A note on the non-geostrophic baroclinic stability problem

By T. J. SIMONS, *Colorado State University*

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ABSTRACT

The baroclinic stability properties of non-geostrophic atmospheric flow in a channel are investigated. The model is formulated so as to include the effects of latitudinal variations of the perturbations and the basic state parameters, which constitute an essential aspect of the non-geostrophic baroclinic stability problem. Spectral and finite-difference techniques are applied to obtain numerical solutions and the results are found in agreement. In order to allow for high horizontal resolution the present calculations are restricted to a particular two-layer model which appears to incorporate the major non-geostrophic aspects of the baroclinic instability problem.

A purely non-geostrophic type of baroclinic instability is found which bears some resemblance to barotropic instability in spite of the absence of latitudinal shear in the zonal flow. This instability is introduced by the twisting term in the vorticity equation and the non-geostrophic advection of temperature in the thermodynamic equation. For low Richardson numbers and short cyclonic waves this instability attains the same order of magnitude as the quasi-geostrophic baroclinic instability. The major instability found in the present model however is not different from the major quasi-geostrophic instability.

Phillips (1964) has suggested that the most important non-geostrophic effects in baroclinic stability studies might be related to the dependence of the perturbations on the horizontal coordinate normal to the basic current. It is the purpose of the present investigation to make a quantitative evaluation of these effects by computing the stability properties of a baroclinic basic current in a channel.

The characteristic equation for the phase speed of the perturbation pertaining to the baroclinic stability problem may be written in non-dimensional form so as to involve the Rossby number $\mathrm{Ro} \equiv V/\Omega L$ where V is a characteristic velocity, L is a characteristic length, and Ω is the angular velocity of the earth's rotation. The quasi-geostrophic equivalent of the characteristic equation may then be obtained by retaining only the zero order terms in Ro and non-geostrophic corrections may be found by considering higher order terms in Ro. This approach was followed by Magata (1964), and Phillips (loc. cit.) pointed out explicitly that the first order Ro terms in the characteristic equation for the non-geostrophic Eady problem (and therefore the non-geostrophic corrections) are directly related to the latitudinal variations of the perturbations.

The main difficulty of the non-geostrophic baroclinic stability problem lies in the fact that the characteristic equation is not separable with respect to the latitudinal and vertical coordinates. Magata (loc. cit.) treated the latitudinal dependence approximately by differentiating the primitive system of equations with respect to the lateral coordinate y, then replacing the coefficients in the equations by constants, and regarding the differentiated perturbation quantities as separate dependent variables. Stone (1966) and Sela & Jacobs (1968) assumed periodic (exponential) solutions in the north-south direction. It should be recognized however that atmospheric circulations tend to be constrained in latitudinal direction and, consequently, it is probably more realistic study the instability of a flow in a channel.

The present study employs numerical methods to investigate the non-geostrophic stability character of a baroclinic channel flow. In order to evaluate the effects of the dependence of the perturbations and the static stability upon the lateral coordinate, it is desirable to retain a large horizontal resolution at the expense of a limited vertical resolution. Therefore a special two-layer model is employed which appears to incorporate all the effects of the north-south dependence present in more sophisticated models, and which is less sensitive to the spurious short wave cut-off to instability found in the more familiar two-level models.

The baroclinic stability problem is concerned with the behavior of small perturbations superimposed on a uni-directional basic current with vertical shear in a hydrostatic, inviscid, and adiabatic fluid. The present two-layer flow is described by equations (44) through (49) presented by Lorenz (1960) but with the balance equation (48) replaced by the divergence equation. Linearizing the equations and assuming periodic solutions in the zonal direction x and the time t of the form $\exp i(kx - \lambda t)$ we obtain

$$D\theta + iku^*\sigma + J(\psi, \bar{\theta}) + J(\tau, \bar{\sigma}) - \nabla.(\bar{\sigma}\nabla\chi) = 0 \quad (1)$$

$$D\sigma + iku^*\theta + J(\tau, \bar{\theta}) + J(\psi, \bar{\sigma}) - [\nabla\bar{\theta}.\nabla\chi] = 0 \quad (2)$$

$$D\nabla^2\psi + J(\psi, f) + iku^*\nabla^2\tau + [u^*\partial(\nabla^2\chi)/\partial y] = 0 \quad (3)$$

$$D\nabla^2\tau + J(\tau, f) + iku^*\nabla^2\psi - \nabla.(f\nabla\chi) = 0 \quad (4)$$

$$[D\nabla^2\chi + J(\chi, f)] - a\nabla^2\theta + \nabla.(f\nabla\tau) = 0 \quad (5)$$

where θ is the vertical-mean potential temperature, σ is the static stability, ψ and τ are the streamfunctions for the vertical-mean wind and the wind shear, and χ is the velocity potential of the lower layer, all of which are perturbation quantities except if denoted by a bar which refers to the basic state. ∇ is the isobaric gradient operator, J is the Jacobian operator, $D \equiv i(k\bar{u} - \lambda)$ where $\bar{u}$ is the vertical-mean basic state wind, and u^* is the basic state windshear measured over 250 mb. The Coriolis parameter f is assumed to be a linear function of the latitudinal coordinate y and the conditions for a balanced basic state may be written $d\bar{\theta}/dy = -bu^*f$, $d\bar{\sigma}/dy = cu^*f$. The numerical values of the constants a, b, and c, are found to be $a = 0.124\ C_p$, $b = 8.295/C_p$, $c = 3.101/C_p$ where C_p is the isobaric specific heat.

The usual quasi-geostrophic counterparts of the equations above are obtained by discarding all the terms within the square brackets and by treating $\bar{\sigma}$ and f as constants in the last terms of (1), (4), and (5). The bracketed terms in the divergence equation (5) constitute the difference between the energy-consistent balanced system of equations and the primitive equations, and therefore introduce the additional class of non-rotational modes of oscillation. Obviously, the latter is the only non-geostrophic effect retained if the perturbations are assumed to be of infinite width.

Equations (1) through (5) are to be solved subject to the boundary condition that the latitudinal velocity component must vanish at the walls of the channel. The equations are reduced to a system of algebraic equations by using series expansions in latitudinal direction and also, for comparison, by replacing the y-derivatives by finite differences. In either case the result is an eigenvalue problem where λ is the eigenvalue. The details will not be presented here, but similar solutions have been given by Rao & Simons (1970). Computations have been carried out for the values of the Coriolis parameter and its variation with y taken at 45 deg of latitude, for a mean zonal wind $\bar{u} = 20$ m/sec, a basic state static stability $\bar{\sigma} = 15$ deg measured over 250 mb, and for various wind shears and wave lengths. Only a few selected results are presented here which are representative of the general findings. The width of the channel is set at $W = 5\,000$ km. However, since the perturbation may in the lateral direction be decomposed into a set of perturbations of "effective width" $W_e \equiv W/n$, $n = 1, 2, 3, \ldots$, as is done explicitly in the spectral solution, we effectively consider a large number of lateral wave numbers.

If $\bar{\sigma}$ and f are treated as constants in the last terms of (1), (4), and (5), the equations have constant coefficients and permit periodic solutions of the form $\exp(i\pi y/W_e)$. In this case the "effective width" W_e of the perturbation refers to one half of the lateral wavelength. Two such periodic solutions are of interest and will be discussed first. The first one applies to the equations obtained by discarding the terms within the square brackets, i.e., the quasi-geostrophic equations. Examples of this are presented in the first column of results in Table 1, denoted by QG. If we restrict the effective widths to $W_e = W/n$, $n = 1, 2, 3, \ldots$, where $W = 5\,000$ km, the lateral scales smaller than the ones shown in the table for each wavelength are stable due to the familiar short wave cut-off to instability. Gates (1961) has presented the quasi-geostrophic stability analysis of the present model for the same basic state parameters, $\bar{u}$ and $\bar{\sigma}$, but for perturbations of infinite width, and showed that the spurious short wave cut-off was considerably reduced as compared to the usual two-layer models.

Table 1

First column of results: instability of baroclinic perturbations periodic in lateral direction for quasi-geostrophic (QG) and non-geostrophic (NG) case, as obtained from equations (1) through (5) with constant coefficients and without last terms of (2) and (3). Last three columns: primary baroclinic instability in two-layer channel model as obtained from spectral (SP) and finite-difference (FD) solutions for a channel of width $W = 5\ 000$ km. Imaginary parts of eigenvalues (in parentheses) representing growth rates per day preceded by real parts of eigenvalues divided by wave number k representing "wave speeds" in m/sec. Vertical shear between layers $2u^* = 25$ m/sec

Perturbation			Solutions for two-layer channel model		
			Without last terms of (2), (3)		
Wave length (km)	Effective width W/n (km)	Solutions periodic in lateral direction	Constant coefficients	y-dependent coefficients	Including last terms of (2), (3)
	5 000	QG 14.4 (±1.02)	SP 14.2 (±1.00)	14.2 (±.98)	14.1 (±.97)
		NG 14.2 (±1.00)	FD 14.2 (±1.00)	14.2 (±.98)	14.1 (±.97)
	2 500	QG 15.5 (±.91)	SP 15.3 (±.87)	15.3 (±.87)	15.3 (±.81)
		NG 15.4 (±.89)	FD 15.3 (±.86)	15.3 (±.86)	15.4 (±.81)
5 000	1 667	QG 15.7 (±.72)	SP 15.7 (±.68)	15.6 (±.70)	15.8 (±.64)
		NG 15.7 (±.71)	FD 15.7 (±.67)	15.5 (±.69)	15.8 (±.63)
	1 250	QG 15.7 (±.52)	SP 15.6 (±.49)	15.4 (±.52)	15.8 (±.43)
		NG 15.6 (±.51)	FD 15.5 (±.45)	15.3 (±.49)	15.7 (±.40)
					+
	1 000	QG 15.3 (±.29)	SP 15.3 (±.26)	15.2 (±.32)	secondary
		NG 15.3 (±.28)	FD 15.2 (±.18)	15.1 (±.25)	instability
	5 000	QG 15.4 (±.92)	SP 15.3 (±.81)	15.4 (±.90)	15.5 (±.60)
		NG 15.4 (±.81)	FD 15.3 (±.80)	15.4 (±.89)	15.5 (±.62)
2 000	2 500	QG 15.3 (±.73)	SP 15.2 (±.62)	15.3 (±.66)	
		NG 15.3 (±.62)	FD 15.2 (±.57)	15.3 (±.62)	+
	1 667	QG 15.1 (±.21)	SP 15.1 (±.08)	15.1 (±.26)	secondary
		NG 15.1 (±.14)	FD —	—	instability

The second periodic solution to be considered differs from the quasi-geostrophic solution in that the bracketed term of (5) is retained. This divergence effect has been discussed repeatedly in the meteorological literature and is in general found to reduce the quasi-geostrophic instability of the shorter wavelengths. Such non-geostrophic solutions have been entered into the first column of Table 1 below the quasi-geostrophic solutions and are identified by NG. Of course, this non-geostrophic effect increases with decreasing Richardson numbers which are associated with less geostrophic flow.

Turning now to the channel solutions subject to the previously stated lateral boundary conditions, it is obvious that the numerical solutions include simultaneously a spectrum of waves of lateral scales $W_e = W/n$, $n = 1, 2, 3, \dots$ Now the periodic solutions discussed above—which do not satisfy these boundary conditions—are very useful to identify the various eigenvalues and to correlate each unstable mode with a specific harmonic component of the y-dependent perturbation amplitude. This will enable us in the following to refer to a particular effective perturbation width in presenting a given unstable root. Recalling that the main purpose of the present study is to investigate the effects of the y-dependence of the perturbations and of the coefficients in the equations (1) to (5), we consider three cases, examples of which are shown in the last three columns of Table 1. For each case and for each perturbation two entries are shown; the upper one represents the spectral solution and the lower one is the finite-difference result. The spectral solution is truncated at 10 terms, resulting into a matrix of order 50, at which point the growth rates as shown in the table do no

Table 2. *Non-geostrophic instability in two-layer channel model resulting from twisting-tilting term in the vorticity equation and non-geostrophic advection of temperature in the thermodynamic equation, i.e., last terms of Eqs. (2) and (3)*

Spectral solutions truncated at $n = 10$. Real parts of eigenvalues divided by wave number k to obtain "wave speeds" in m/sec; imaginary parts of eigenvalues (in parentheses) represent growth rates per day. Width of channel $W = 5\ 000$ km, wave length of perturbation $L = 1\ 000$ km, shear between layers $2u^* = 25$ m/sec

Latitudinal wave number n									
1	2	3	4	5	6	7	8	9	10
Without last terms of (2) and (3)									
9.7	9.5	9.4	9.3	9.2	9.0	8.9	8.8	8.6	8.0
19.1	19.2	19.2	19.3	19.3	19.4	19.5	19.5	19.6	19.7
31.1	31.1	31.1	31.2	31.3	31.4	31.5	31.6	31.7	32.0
−20.6	−24.0	−27.3	−30.5	−33.8	−36.7	−38.0	−40.7	−45.9	−51.9
59.1	62.9	66.3	69.6	72.9	76.2	79.5	83.0	87.3	95.0
Including last terms of (2) and (3)									
9.4	9.3 (±.17)		9.1 (±.32)		8.9 (±.44)		8.8 (±.54)		8.2
19.4 (±.07)		19.4 (±.20)		19.4 (±.30)		19.5 (±.37)		19.6 (±.44)	
31.1	31.2 (±.05)		31.3 (±.11)		31.5 (±.16)		31.6 (±.21)		31.6
−20.5	−24.0	−27.3	−30.5	−33.8	−36.7	−38.1	−40.8	−45.9	−51.9
59.1	62.9	66.3	69.5	72.9	76.2	79.5	82.9	87.3	95.0

longer vary with increased resolution. The finite difference results refer to a resolution of 20 points, resulting into a matrix of order 100, at which point the growth rates still exhibit a slight increase with increasing resolution, which indicates a convergence toward the spectral results.

Considering the three types of channel solutions in turn, we first obtain the numerical solutions of equations (1) to (5) with constant $\bar{\sigma}$ and f in the last terms of (1), (4), and (5), and without the last terms of (2) and (3). The results shown in the second column of results in Table 1 may be compared with the periodic solutions denoted by NG which apply to the same equations but without lateral boundary conditions. Obviously, for this system of equations the effects of the boundary conditions are negligible which is also a known feature of the quasi-geostrophic system of equations. The next column of Table 1 shows the effects of the y-dependence of the coefficients $\bar{\sigma}$ and f in the last terms of (1), (4), and (5). For practical purposes this effect is negligible. The last column presents results for the third case which includes also the effects of the last terms of (2) and (3). These are therefore solutions of the complete equations (1) through (5). However, the numerical solutions for these equations produce two distinct classes of unstable modes to which we will refer as primary and secondary instability, respectively. The primary instabilities clearly correspond to the quasi-geostrophic unstable modes and only these are included in the last column of Table 1. For the larger scales of motion the differences between these unstable modes and those presented in the other columns of Table 1 are small. For the smaller scales the quasi geostrophic type of instability is reduced considerably, but here we observe a smooth transition toward the secondary instabilities which will be discussed presently.

The last terms of (2) and (3) obviously cause a strong coupling between the lateral harmonic components of the perturbation, and in effect seem to introduce a number of unstable modes, the nature of which does not resemble that of the familiar baroclinic instability. This may be illustrated with the help of Table 2. This table presents all eigenvalues corresponding to a wavelength $L = 1\ 000$ km as obtained from a spectral model truncated at 10 terms of the series. Again the collection of roots may be partitioned in sets of five roots belonging to a specific latitudinal wave number n by comparison with the periodic solutions. The upper part of Table 2 presents the eigenvalues for the equations without the last terms of (2) and

(3). As in the quasi-geostrophic case, all solutions are stable for $L = 1\ 000$ km. The lower part of the table presents the solutions of the complete equations for which virtually all roots are unstable except for the gravitational modes of oscillation represented by the last two rows. To identify the nature of the secondary instability we must recall that the quasi-geostrophic baroclinic instability results from the confluence of two eigenvalues associated with the same latitudinal wave number. However, the non-geostrophic instability in the lower part of Table 2 finds its origin in the confluence of eigenvalues within a horizontal row of Table 2 rather than within a vertical column. It results from the coupling of two latitudinal wave numbers and, as such, this instability is of a barotropic nature, which is of course precluded by the absence of latitudinal shear in the basic flow.

The secondary baroclinic instability is present in both the spectral and the finite-difference solutions and increases with shear as expected from the shear-dependence of the last terms of (2) and (3). The convergence of the secondary instabilities is rather poor, and with increasing horizontal resolution more and more unstable modes tend to be generated. The secondary instability for the shorter waves is of the same order of magnitude as the primary instability, but the non-geostrophic instability for the longer waves is completely overshadowed by the quasi-geostrophic type of instability. While it remains doubtful whether the secondary instability will exist in a continuous atmosphere, its existence cannot be discarded in view of the current use of numerical models of the atmosphere. Furthermore, the secondary instability is very difficult to separate from the primary instability in the cases where both exist together, thus causing a serious problem for numerical studies of dynamic instability.[1]

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REFERENCES

Gates, W. L. 1961. The stability properties and energy transformations of the two-layer model of variable static stability. *Tellus 13*, 460–471.

Lorenz, E. N. 1960. Energy and numerical weather prediction. *Tellus 12*, 364–373.

Magata, M. 1964. On the baroclinic instability and the Rossby number. *J. Meteorol. Soc. Japan 42*, 109–118.

Phillips, N. A. 1964. An overlooked aspect of the baroclinic stability problem. *Tellus 16*, 268–270.

Rao, D. B. & Simons, T. J. 1970. The stability of a sloping interface in a rotating two-fluid system. *Tellus* (to be published).

Sela, J. & Jacobs, S. J. 1968. On baroclinic and barotropic ageostrophic stability. Univ. of Michigan, Dept. of Meteorology and Oceanography, Technical Report, 98 pp.

Stone, P. 1966. On non-geostrophic baroclinic stability. *J. Atmos. Sci. 23*, 390–400.

ЗАМЕЧАНИЕ К ЗАДАЧЕ О НЕГЕОСТРОФИЧЕСКОЙ БАРОКЛИННОЙ УСТОЙЧИВОСТИ

Исследуются свойства бароклинной устойчивости негеострофического атмосферного потока в канале. Модель включает эффекты широтных изменений возмущений и основные параметры состояния потока, что является существенным аспектом задачи о негеострофической бароклинной устойчивости. Используются методы спектральных разложений и конечных разностей для получения численных решений, результаты которых находятся в согласии друг с другом. Для получения высокого разрешения по горизонтали эти вычисления ограничиваются определенной двухслойной моделью, которая представляется включающей в себя основные негеострофические аспекты задачи о бароклинной неустойчивости. Найден чисто негеострофический тип бароклинной неустойчивости, напоминающий баротропную неустойчивость, несмотря на отсутствие широтного градиента в зональном потоке. Эта неустойчивость вводится членом с кручением в уравнении вихря и негеострофической адвекцией температуры в термодинамическом уравнении. Для малых значений числа Ричардсона и коротких циклонических волн эта неустойчивость достигает того же порядка величины, что и квазигеострофическая бароклинная неустойчивость. Однако, основная неустойчивость, найденная в данной модели, не отличается от основной квазигеострофической неустойчивости.

[1] *Note added in proof:* Recently, a solution of the present problem for a continuous atmosphere has been given by J. Derome and C. L. Dolph (1970): "Three-dimensional non-geostrophic disturbances in a baroclinic zonal flow". Geophys. Fluid Dyn., vol 1, pp 91–123.