problem, the wave tilt with height is immediately calculated from the phase angle of the shear wave, θ_β. If, moreover, one assumes that the wave vectors α and β are equal—which does *not* imply that θ_α and θ_β are equal—then the wave has only one component, since $P_\alpha = P_\beta$ where P_α represents the normalized Legendre Polynomial for wave vector α, and no horizontal exchanges will exist. Such a condition applied to the barotropic problem would yield trivial solutions.

Although the equations for the individual phase angles are reasonably complex, we may draw certain conclusions from θ, their difference. For convenience we repeat here the variation of θ with time,

$$\tan\theta = \frac{\dot{B}_n}{G}$$

$$G \equiv (d_1 + d_2 B_n) B_n + d_0 \tag{38}$$

The variation of B_n has already been established, and since it is periodic with period T, it has a limited range. There are three possibilities for the variation of $G(B_n)$ in the range $mT \leqslant t \leqslant (m+1)T$ where m is any positive integer, and they depend on whether G has zero, one, or two roots in the allowed range of B_n. Since we have selected $\theta = 0$ at $t = mT$, it will also be zero at $t = (m + \frac{1}{2})T$. The range of θ may be summarized in terms of the number of roots of G and is listed in Table 4. We have noted that θ must pass through zero at $t = (m + \frac{1}{2})T$; thus θ remains either positive or negative during one half period and reverses

Table 4. *Range of θ in terms of the number of roots of G in one nonlinear period*

No. of Roots	0	1	2
Range	$-\frac{\pi}{2}<\theta<\frac{\pi}{2}$	$0\leqslant\theta\leqslant 2\pi$	$-\pi<\theta<\pi$

its sign during the other half. The direction of θ at the beginning of the period is specified as follows:

$$\theta(mT+\varepsilon)>0 \begin{cases} B_n(mT)\ \text{min.};\ G(mT)>0 \\ B_n(mT)\ \text{max.};\ G(mT)<0 \end{cases}$$

$$\begin{matrix}\theta(mT+\varepsilon)<0 \\ \varepsilon<T\end{matrix} \begin{cases} B_n(mT)\ \text{max.};\ G(mT)>0 \\ B_n(mT)\ \text{min.};\ G(mT)<0 \end{cases} \tag{39}$$

Aside from the information on the position of the wave components relative to each other during the period which Table 4 and (39) yield, we shall see how the variation of θ is related to the variation of other flow characteristics.

Phase frequencies may be calculated directly from the differentiated form of (38) once B_n is known. These frequencies include the Rossby-Haurwitz contribution, a linear contribution based on the jet configuration, and a nonlinear term. The frequencies undergo a periodic variation with the same period as B_n. They have maximum and minimum values at the same times as B_n, but they may have other relative extremes depending on whether or not $\ddot{\theta}_{\alpha,\beta}=0$ in the allowed range of B_n. Differentiating (38) twice with respect to B_n and setting the resulting equation to zero will yield a quartic equation, the roots of which will represent the values of B_n at which the phase frequency will have extremes. These values may then be compared with the allowed range of B_n to determine whether extra maxima or minima do indeed exist.

Let us now consider the horizontal distribution of the wave at any time. We shall focus our attention on the barotropic problem since the wave distribution is more complex, but the results can easily be extended to the mean wave field of the baroclinic model. We have from (4) and the equation above (7) that

$$\begin{aligned}\psi'(\lambda,\mu,t) &= 2^{\frac{1}{2}}(B_\alpha P_\alpha(\mu)\cos(l\lambda+\theta_\alpha) \\ &\quad + B_\beta P_\beta(\mu)\cos(l\lambda+\theta_\beta))\end{aligned} \tag{40}$$

from which we may determine the meridional velocity component after substitution of the definition,

$$\varepsilon \equiv l\lambda+\theta_\beta \tag{41}$$

and recalling the definition of $\theta=\theta_\alpha-\theta_\beta$. The resulting equation for meridional velocity v is,

$$\begin{aligned} v &= (1-\mu^2)^{-\frac{1}{2}}\frac{\partial\psi'}{\partial\lambda} \\ &= -2^{\frac{1}{2}}l(1-\mu^2)^{-\frac{1}{2}}((B_\beta P_\beta+B_\alpha P_\alpha\cos\theta) \\ &\quad \times\sin\varepsilon+B_\alpha P_\alpha\sin\theta\cos\varepsilon)\end{aligned} \tag{42}$$

Since the trough or ridge line is defined as the longitude at which the motion is zonal, the trough (ridge) longitude, λ_{tr}, is given from (42) by

$$\tan\varepsilon_{tr}(\mu,t) = -\frac{B_\alpha P_\alpha\sin\theta}{B_\alpha P_\alpha\cos\theta+B_\beta P_\beta} \tag{43}$$

We see immediately that the latitudinal dependence of λ_{tr} is related to θ, for when B_n is at an extreme, and hence $\theta=0$ or π, the trough (ridge) is north–south. The slope of the line is related both to the magnitude and sign of θ and can be evaluated as follows. Differentiating (43) with respect to μ, solving for the variation of λ_{tr} and using the first of (6), we find

$$\begin{aligned}\frac{d\lambda_{tr}}{d\mu} &= \frac{1}{l(1+\tan^2\varepsilon_{tr})}\frac{d\tan\varepsilon_{tr}}{d\mu} \\ &= \frac{\dot{B}_n}{la_n}\frac{P_\alpha P'_\beta-P_\beta P'_\alpha}{|B_\alpha P_\alpha e^{i\theta_\alpha}+B_\beta P_\beta e^{i\theta_\beta}|^2}\end{aligned} \tag{44}$$

Eq. (44) indicates that the trough slope goes through a periodic cycle, beginning with no slope at $t=mT$ (north–south), bending to one side for one-half period, returning to zero at $t=(m+\frac{1}{2})T$, and then bending to the other side for the second half period. Since the numerator of (44) may have roots in the allowed range of μ, the direction of the trough line at any time will be a function of latitude and may change sign.

The transport of momentum across a latitude circle has been shown to depend closely on the slope of troughs and ridges (Starr, 1948) and thus to the development of jets and barotropic instability. Continuing our discussion of the barotropic problem, the momentum transport may be defined as

$$M = \frac{1}{2\pi}\int_0^{2\pi} uv(1-\mu^2)^{\frac{1}{2}} d\lambda$$

$$= -\frac{(1-\mu^2)^{\frac{1}{2}}}{2\pi}\int \frac{\partial\psi}{\partial\lambda}\frac{\partial\psi}{\partial\mu} d\lambda \tag{45}$$

We again use the wave stream function ψ' from (40) since it is apparent that the zonal components will make no contribution to the integral. Substituting ε and θ for θ_α and θ_β and performing the required differentiations and integration as indicated by (45) we find

$$M = l\frac{\dot{B}_n}{a_n}(P_\alpha P'_\beta - P_\beta P'_\alpha)(1-\mu^2)^{\frac{1}{2}} \tag{46}$$

The momentum transport is therefore periodic with period T, zero at $t = mT$, $(m+\frac{1}{2})T$, and opposite in direction during the two half cycles. Rather than discuss the details of (46), we shall show that the momentum transport is directly related to the tilt of the trough. By combining (46) with (44) we have,

$$M = \frac{d\lambda_{tr}}{d\mu}(l^2(2-\mu^2)^{\frac{1}{2}} |B_\alpha P_\alpha e^{i\theta_\alpha} + B_\beta P_\beta e^{i\theta_\beta}|^2) \tag{47}$$

The results of this analysis, which show the momentum transport linearly related to trough slope with the same sign, are in complete agreement with general theory.

Finally, we may consider the mean square vorticity for both the barotropic and baroclinic models. For the baroclinic problem, the conservation condition must be converted to conservation of potential vorticity. In spectral terms, this modification is trivial, implying merely that we substitute d_γ for c_γ as eigenvalues. From the basic Eqs. (6) we find, therefore, the mean square vorticity (the squared vorticity integrated over the entire domain) to be given by the relation,

$$F = \sum d_\gamma^2 B_\gamma^2 + c_\alpha^2 B_\alpha^2 + d_\beta^2 B_\beta^2 \tag{48}$$

This parameter must be conserved for the flows considered herein, and the constraint thus imposed has already been implied in our discussion of energy exchanges in Section 3.

6. Conclusion

Our understanding of nonlinear energy exchange processes in the atmosphere is severly restricted by the complexity of observed flows. Hopefully, insight into more simple flows and their exchange processes may yield clues to the more general flow problem. To this end, we have considered the nonlinear interactions of a single planetary wave with an arbitrary zonal flow based on a physical system described by the two level potential vorticity equation with constant static stability. Fortunately, when the equations for this system are written in spectral form and the suggested truncation imposed, the resulting equations have exact solutions which show periodic properties with time. Thus the nonlinear energy exchange may be discussed in terms of initial conditions and zonal and wave profiles.

Because of the periodic nature of the solutions, all energy components (both wave and zonal) are characterized by a maximum variation during a time cycle. This range is directly related to the initial energy distribution and may be used predictively to establish the extent of nonlinear exchange. At any time during the cycle, an energy diagram may be drawn showing the direction of energy transfer amongst the components for both kinetic and potential energy. This diagram shows properties similar to those derived from numerical models and observational data. Although the spectral truncation is severe (only one planetary wave is allowed) the system suggests that the zonal kinetic energy should be less than one half of the zonal potential energy, a well established observation of atmospheric flow. The direction of transfer of energy amongst the allowed components is not entirely arbitrary; one finds that the transfer is restricted by a theorem comparable to that derived by Fjortoft (1953) for barotropic flow.

Linear analysis of our truncated (low-order) system shows stability criteria for both barotropic and baroclinic flows comparable to those derived for non-spectral models. For stable initial configurations, the maximum energy exchange during a time cycle is shown to be linearly proportional to the initial energy in the wave relative to the total energy. Provided this ratio remains small, the linear solution compares favorably to the exact (nonlinear) solution for various total energy levels of the system.

Finally, although our truncation is severe, the time variation of the phase of the wave allows for a continually changing flow configuration. The transport of momentum and the distribution with latitude or height of the troughs and ridges is given in relation to the time dependence of the phase and the correspondence of these processes agrees with more general theory.

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ЭНЕРГЕТИКА СПЕКТРАЛЬНОЙ СИСТЕМЫ НИЗКОГО ПОРЯДКА

Уравнение потенциального вихря и баротропное уравнение вихря, записанные в спектральной форме с помощью сферических гармоник, аппроксимируются так, чтобы описать взаимодействие только между одной планетарной волной и произвольным зональным потоком. Получаемые для таких систем аналитические решения исследуются с точки зрения нелинейного переноса энергии. Составлена диаграмма потока энергии, включающая потенциальную и кинетическую энергии, которая показывает зависимость направления и величины обмена от начальных условий. Обсуждается обмен энергией между волновыми компонентами. Рассматриваются линеаризированные решения и сравниваются с точными. Исследуются также фазовые скорости волн и перенос количества движения.