Vertical Diffusion in Shallow Waters

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ABSTRACT

From tracer measurements in stratified, shallow waters with a vertical current shear the vertical diffusion coefficient is determined. Two different models are used, treating separately the cases of strong and weak vertical stratification. The diffusion coefficient is related to the stratification, the frictional stress, and the shear. With a persisting wind of strength above 4–5 m/s, the stress in the upper layer is assumed to be proportional to the wind stress. In this special case the theoretical and experimental results can be compared, and a good agreement is found.

Key to symbols

A	vertical eddy viscosity (cm^2/s)
C	concentration
c_d	drag coefficient
g	acceleration of gravity
$2h$	thickness of rhodamine layer
K	vertical turbulent diffusion coefficient (cm^2/s)
K_H	vertical turbulent diffusion coefficient of heat (cm^2/s)
N^2	stratification parameter (s^{-2})
q	horizontal velocity vector
Ri	Richardson number
Rf	flux Richardson number
ϱ	density
ϱ_a	density of air
t	time
τ	frictional stress
u, v, w	velocity components in directions x, y, z
W	wind velocity
x, y, z	coordinates, pos. eastwards, northwards, downwards

1. Observations

The diffusion is studied by means of continuous *in situ* measurements of the concentration distribution in a spot of rhodamine B injected at a distance from the surface (Kullenberg, 1969, in the following called I). The experimental areas, the depth to the bottom, and the depth of injection are stated in Table 1.

The injection of the dye solution is accomplished either by using a barrel, which is lowered to the depth of injection and there opened, or by pumping the solution from a container on deck out through a hose connected to a tube with many small holes, and closed at the lower end. The first type of injection is regarded as instantaneous. When using the second technique it takes 3–5 minutes to release the solution, and the initial spot is larger than in the first case. Before the injection, the density of the dye solution is adjusted to the density at the depth of injection by adding concentrated NaCl solution.

The tracing is carried out with a specially developed *in situ* fluorometer, described in I. The instrument makes it possible to detect thin layers of rhodamine, and to observe steep variations of the concentration. The output signal, yielding a measure of the volume concentration, is recorded continuously as the ship cruises over the area. The spot is crossed several times in different directions, and the concentration distribution is determined. The navigation is performed with reference to a parachute drogue, placed in the depth interval where the rhodamine is injected, and connected to a surface buoy. It is released before the injection, which in turn is made close to the buoy. The position of the rhodamine spot is determined relative to the drogue, which acts as a moving origo. When the spot has grown large

Table 1. *Experimental areas and depths*

Date	Area	Depths (m) of injection	Depths (m) of bottom
1965, 21.6.	N 56°09′ E 12°31′	14	24
17.11.	N 55°47′ E 12°45′	12–13	19
1966, 24.10.	N 57°41′ E 11°35′	12–15	30
1967, 16.8.	N 57°40′ E 11°35′	21–25	30
7.12.	N 55°47′ E 12°42′	9–10	14
8.12.	N 55°47′ E 12°42′	9–10	14
20.12.	N 57°42′ E 11°34′	19–23	30
21.12.	N 57°43′ E 11°33′	17–21	30
1968, 20.3.	N 55°47′ E 12°41′	9–10	14
25.3.	N 55°46′ E 12°52′	2	20

enough, Decca navigation is used. The tracing period varies between 5 and 15 hours.

In connection with the tracing, the salinity, temperature, and current profiles are observed. If possible, these observations are carried out before, during, and after the tracing. In some cases (21.6., 8.12., 20.3.) only one ship was available, why the observations of the current could not be made during the tracing. The current is measured during several hours at intervals of 1–2 m, using Ekman current meters. The measurements are concentrated at the depth where the rhodamine is injected. From the observations are determined average values of the stratification parameter $N^2 = g \cdot \Delta\varrho / \varrho\Delta z = (g\Delta\sigma_t/\varrho\Delta z)\cdot 10^{-3}$, $\sigma_t = (\varrho - 1)\cdot 10^3$, and the vertical gradient of the horizontal current vector $|\Delta q/\Delta z|$, i.e. the shear. These parameters are calculated for the depth layer where the rhodamine is found. From the total set of observations mean values are determined (Table 2), to be used in subsequent calculations. The wind is measured at intervals, and a mean value (Table 2) is used in the calculations. For the primary data it is referred to I.

2. The vertical diffusion

This is investigated by means of the simplified diffusion equation

$$\frac{\partial C}{\partial t} = K\frac{\partial^2 C}{\partial z^2} \tag{1}$$

The turbulent diffusion coefficient K is regarded as a constant in each experiment. Two separate cases are studied, viz. situations with weak (A) and with strong (B) diffusion. In case (A) the diffusion is determined from observations on horizontally extended, thin layers of rhodamine, and the neglect of other terms in the equation is regarded as a plausible approximation. In case (B) the influence of the horizontal diffusion is small compared with the vertical diffusion, and the observations are made on relatively small spots.

(A) The first case is characterized by a strong stratification. Thin layers of rhodamine, 30–70 cm thick, are observed, having virtually constant thickness during limited periods of time, but with decreasing concentration. The experiments are performed in regions of considerable vertical mean shear. The shear acts as a stretching agency on the rhodamine spot, decreasing the thickness of the dyed layer. The vertical diffusion increases the thickness, and an approximate balance between the two mechanisms is visualized. The decrease of the concentration is attributed to the vertical diffusion only. It seems plausible that these layers, with high rhodamine concentration, observed during considerable time after release, are preferably situated in depth layers where the turbulence is weakly developed. Whether this depends upon an enhanced density gradient or a weak vertical shear in the layer, cannot be established from the present observations, since the measurements are not made on a small enough vertical scale. Due to the action of the current shear outside the layer, continuously sweeping clean water along the layer, the concentration outside is kept at a level below the limit of detection.

Mathematically this is formulated by solving (1) subject to the conditions: at time $t = 0$, $C = C_0$ in the layer $-h < z < +h$, and $C = 0$ for $[z] \geq h$; at time $t > 0$, $C = 0$ for $[z] \geq h$.

The solution is

$$C = \frac{4C_0}{\pi}\sum_0^\infty \frac{(-1)^n}{2n+1} \exp - K(2n+1)^2\frac{\pi^2 t}{4h^2} \times \cos\frac{(2n+1)\pi}{2h}z \tag{2}$$

This solution is used in I. The series converges rapidly, and with the present values the second term is estimated to be at most 10% of the first. Considering the individual scatter of the

Table 2. *Diffusion coefficient K, stratification N^2, wind W, shear, $|dq/dz|$, and parameter $(W^2/N^2)\,|dq/dz|$*

Date	Mean K (cm²/s)	Interval K (cm²/s)	N^2 (s^{-2})	W (m/s)	$\left\|\frac{dq}{dz}\right\| \cdot 10^2$ (s^{-1})	$\frac{W^2}{N^2}\left\|\frac{dq}{dz}\right\|$ (cm²/s)
1965, 21.6.	0.09	0.05–0.12	$5.6\cdot10^{-2}$	10	5.3	$9.5\cdot10^{5}$
17.11.	0.08	0.05–0.1	$7.4\cdot10^{-2}$	10	9.7	$1.3\cdot10^{6}$
1966, 24.10.	0.2	0.1–0.3	$4.9\cdot10^{-3}$	4–5	4.8	$2.0\cdot10^{6}$
1967, 16.8.	0.35	—	$9.6\cdot10^{-3}$	8–9	6.0	$4.5\cdot10^{6}$
7.12.	35	25–54	$8.0\cdot10^{-5}$	8	5.3	$4.3\cdot10^{8}$
8.12.	60	50–110	$2.2\cdot10^{-5}$	6	4.1	$6.7\cdot10^{8}$
20.12.	0.7	0.45–1.0	$1.5\cdot10^{-2}$	5	4.8	$8.0\cdot10^{6}$
21.12.	1.5	0.85–2.1	$2.5\cdot10^{-3}$	8	7.7	$2.0\cdot10^{7}$
1968, 20.3.	0.9	—	$4.9\cdot10^{-3}$	9	5.0	$8.3\cdot10^{6}$
25.3.	5	—	$2.6\cdot10^{-4}$	5	7.0	$6.7\cdot10^{7}$

values, only the first term is used for the calculation of the diffusion coefficient. However, if instead of a homogeneous initial distribution the initial distribution

$$C = C_0 \cos \frac{\pi z}{2h}$$

is used, the simpler, and finite expression

$$C = C_0 \exp - K\frac{\pi^2 t}{4h^2} \times \cos \frac{\pi z}{2h} \qquad (3)$$

is obtained for the concentration distribution in the layer. Outside the layer, and at the boundaries, the concentration is always zero. This is accomplished by assuming that the amount of substance diffusing out of the layer is immediately carried away by the shear. The initial distribution is probably neither homogeneous nor of a pure cosine form, but the use of the latter is justified because it gives the simple solution (3). The concentration is evaluated in the center of the layer, i.e. at $z = 0$, which gives

$$C = C_0 \exp - K\frac{\pi^2 t}{4h^2} \qquad (4)$$

Observations of the concentrations (C_1 and C_2) at two different times (t_1 and t_2) in the experiment are now used to eliminate the initial concentration C_0, yielding

$$\frac{C_2}{C_1} = \exp - K\frac{\pi^2}{4h^2}(t_2 - t_1)$$

$$K = \frac{4h^2}{\pi^2(t_2 - t_1)} \ln \frac{C_1}{C_2} \qquad (5)$$

The time interval is of the order of magnitude 0.5–1.5 hours. The model is an attempt to determine the diffusion coefficient from small scale, short time observations in conditions of strong stratification and shear flow. The vertical exchange is not necessarily a continuous process, but by using a model of this kind we obtain, through the diffusion coefficient, a measure of the mean vertical exchange, which can be related to the surrounding conditions. Probably the mean picture in space is not much altered by assuming the process to be continuous.

Using the value of $2h$ for the thickness of the spot at time $t = 0$ is only approximately correct. This is the thickness observed at times t_1 and t_2, and the initial thickness is usually larger. It is further necessary to observe that the thickness of the spot varies within certain limits, which explains most of the scatter of the K values (Table 2, four first values).

In the model no regard is paid to the effect of the horizontal diffusion. This effect is not very important when the measurements are carried out in the centre of a large spot. However, this is not always the case. The spot is of limited extent, the observations are made in different areas of the spot, and several observations are used to derive a mean value. The purely horizontal diffusion is very weak, see I.

(*B*) In the case of weak stratification the diffusion is stronger, and the thickness of the rhodamine layer increases with time. The stretching action of the current shear is not so marked in this case.

If homogeneous initial concentration is assumed the solution of (1) appropriate to extended initial distributions is

$$\frac{C}{C_0} = \frac{1}{2}\left[\operatorname{erf}\frac{1-z/h}{2\sqrt{Kt/h^2}} + \operatorname{erf}\frac{1+z/h}{2\sqrt{Kt/h^2}}\right] \tag{6}$$

The primary data are given in I and in Kullenberg (1971).

The mean values of K (Table 2, six last values) are determined from several observations in each experiment. The range of the values is stated in the table. In some cases the determination of the initial concentration has been difficult, which explains part of the scatter. In general the thickness of the spot can only be determined within $\pm 10\%$.

3. Vertical diffusion in relation to oceanographical conditions

In I it was found empirically that K could be related to the stratification, the shear and the wind, and the dimensionally correct form

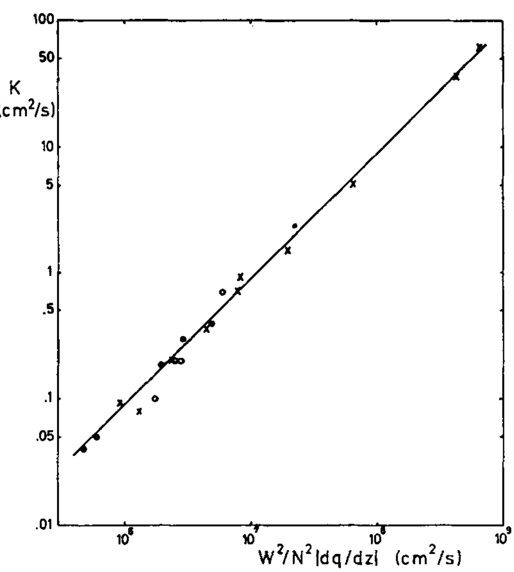

Fig. 1. The vertical diffusion coefficient K vs. the parameter $(W^2/N^2)\,|dq/dz|$. x, present values; ●, Jacobsen; ○, Hansen.

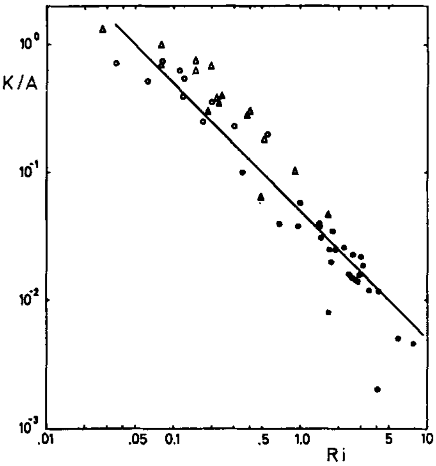

Fig. 2. Ratio K/A of vertical diffusion coefficient and momentum transport coefficient vs. Richardson number Ri. ●, Lofquist; ▲, Bowden; ○, Ellison & Turner, 1.0 cm; △, Ellison & Turner, 1.5 cm; the line indicates $Rf = 0.05$.

$$K = \text{const. } W^2 N^{-2}\left|\frac{dq}{dz}\right| \tag{7}$$

was proposed.

The diffusion coefficient is related to shear, stratification, and internal stress as follows. The turbulent diffusive transport of mass in a layer of thickness Δz is studied. The z-axis is positive downwards and the density stratification is stable. The diffusion increases the potential energy. The work against gravity per unit time and volume which must be carried out to accomplish this is

$$gK\frac{d\varrho}{dz} \tag{8}$$

which is found from the diffusion equation. The energy supplied per unit time and volume by the shear stress τ is

$$-\frac{d}{dz}(q\tau) = -\tau\frac{dq}{dz}$$

assuming τ to be constant in the layer.

By introducing the flux Richardson number Rf, defined as the ratio of the energy consumed by work against buoyancy to the total turbulent energy (Bowden, 1960), it is found that

$$K = \cdot \mathrm{Rf}\frac{\tau}{\varrho}\frac{dq}{dz}N^{-2} \tag{9}$$

Table 3. *Diffusion coefficient K, stratification N^2, wind W, shear $|dq/dz|$, and parameter $(W^2/N^2)\,|dq/dz|$*

Depth (m)	K (cm²/s)	N^2 (s⁻²)	W (m/s)	$\|dq/dz\|\cdot 10^2$ (s⁻¹)	$(W^2/N^2)\,\|dq/dz\|$ (cm²/s)
2.5	0.3	$8.6\cdot 10^{-4}$	5	1.0	$2.9\cdot 10^6$
5	0.4	$8.6\cdot 10^{-4}$	5	1.7	$4.9\cdot 10^6$
7.5	0.18	$2.8\cdot 10^{-3}$	5	2.2	$2.0\cdot 10^6$
10	0.05	$1.0\cdot 10^{-2}$	5	2.4	$6.0\cdot 10^5$
12.5	0.04	$1.0\cdot 10^{-2}$	5	1.9	$4.7\cdot 10^5$

This equation relates K to the stress, the shear, and the stratification.

In I it was found that for wind speeds above 4 m/s, the wind has a marked influence on the diffusion. In some cases the wind effect can be introduced in (9) by assuming the stress τ to be proportional to the wind stress τ_0. Considering the shallowness of the experiments it may be permissible to put τ equal to τ_0 (Phillips, 1966). Introducing a wind stress of the form $\tau_0 = c_d \cdot \varrho_a \cdot W^2$ in (9) gives

$$K = -\mathrm{Rf}\, c_d \frac{\varrho_a}{\varrho} W^2 N^{-2} \frac{dq}{dz} \qquad (10)$$

In Fig. 1 the values of K are given vs.

$$W^2 N^{-2} \left|\frac{dq}{dz}\right|$$

(Table 2). The least square calculated line is

$$K = 8.1 \cdot 10^{-8} \left[W^2 N^{-2} \left|\frac{dq}{dz}\right| \right]^{1.0} \qquad (11)$$

In order to be able to compare the experimental numerical coefficient in (11) with the theoretical one in (10) the value of Rf must be known. From investigations with simultaneous determinations of K, A, and Ri the range of Rf can be determined, which has a bearing on the present conditions of stability.

Lofquist (1970) made laboratory measurements in shear flow from which simultaneous values of K/A and Ri were determined (Fig. 2), (Sjöberg, 1967). The mean value of Rf was 0.045.

Ellison & Turner (1960) carried out an investigation of mixing of dense fluid in a turbulent pipe flow. From the measurements both K/A and Ri were found (Fig. 2). Two sets of experiments, with the depth of the layer of dense fluid being 1.0 and 1.5 cm, yield for Rf mean values of 0.057 and 0.092, respectively.

Bowden (1962) made measurements of both mixing and stability in a tidal current, from which K/A and Ri were found (Fig. 2). The mean value of Rf is 0.055.

Webster (1964) investigated turbulence in a density stratified shear flow in a wind tunnel. He determined Ri and the ratio K_H/A. These covered a large range, some of the values being very high. They will not be used here.

With reference to Fig. 2 we chose as a mean value $Rf = 0.05$, and assume this to cover the whole range of the present conditions of stabil-

Table 4. *Diffusion coefficient K, stratification N^2, wind W, shear $|dq/dz|$, and parameter $(W^2/N^2)\,|dq/dz|$*

Date	K (cm²/s)	N^2 (s⁻²)	W (m/s)	$\|dq/dz\|\cdot 10^2$ (s⁻¹)	$(W^2/N^2)\,\|dq/dz\|$ (cm²/s) Mean	Interval
1967, 14.9.	0.2	$3.4\cdot 10^{-3}$	5	5.0	$2.5\cdot 10^6$	$(1.3–3.7)\cdot 10^6$
14.12.	0.2	$4.0\cdot 10^{-3}$	5	6.2	$2.7\cdot 10^6$	$(1.5–3.9)\cdot 10^6$
1968, 22.3.	0.1	$(1.4–1.7)\cdot 10^{-2}$	7–10	3.2	$1.8\cdot 10^6$	$(1.2–2.3)\cdot 10^6$
29.3.	0.7	$(1.1–0.4)\cdot 10^{-2}$	5	13.9	$6.0\cdot 10^6$	$(3.2–8.7)\cdot 10^6$

ity. From investigations of the drag coefficient c_d, in largely landlocked areas with shallow water, a range of $0.95 \leq c_d \cdot 10^3 \leq 1.5$ is found (Deacon & Webb, 1960). With these estimates, and the value of $\varrho_a/\varrho = 1.2 \cdot 10^{-3}$, we find $5.7 \cdot 10^{-8} \leq Rf \cdot c_d \cdot \varrho_a/\varrho \leq 9.0 \cdot 10^{-8}$, which is consistent with the experimental coefficient in (11). Of course the values of Rf and c_d may vary, though within rather narrow limits, causing a corresponding variation of the coefficient. Nevertheless the theoretical and the experimental values agree well, and it is concluded that the experimental results support the theoretical expression (10), and indirectly (9).

4. Comparison with other investigations

To check the validity of the present results it is of interest to compare with other investigations in the same areas.

Jacobsen (1913) determined the vertical diffusion coefficient from long time observations of current and salinity in the Kattegatt (Table 3). We determine from the daily observations at the light vessel *Schultz Grund* (Naut. Met. Ann. 1910, 1911) the corresponding mean values of the stratification and the wind (Table 3), and calculate the mean shear from the current measurements, which are presented by Jacobsen. The data are plotted in Fig. 1. Despite the general agreement with the present results, some caution is necessary since the data of Jacobsen are long time mean values. However, the agreement is such as to support the theoretical model.

In Table 4 are given values observed by Hansen (1968) in the Öresund. From tracer measurements with a radioactive isotope the diffusion coefficients were calculated using a Fickian model of the diffusion, which is appropriate for a continuous point source (Harremoes, 1967). Density and current profiles were observed simultaneously and from these observations the mean values of stratification and shear have been determined (Table 4). The wind was also observed. The data are plotted in Fig. 1. This material is similar to the present, and the general agreement conforms with the present results. Using the complete set of data the equation

$$K = 8.9 \cdot 10^{-8} W^2 N^{-2} \left|\frac{dq}{dz}\right|$$

is calculated.

5. Conclusions

Despite the shortcomings of the material the experiments clearly manifest the dependence of the vertical diffusion on stratification, shear, and wind. The proposed theoretical model is supported by the observations.

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REFERENCES

Bowden, K. F. 1960. Turbulence. In *The Sea. I* (ed. M. N. Hill), pp. 802–825.

— 1966. Stability effects on turbulent mixing in tidal currents. In *Boundary layers and turbulence* (ed. K. F. Bowden, F. N. Frenkiel, and I. Tani), pp. 278–280. IUGG-IUTAM Symposium.

Deacon, E. L. & Webb, E. K. 1960. Small-scale interactions. In *The Sea. I* (ed. M. N. Hill), pp. 43–87.

Ellison, T. H. & Turner, J. S. 1960. Mixing of dense fluid in a turbulent pipe flow, part I and II, *J. Fluid Mech.*, *8*, 514–544.

Hansen, J. 1968. Recipient investigations in the Øresund outside Skovshoved Harbour, *Rep. No. 2* (in Danish). The Danish Isotope Center, Copenhagen, 36 pp.

Harremoes, P. 1966. Recipient investigations for the municipalities of Helsingør and Helsingborg, *Rep. No. 5* (in Danish). The Danish Isotope Center, Copenhagen, 29 pp.

Jacobsen, J. P. 1913. *Beitrag zur Hydrographie der dänischen Gewässer*, Medd. Komm. Havunders., Ser. Hydrogr. 1, Nr. 2, 94 + 14 pp.

Kullenberg, G. 1969. Measurements of horizontal and vertical diffusion in coastal waters. *Kungl. Vetenskaps- och Vitterhets-Samhället Acta Geoph. 2, Göteborg*, 52 pp.

Kullenberg, G. 1971. Results of diffusion experiments in the upper region of the sea. *Rep. No. 12.* Inst. Phys. Oceanography, University of Copenhagen, 50 pp.

Lofquist, K. 1960. Flow and stress near an inter-

face between stratified liquids. *The Physics of Fluids*, 3, 2, pp. 158–175.
Nautical-Meteorological annual 1910, 1911. The Danish Met. Inst., Copenhagen, 222 pp., 204 pp.
Phillips, O. M. 1966. *The dynamics of the upper ocean*. Cambridge University Press, Cambridge. 261 pp.
Sjöberg, A. 1967. Diffusive properties of interfacial layers. *Contrib. to the XII IAHR congr.*, 8 pp.
Webster, C. A. G. 1964. An experimental study of turbulence in a density stratified shear flow. *J. Fluid Mech. 19*, 221–245.

ВЕРТИКАЛЬНАЯ ДИФФУЗИЯ В МЕЛКИХ ВОДАХ

Из трассерных измерений определяется коэффициент вертикальной диффузии в стратифицированной мелкой воде с вертикальным градиентом горизонтальной скорости. Используются две различные модели, отдельно рассматривающие случаи сильной и слабой вертикальной стратификации. Коэффициент диффузии связан со стратификацией, напряжением трения и с градиентом скорости. При продолжительном ветре со скоростью выше 4–5 м/сек напряжение трения в верхнем слое предполагается пропорциональным напряжению ветра. В этом специальном случае теоретические и экспериментальные результаты могут быть сравнены и было найдено их хорошее согласие.