On the motion of various vertical modes of transient, very long waves*

Part I. *Beta Plane Approximation*

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ABSTRACT

The motion and structure of very long, transient waves are investigated by solving the perturbation problem using a basic state characterized by no motion and a constant lapse rate. It has been assumed that the motion is adiabatic and frictionless, and a beta-plane geometry has been used.

Using the method of separation of variables it has been shown that the solution consists of a discrete set of vertical structure functions each characterized by an equivalent depth which decreases as the eigenvalue increases. The existence of the smallest eigenvalue depends on the use of a boundary condition of $\omega \neq 0$ at the surface of the earth, while the higher eigenvalues are influenced only slightly if the boundary condition $\omega = 0$ is used.

The phase speed corresponding to the smallest eigenvalue is in good agreement with the results obtained from observational studies from 500 mb data or for the first vertical mode, and the wave corresponding to this eigenvalue behaves in a way similar to waves in a homogeneous fluid with a free surface. For the larger eigenvalues we obtain phase speeds which agree with the motion observed for the higher vertical modes.

It is finally shown that the equations based on Burger's assumption give good results for the higher vertical modes, but not for the mode corresponding to the smallest eigenvalue.

1. Introduction

The purpose of this paper is to discuss certain aspects of the dynamics of very long, transient waves in the atmosphere. A considerable amount of research has been done during the last few years on this subject. The interest in the behavior of the very long atmospheric waves has largely been created because of errors in numerical forecasts on the largest scale in the atmosphere (Cressman, 1958). However, in order to solve the forecast problem it has been necessary to obtain a better knowledge of the motion and structure of these waves from meteorological data. Several studies (Deland, 1964; Deland, 1965*a*; Eliasen & Machenhauer, 1965; Deland & Lin, 1967; Deland & Johnson, 1968; Bradley & Wiin-Nielsen, 1968; Eliasen & Machenhauer, 1969) have analyzed various aspects of the behavior of very long, transient waves using samples of atmospheric data. Some of the studies mentioned above include also prediction experiments in order to investigate whether or not a given atmospheric model is capable of predicting the motion of the very long, transient waves. In particular, Deland & Lin (1967) and Eliasen & Machenhauer (1969) have arrived at the conclusion that the transient, very long waves at 500 mb to some approximation can be described using a divergent barotropic model formulated as a homogeneous fluid with a free surface. The coefficient giving the intensity of the divergence was determined empirically by the above authors. However, Diky & Golitsyn (1968) showed, using Laplace's tidal equation, that they were able to reproduce the observed phase speeds by Eliasen & Machenhauer (1965) with excellent accuracy for most spherical harmonics.

While the studies mentioned above deal with the transient, very long waves at 500 mb,

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corresponding approximately to the vertically averaged flow, there are other studies such as Deland & Johnson (1968) and Bradley & Wiin-Nielsen (1968), which describe the vertical structure of the waves. It is shown that the transient modes have a considerable tilt in the vertical direction, and that their structure therefore is of a baroclinic nature. In particular, Bradley & Wiin-Nielsen (1968) find that three vertical pressure modes explain 98% or more of the variance of the geopotential. The three modes are for practical purposes identical to those determined by Obukhov (1960) and Holmström (1963). It was furthermore shown that the vertical mode 1 normally moves with a mean speed near those determined from a barotropic, divergent model, while vertical mode 2 has a mean speed from west to east. Comparing these results with the experiments in predicting the transient, very long waves using a divergent, barotropic model it is apparent that it is the vertical mode 1 which can be predicted using such a model, while the vertical mode 2 is of an entirely different nature. The reason that Eliasen & Machenhauer (1969) and Deland & Lin (1967) did not have to deal with vertical mode 2 is simply that they used 500 mb data in the prediction experiments, and vertical mode 2 is very small at this level. It is thus apparent that while the motion of vertical mode 1 is reasonably well described by a divergent, barotropic model, there exists no satisfactory description of the dynamics of vertical mode 2.

Burger (1958) and Murakami (1963) showed that in slow moving very long waves the vorticity equation and the divergence equation both become diagnostic. Deland (1965*b*), on the other hand, showed that if one assumes a wave speed of the order of the Rossby-Haurwitz speed one finds that the vorticity equation still has prognostic value. As pointed out by Bradley & Wiin-Nielsen (1968) vertical mode 1 obeys Deland's assumption, while vertical mode 2 may satisfy the assumption made by Burger (1958) and Murakami (1963) although its speed is not zero, but at least small.

The preliminary study of the dynamics of transient, very long waves made by Wiin-Nielsen (1961) was based on Burger's assumption. It appears now, after the diagnostic studies of transient, very long waves, that the results of that study may apply to vertical mode 2.

It is the purpose of this study to determine from a simple atmospheric model for which mode Burger's equations can be used to study the behavior of transient, very long waves. In addition we want to investigate the vertical structure of the various modes which will appear as solutions to the problem.

2. The model

The main purpose of the present study is to determine the nature of the various vertical modes and to make comparisons between various model assumptions. We shall not try in this paper to include various complications which greatly increase the mathematical complexity of the problem. In agreement with this attitude we shall adopt a beta-plane geometry as for example used by Lindzen (1967). The beta-plane which we shall use will be centered at 45° N where $f_0 = 10^{-4}\ \mathrm{sec}^{-1}$ and $\beta = 16 \times 10^{-12}\ \mathrm{m}^{-1}\ \mathrm{sec}^{-1}$. If a spherical geometry were used we would get a problem similar to the one treated by Diky & Golitsyn (1968) leading to the Laplace tidal equation and to solutions in terms of Hough functions. However, having the solution presented by Diky & Golitsyn (1968) for the first vertical mode we may compare one of our solutions with the more exact solution obtained by them.

We shall furthermore restrict the problem to a consideration of perturbation solutions using a basic state characterized by no motion. It is relatively easy to incorporate a constant basic current resulting in a constant additive term in the equation for the phase speed. The incorporation of a basic current having vertical and/or horizontal shear leads to much greater complications, and the solution of that problem will not be attempted in this paper. However, a numerical solution of the problem based upon Burger's assumptions will be presented in the near future. As we shall demonstrate here such a solution is a good approximation to the higher vertical modes.

The perturbation equations for a basic state of no motion are

$$\frac{\partial u}{\partial t} = -\frac{\partial \phi}{\partial x} + fv$$

$$\frac{\partial v}{\partial t} = -\frac{\partial \phi}{\partial y} - fu$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial \omega}{\partial p}=0$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial p}\right)+\bar{\sigma}\omega=0 \tag{2.1}$$

where (u, v, ω) are the eastward, northward and vertical component of the three dimensional velocity vector in the usual Cartesian coordinate system, where the positive x-direction is toward the east, the y-direction toward north and the vertical coordinate is pressure (p), ϕ is the geopotential, f the Coriolis parameter, and $\bar{\sigma} = -(\bar{\alpha}/\bar{\theta})\,(\partial\bar{\theta}/\partial p)$ a measure of static stability in the basic state. We shall assume that the perturbations have the form

$$u=\hat{u}(y,p)\,e^{ik(x-ct)}$$

$$v=\hat{v}(y,p)\,e^{ik(x-ct)}$$

$$\omega=\hat{\omega}(y,p)\,e^{ik(x-ct)}$$

$$\phi=\hat{\phi}(y,p)\,e^{ik(x-ct)} \tag{2.2}$$

where the amplitude is a function of y and p, $k=2\pi/L$ is the wave number, L the wave length and c the phase speed. Introducing (2.2) in (2.1) we get the following set of equations:

$$ikc\,\hat{u}+f\hat{v}=ik\hat{\phi}$$

$$-f\hat{u}+ikc\,\hat{v}=\frac{\partial\hat{\phi}}{\partial y}$$

$$ik\,\hat{u}+\frac{\partial\hat{v}}{\partial y}+\frac{\partial\hat{\omega}}{\partial p}=0$$

$$-ikc\frac{\partial\hat{\phi}}{\partial p}+\bar{\sigma}\hat{\omega}=0 \tag{2.3}$$

In order to reduce the system (2.3) to a single differential equation in one dependent variable we start by solving the first two equations for $\hat{u}$ and $\hat{v}$. We get

$$\hat{u}=-\frac{1}{\Delta}\left(k^2c\hat{\phi}+f\frac{\partial\phi}{\partial y}\right)$$

$$\hat{v}=\frac{ik}{\Delta}\left(f\hat{\phi}+c\frac{\partial\hat{\phi}}{\partial y}\right) \qquad \Delta=f^2-k^2c^2 \tag{2.4}$$

The expressions (2.4) are substituted in the third equation of (2.3), and we obtain

$$-\frac{ik}{\Delta}\left(k^2c\hat{\phi}+f\frac{\partial\hat{\phi}}{\partial y}\right)+ik\frac{\partial}{\partial y}\times\left(\frac{1}{\Delta}\left(f\hat{\phi}+c\frac{\partial\hat{\phi}}{\partial y}\right)\right)+\frac{\partial\hat{\omega}}{\partial p}=0 \tag{2.5}$$

We get a single equation for $\hat{\phi}$ by solving the fourth equation in (2.3) for $\hat{\omega}$ and substituting in (2.5). The result is after some manipulations:

$$c\frac{\partial}{\partial p}\left(\frac{1}{\bar{\sigma}}\frac{\partial\hat{\phi}}{\partial p}\right)+\frac{c}{\Delta}\frac{\partial^2\hat{\phi}}{dy^2}-\frac{2\beta f}{\Delta^2}c\frac{\partial\hat{\phi}}{dy}-\left[\frac{\beta(f^2+k^2c^2)}{\Delta^2}+\frac{k^2c}{\Delta}\right]\hat{\phi}=0 \tag{2.6}$$

where we have used the assumption

$$f=f_0+\beta y \tag{2.7}$$

where f_0 and β are constants.

Equation (2.6) represents an eigenvalue problem where $\hat{\phi}\,(y,p)$ is the eigenfunction and c the eigenvalue. The equation must be solved with suitable boundary conditions to which we shall return in a moment. The types of wave solutions to (2.6) may be characterized as inertial waves, gravity waves and Rossby waves. In this problem we are mainly interested in the last category, the Rossby waves, which are characterized by relatively small values of the phase speed. We shall therefore simplify the problem in two ways. First of all we limit our interest to waves for which

$$kc<f \tag{2.8}$$

which means that we look for wave solutions where the speed is small compared with the speed of pure inertial waves. Secondly, we shall use the approximation, which has become practice on the beta plane, that $f=f_0=\text{const.}$ where f appears undifferentiated. (2.8) can then be written

$$c<\frac{f_0}{k}=\frac{f_0}{2\pi}L \tag{2.9}$$

and it is seen that (2.9) will be better satisfied the larger the wave length is. This is consistent with our desire to consider transient, very long waves. Using (2.8), which means that $\Delta\approx f_0^2$, (2.6) reduces to

$$c\frac{\partial}{\partial p}\left(\frac{1}{\bar{\sigma}}\frac{\partial\hat{\phi}}{\partial p}\right)+\frac{c}{f_0^2}\frac{\partial^2\hat{\phi}}{\partial y^2}-\frac{2\beta}{f_0^3}c\frac{\partial\hat{\phi}}{dy}-\left[\frac{\beta}{f_0^2}+\frac{k^2c}{f_0^2}\right]\hat{\phi}=0 \tag{2.10}$$

(2.10) can be solved by a separation of variables. Let us assume that

$$\hat{\phi}=F(p)\,G(y) \tag{2.11}$$

Substituting (2.11) in (2.10) we find the equation

$$\frac{d}{dp}\left(\frac{1}{\bar{\sigma}}\frac{dF}{dp}\right)+\frac{1}{gh}F=0 \tag{2.12}$$

$$\frac{d^2G}{dy^2}-\frac{2\beta}{f_0}\frac{dG}{dy}-\left[\frac{\beta}{c}+k^2+\frac{f_0^2}{gh}\right]G=0 \tag{2.13}$$

where the constant of separation is $(gh)^{-1}$. It is convenient to express this constant in such a form, where g is the acceleration of gravity, and h has the dimension of a length, called the equivalent depth (Diky & Golitsyn, 1968; Lindzen, 1967). (2.12) is the vertical structure equation, while (2.13) is the beta-plane counterpart of Laplace's tidal equation.

Considering now the boundary conditions we shall use $\hat{\omega}=0$, $p=0$. It is seen from the last equation in (2.3) that this condition corresponds to

$$\frac{dF}{dp}=0, \quad p=0 \tag{2.14}$$

At the lower boundary we shall use $gW=d\phi/dt=0$, or with the usual approximation of applying this condition at a pressure surface $p=p_0=\text{const.}$

$$\omega=\varrho_0\frac{\partial\phi}{\partial t}, \quad p=p_0 \tag{2.15}$$

It is seen that (2.15) corresponds to

$$\hat{\omega}=-ikc\varrho_0\hat{\phi}, \quad p=p_0 \tag{2.16}$$

Using again the last equation in the system (2.3) we find that (2.16) corresponds to

$$\frac{dF}{dp}+\frac{\bar{\sigma}_0 p_0}{RT_0}F=0, \quad p=p_0 \tag{2.17}$$

where we have used $\varrho_0=p_0/(RT_0)$ from the gas equation.

Introducing now a nondimensional vertical coordinate

$$p_*=\frac{p}{p_0}, \quad p_0=\text{const.} \tag{2.18}$$

we can express (2.14) and (2.17) in the form

$$\frac{dF}{dp_*}=0, \quad p_*=0$$

$$\frac{dF}{dp_*}+\frac{\sigma_0 p_0^2}{RT_0}F=0, \quad p_*=1 \tag{2.19}$$

The lateral boundary conditions will be formulated by assuming that the region is bounded to the south and north by rigid walls, separated by a distance d. At these walls we have: $\hat{v}=0$. It is then seen from the first two equations in the system (2.3) that $\hat{v}=0$ leads to

$$\frac{\partial\hat{\phi}}{\partial y}+\frac{f_0}{c}\hat{\phi}=0 \tag{2.20}$$

or

$$\frac{dG}{dy}+\frac{f_0}{c}G=0 \quad \text{at} \quad y=0,\ y=d \tag{2.21}$$

3. The vertical structure

Turning our attention to the solutions of (2.12) and (2.13) with the boundary conditions (2.19) and (2.21) it should be pointed out that Jacobs & Wiin-Nielsen (1966) have shown that (2.12) can be solved in terms of Bessel functions if $\bar{\sigma}$ is evaluated for a basic state characterized by a constant lapse rate. We shall use the same approach in this problem. Assuming therefore that the basic state is characterized by a constant lapse rate $\gamma=-\partial T/\partial z$ we find from the definition of $\bar{\sigma}$ and hydrostatic considerations that

$$\bar{\sigma}=\sigma_0 p_*^{-(2-(R\gamma/g))}, \sigma_0=\frac{RT_0}{P_0^2}\left(\frac{R}{c_p}-\frac{R\gamma}{g}\right) \tag{3.1}$$

Denoting

$$\frac{g}{R\gamma}=n+1 \tag{3.2}$$

we find that

$$\bar{\sigma}=\sigma_0 p_*^{-[(2n+1)/(n+1)]} \tag{3.3}$$

and (2.12) becomes:

$$\frac{d}{dp_*}\left(p_*^{(2n+1)/(n+1)}\frac{dF}{dp_*}\right)+\frac{\sigma_0 p_0^2}{gh}F=0 \quad (3.4)$$

Denoting

$$S^2=\frac{\sigma_0 p_0^2}{gh} \quad (3.5)$$

the solution satisfying (3.4) and the boundary condition at $p_*=0$ is

$$F=Ap_*^{-[(n)/(2(n+1))]}J_n[2(n+1)\,Sp_*^{(1)/(2(n+1))}] \quad (3.6)$$

where A is an arbitrary constant.

The eigenvalue S is now determined from the boundary condition at $p_*=1$ as expressed in (2.19). From (3.6) we find

$$\frac{dF}{dp^*}=-Sp_*^{-[(3n+1)/(2(n+1))]}J_{n+1}$$
$$\times[2(n+1)\,Sp_*^{(1)/(s(n+1))}] \quad (3.7)$$

Substituting in the second eqation in (2.19) we find

$$-SJ_{n+1}[2(n+1)\,S]+r^2J_n[2(n+1)\,S]=0, r^2=\frac{\sigma_0 p_0^2}{RT_0} \quad (3.8)$$

The eigenvalue S must be determined from the transcendental equation (3.8) for a given value of n. Adopting the two cases treated by Jacobs & Wiin-Nielsen (1966), i.e., $n=3$, corresponding to a lapse rate $\gamma=8.54°\mathrm{K\ km^{-1}}$, and $n=4$, corresponding to $\gamma=6.82°\mathrm{K\ km^{-1}}$, we get from (3.8)

$$-\tfrac{1}{8}\mu J_4(\mu)+r^2J_3(\mu)=0,\ \mu=8S,\ n=3,\ r^2=\frac{\sigma_0 p_0^2}{RT_0} \quad (3.9)$$

$$-\tfrac{1}{10}\mu J_4(\mu)+r^2J_3(\mu)=0,\ \mu=10S,\ n=4,\ r^2=\frac{\sigma_0 p_0^2}{RT_0} \quad (3.10)$$

(3.9) and (3.10) were solved using methods described in the appendix of this paper. Only the smallest eigenvalues have been determined. For $n=3$ we have:

$$\sigma=\sigma_0 p_*^{-1.75};\ \sigma_0=0.3 \quad (3.11)$$

$$F(p_*)=Ap_*^{-\frac{3}{8}}J_3[\mu p_*^{\frac{1}{8}}]$$

Table 1. *Eigenvalues, equivalent depths and levels of nondivergence for $n=3$ (see eq. (2.32))*

μ	S	$gh(\mathrm{m^2\,sec^{-2}})$	$h(m)$	p_*, ND
1.50	0.188	8.611×10^4	8 780	None
7.62	0.952	0.334×10^4	340	0.242
11.09	1.386	0.158×10^4	161	0.012, 0.360
14.40	1.800	0.093×10^4	95	0.002, 0.044 0.445

while the corresponding expressions for $n=4$ are:

$$\sigma=\sigma_0 p_*^{-1.80};\ \sigma_0=0.7 \quad (3.12)$$

$$F(p_*)=Ap_*^{-(2/5)}J_4[\mu p_*^{(1/10)}]$$

Table 1 contains a summary of the results obtained for $n=3$, while Table 2 gives the corresponding information for $n=4$. Included in the tables are the values of the equivalent depth computed from the formula

$$gh=\frac{\sigma_0 p_0^2}{S^2} \quad (3.13)$$

and the levels of nondivergence. These levels are determined as those at which $F(p_*)=0$. As seen from (2.5) the horizontal divergence is proportional to $F(p_*)$.

It is seen from Table 1 and Table 2 that the basic mode, corresponding to smallest eigenvalue, is characterized by a large value of the equivalent depth and no level at which the horizontal divergence vanishes. For the next eigenvalue there is one level of nondivergence located rather high in the troposphere, and the equivalent depth for this mode has decreased considerably as compared with the basic mode. The next modes have an increasing number of levels of nondivergence with decreasing values of the equivalent depth.

Table 2. *Eigenvalues, equivalent depths and levels of nondivergence for $n=4$ (see eq. (2.33))*

μ	S	$gh(\mathrm{m^2sec^{-2}})$	$h(m)$	p_*, ND
2.82	0.282	9.094×10^4	9 273	None
8.86	0.886	0.921×10^4	939	0.208
12.41	1.241	0.470×10^4	479	0.007, 0.318

Table 3. *Eigenvalues, equivalent depths and levels of nondivergence for $n = 3$ (boundary condition (3.14))*

μ	S	$gh(\mathrm{m}^2\,\mathrm{sec}^{-2})$	$h(m)$	p_*, ND
7.59	0.949	0.337×10^4	343	0.250
11.07	1.388	0.157×10^4	160	0.012, 0.367
14.37	1.797	0.094×10^4	95	0.002, 0.045, 0.452

The only difference between the modes determined here and those given by Jacobs & Wiin-Nielsen (1966) is the boundary condition at $p_* = 1$. In this study we have used the condition (2.15), while the earlier study used the condition $\hat{\omega} = 0$, $p_* = 1$ which is equivalent to

$$\frac{dF}{dp_*} = 0, \quad p_* = 1 \tag{3.14}$$

It is seen from (3.7) that (3.14) corresponds to

$$J_{n+1}[2(n+1)S] = 0, p_* = 1 \tag{3.15}$$

The zeroes of the Bessel function are tabulated, and it is a straightforward matter to prepare tables corresponding to $n = 3$ and $n = 4$ and therefore equivalent to Table 1 and Table 2. The results are presented as Table 3 and Table 4. A comparison between these tables and Tables 1 and 2, respectively, shows that the main effect of the condition (3.14) is to eliminate the lowest eigenvalue. This effect of the boundary condition $\omega = 0$, $p_* = 1$, is of course well known. It is furthermore seen that the condition (3.14) has only minor effects on the remaining modes, and one could safely use the condition (3.14), if the interest were limited to the higher modes.

An investigation of the vertical structure of the various modes using the expressions (3.11) and (3.12) for $F(p_*)$ reveals that the basic mode ($\mu = 1.50$ for $n = 3$, $\mu = 2.82$ for $n = 4$) in-

Table 4. *Eigenvalues, equivalent depths and levels of nondivergence for $n = 4$ (boundary condition (3.14))*

μ	S	$gh(\mathrm{m}^2\mathrm{sec}^{-2})$	$h(m)$	p_*, ND
8.77	0.877	0.940×10^4	959	0.235
12.34	1.234	0.475×10^4	484	0.008, 0.336

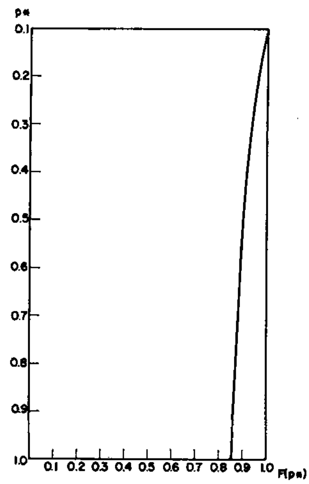

Fig. 1. The vertical structure function $F(p)$ for the smallest eigenvalue in the case $n = 4$ normalized to the value 1 for $p_* = 0.1$.

creases very slowly with decreasing pressure except when the pressure p_* becomes very small. This mode corresponds clearly to the type of wave motion which would be present in a homogeneous fluid with a free surface as investigated by Eliasen & Machenhauer (1965, 1969) and Deland et al. (1967, 1968). Due to this analogy we shall refer to the mode as the divergent barotropic mode. The other modes which are influenced by the boundary condition at $p_* = 1$ to a minor extent we shall call the internal modes. They are characterized by having one or more levels of nondivergence which also in this model are levels at which $F(p_*) = 0$. No empirical investigation of the motion and structure of the internal modes has been done except for the investigation by Bradley & Wiin-Nielsen (1968).

It should be pointed out that the vertical structure of the basic, divergent, barotropic

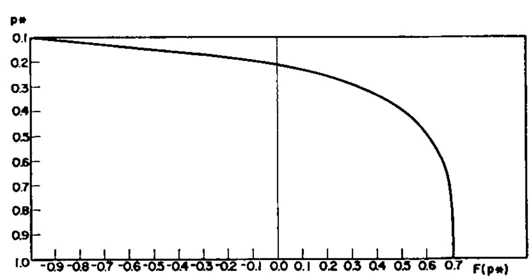

Fig. 2. The vertical structure function $F(p_*)$ for the second eigenvalue in the case $n = 4$ normalized to the value -1 for $p_* = 0.1$.

Table 5. *Phase speeds in msec^{-1} for transient waves with longitudinal wave numbers $N = 1, 2, 3$. based on values of gh taken from Table 1 for c and Table 3 for c_S ($n = 3$)*

Wave no.	$N=1$			$N=2$			$N=3$		
Mode	1	2	3	1	2	3	1	2	3
c	−54.9	−5.0	−2.5	−36.2	−4.8	−2.4	−23.1	−4.5	−2.3
c_S	—	−5.1	−2.4	—	−4.9	−2.4	—	−4.5	−2.3

mode is similar to the first mode as found by Holmström (1963) and Bradley & Wiin-Nielsen (1968) because $F(p_*)$ for this mode increases with decreasing pressure and has no zeroes. $F(p_*)$ for the next two modes as found here bear some resemblance to the second and third modes as found by the same authors, because $F(p_*)$ has the same number of zeroes, although it should be stressed that the theoretical modes are different from the empirical modes in the details. The difference is probably due to the fact that our theoretical model has an extremely simple basic state characterized by no motion and a very idealized stratification of which the latter fails to model the atmosphere correctly because there is no stratosphere in the model. However, if we are justified in identifying the theoretical vertical modes found here with those found in the empirical study by Bradley & Wiin-Nielsen (1968) we must be able to find a rough agreement also in the phase speeds for the modes. Since the empirical investigation gave zonal phase speeds for the first two transient modes, we shall also restrict our theoretical investigation to these two modes of which the basic, divergent, barotropic mode moves with a rather large speed from west to east, while the first internal mode moves with a speed from west to east which is comparable to the zonal current. In the next section we shall derive the phase speed for the various modes. The vertical structure $F(p_*)$ for the first two modes is shown in Fig. 1 and Fig. 2. These modes may be compared with those obtained by Holmström (1963).

4. The horizontal structure and phase speeds

In this section we shall solve equation (2.13) with the boundary conditions (2.21) at the two lateral walls. With the application of the beta plane approximation equation (2.13) is an ordinary second-order differential equation with constant coefficients. The solution is most easily analyzed if we first make the transformation

$$G = \exp \frac{\beta}{f_0} y \, G_* \tag{4.1}$$

which brings (2.13) into the form

$$\frac{d^2 G_*}{dy^2} + \delta^2 G_* = 0; \ \delta^2 = -\left(\frac{\beta^2}{f_0^2} + \frac{\beta}{c} + k^2 + \frac{f_0^2}{gh}\right) \tag{4.2}$$

while (2.21) becomes

$$\frac{dG_*}{dy} + \left(\frac{\beta}{f_0} + \frac{f_0}{c}\right) G_* = 0, \ y = 0 \quad \text{and} \quad y - d \tag{4.3}$$

The solution to (4.2) is

$$G_* = C \cos \delta y + D \sin \delta y \tag{4.4}$$

Table 6. *Phase speeds in msec^{-1} for transient waves with longitudinal wave numbers $N = 1, 2, 3$, based on values of gh taken from Table 2 for c and Table 4 for c_S ($n = 4$)*

Wave no.	$N=1$			$N=2$			$N=3$		
Mode	1	2	3	1	2	3	1	2	3
c	−56.0	−12.6	−6.9	−36.7	−11.3	−6.5	−23.3	−9.6	−5.9
c_S	—	−12.9	−7.0	—	—11.5	−6.1	—	−9.7	−5.6

Table 7. *Observed phase velocities of transient waves, January–March 1963, msec^{-1} relative to the zonal current (= 20 msec^{-1})*

Wave no.	1		2		3	
Mode	1	2	1	2	1	2
c	−49.0	−5.6	−35.0	−6.5	−23.0	−9.2

Substituting (4.4) in (4.2) and setting $y = 0$ and $y = d$ we get the equations

$$\delta D + \left(\frac{\beta}{f_0} + \frac{f_0}{c}\right) C = 0, \ y = 0 \qquad (4.5)$$

$$\delta d = m\pi, \ y = d, \ m = \pm 1, \ \pm 2, \ldots \qquad (4.6)$$

Using the definition of δ, given in (4.2), and the condition (4.6) we can solve for the phase speed c, and we obtain

$$c = -\frac{\beta}{\dfrac{\beta^2}{f_0^2} + k^2 + \dfrac{m^2\pi^2}{d^2} + \dfrac{f_0^2}{gh}} \qquad (4.7)$$

Using (4.1), (4.4), (4.5) and (4.6) we may write the solution for $G(y)$ in the form

$$G(y_*) = C \exp\frac{\beta}{f_0} dy_* \times \left[\cos(m\pi y_*) - \frac{d}{\pi}\left(\frac{\beta}{f_0} + \frac{f_0}{c}\right)\sin(m\pi y_*)\right] \qquad (4.8)$$

where
$$y_* = \frac{y}{d} \qquad (4.9)$$

The expression (4.7) can be used to calculate the phase speed of the various modes. In the numerical calculations we shall use $f_0 = 10^{-4}$ sec^{-1}, $\beta = 16 \times 10^{-12}$ m^{-1} sec^{-1}, $d = 10^7$ m, corresponding to the distance from equator to pole, and $m = 1$. Since we are mainly interested in the very long, transient waves we shall make calculations for longitudinal waves with wave numbers 1, 2, and 3 measured at 45° N. The values of gh in (4.7) are obtained from Tables 1, 2, 3, and 4. When we use values from either Table 3 or Table 4 we shall denote the corresponding value of the phase speed by c_S, corresponding the simplified lower condition. The results of these calculations are given in Table 5 ($n = 3$) and Table 6 ($n = 4$).

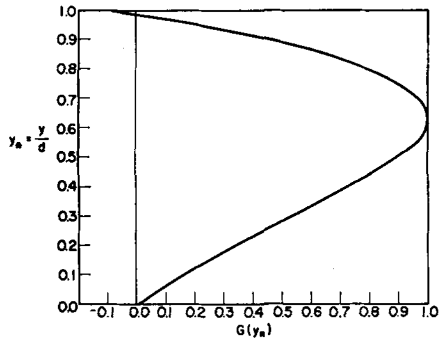

Fig. 3. The horizontal structure function $G(y_*)$ for the smallest eigenvalue in the case $n = 4$ normalized in such a way that the maximum value is 1. ($c = -56.0$ msec^{-1}.)

It is seen from Table 5 and Table 6 that the basic mode (mode 1) moves with a considerable speed from west to east. The speed depends rather strongly on the wave number N decreasing in absolute value as the wave number increases. The phase velocities given in Tables 5 and 6 are in reasonable agreement with those found in the observational study by Bradley & Wiin-Nielsen (1968) when it is noted that the values given in these tables are obtained for a model in which the motion in the basic state vanishes, while those obtained in the study based on data are relative to the earth. As noted in section 2 of this paper a constant zonal velocity in the basic state will only give an additive constant in the expression (4.7) for the phase velocity. Assuming a basic current of

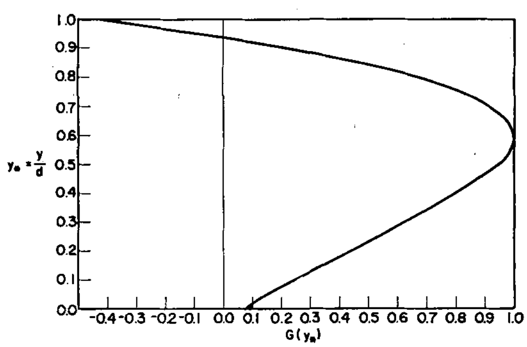

Fig. 4. The horizontal structure function $G(y_*)$ for the second eigenvalue in the case $n = 4$ normalized in such a way that the maximum value is 1. ($c = -12.6$ msec^{-1}.)

20 msec^{-1} we may convert the values obtained by Bradley & Wiin-Nielsen (1968) to numbers which are comparable to those given in Table 5 and Table 6. These values are listed in Table 7 for modes 1 and 2. In comparing Tables 5 and 6 with Table 7 it should be remembered that the latter table was determined from data for a relatively short period, and that the numerical values were obtained as a mean of daily values. We note further that vertical mode 2 moves much slower than mode 1, and that the phase speed depends much less on the longitudinal wave number. We note finally that only minor modifications are found in the phase speed when we use the simplified boundary condition $\omega = 0$, $p_* = 1$ for modes 2 and 3. The main reason that the higher vertical modes show less dependence on the longitudinal wave number and the width of the region is the small values of the equivalent depth for the higher modes as seen from Tables 1 to 4.

It should finally be noted the phase speeds derived in this study are comparable to those found from an observational study by Eliasen & Machenhauer (1969) from 500 mb data and for the components of the largest horizontal scale. Having data from only one level makes it difficult to decide to which vertical mode the observed phase velocity belongs. It seems, however, rather safe to assume that the largest negative phase velocity belongs to the first vertical mode. Neglecting the other phase velocities which always correspond to modes with a meridional scale smaller than the one considered here we have constructed Table 8 which applies to vertical mode 1 only.

The theoretical and observational results displayed in this section show that it is possible to describe in a qualitative and quantitative way the observed motion of transient, very long waves using a realistic boundary condition at the surface of the earth. Such a boundary condition is particularly important for the first vertical mode, while a simpler boundary condition $\omega = 0$, $p_* = 1$ may be used for the higher modes without a major error in the computed structure and the computed phase speed.

Fig. 3 and Fig. 4 show the function $G(y_*)$ which gives the latitudinal variation of the perturbation geopotential, computed from (4.8). The values for c were from Table 6. Fig. 3 shows $G(y_*)$ for the longitudinal wave number $N = 1$ and the first vertical mode, while Fig. 4 corresponds to the same value of N, but the second vertical mode. Each of the curves are normalized in such a way that the maximum value is unity.

5. Burger's assumption

The main assumption made by Burger (1958) in his scale analysis for waves with a wave length comparable in magnitude to the radius of the earth is that these waves move very slowly. It can be seen from this study and from the results obtained from observations by Eliasen & Machenhauer (1965, 1969) and Deland et al. (1967) that this assumption is invalid for certain components of the very long, transient waves, in particular the first vertical mode (the divergent barotropic mode) for the small longitudinal wave numbers. A modified scale analysis in which an observed value of the phase speed of these waves were taken into account has been presented by Deland (1965*b*).

The theoretical results presented in Tables 5 and 6 as well as the observational results in Table 7 indicate that the higher vertical modes move with a relatively small speed relative to the basic current. It is therefore likely that the speed of these modes may be predicted correctly from equations based on Burger's assumption. In order to investigate this possibility we shall repeat the analysis given in the previous sections of this paper when the basic equations (2.1) are replaced by the following set:

$$
\begin{aligned}
0 &= -\frac{\partial \phi}{\partial x} + fv \\
0 &= -\frac{\partial \phi}{\partial y} - fu \\
&\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \\
&\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial p}\right) + \bar{\sigma}\omega = 0
\end{aligned}
\tag{5.1}
$$

The difference between (5.1) and (2.1) is that the two equations of motion in (5.1) reduce to the geostrophic relations. Making the same assumptions as in (2.2) and (2.7) we now get the following equation, corresponding to (2.6):

$$c\frac{\partial}{\partial p}\left(\frac{1}{\bar{\sigma}}\frac{\partial \hat{\phi}}{\partial p}\right) - \frac{\beta}{f_0^2}\hat{\phi} = 0 \tag{5.2}$$

Table 8. *Observed phase velocities of transient waves in msec^{-1} from Eliasen & Machenhauer (1968) for 500 mb*

N	1	2	3
c	−62	−40	−22

Equation (5.2) may be brought into the same form as (2.12), the equation for the vertical modes, by writing

$$\frac{\partial}{\partial p}\left(\frac{1}{\bar{\sigma}}\frac{\partial\hat{\phi}}{\partial p}\right)+\frac{1}{gh}\hat{\phi}=0 \tag{5.3}$$

where, by definition:

$$gh=-\frac{f_0^2}{\beta}c_B \tag{5.4}$$

and we use the notation c_B for the phase speed determined from the equations (5.1). In view of the formal analogy between (5.3) and (2.12), we can adopt all the results obtained in section 3 for the function $F(p_*)$ and, in particular, the values of gh listed for the various cases in Tables 1–4. Having these values it is possible to compute c_B from (5.4). We may naturally also compute a value c_{BS} when we use the simplified lower boundary condition (3.14). The relation for the phase speed c_B may be written

$$c_B=\frac{\beta}{f_0^2/gh} \tag{5.5}$$

A comparison between (5.5) and (4.7) shows that c_B is a good approximation to c, whenever

$$\frac{\beta^2}{f_0^2}+k^2+\frac{m^2\pi^2}{d^2}<\frac{f_0^2}{gh} \tag{5.6}$$

Table 9. *Values of c_B and c_{BS} computed from (5.5) using values of gh from Table 1 and Table 3 ($n=3$)*

Mode	1	2	3
c_B	−137.8	−5.3	−2.5
c_{BS}	—	−5.4	−2.5

Table 10. *Values of c_B and c_{BS} computed from (5.5) using values of gh from Table 2 and Table 4 ($n=4$)*

Mode	1	2	3
c_B	−145.5	−14.7	−7.5
c_{BS}	—	−15.0	−7.6

It is seen that (5.6) is satisfied if gh is small. Table 9 contains values of c_B and c_{BS} computed on the basis of values for gh obtained from Table 1 and Table 3 ($n=3$). As expected we find that c_B is not a good approximation to c for the first vertical mode. However, a comparison between Table 9 and Table 5 shows that there is good agreement between c_B, c_{BS}, c_S, and c for the vertical modes 2 and 3. Table 10 gives c_B and c_{BS} computed on the basis of values for gh obtained from Table 2 and Table 4 ($n=4$). The agreement between Table 10 and Table 6 is not as good as the comparison between Table 9 and Table 5 because of the somewhat larger values of gh found in the case of $n=4$.

The main conclusion from this section is that it seems permissible to use Burger's assumption to investigate the dynamics of the higher vertical modes. In addition, it can be concluded that no great error is committed by applying the simplified lower boundary condition $\omega=0$, $p_*=1$. These conclusions greatly simplify investigations of the dynamics of baroclinic, transient, very long waves.

6. Concluding remarks

The primary purpose of this study has been to investigate very long transient waves. Using a simple basic state characterized by no motion and a constant lapse rate it has been shown that there exists a discrete set of eigenvalues and eigenfunctions. The smallest eigenvalue corresponds to a vertical eigenmode having a large equivalent depth and a large negative phase speed. The larger eigenvalues correspond to vertical eigenmodes with progressively smaller equivalent depths and smaller negative phase speeds. In order to include the smallest eigenvalue it is necessary to treat the boundary condition at the surface of the earth with some care, while good approximations to the larger eigenvalues can be obtained with the simpler boundary condition $\omega = 0$, $p = p_0$.

It is furthermore shown that Burger's assumption may be used to obtain the larger eigenvalues with good accuracy, while the assumption fails completely for the smallest eigenvalue, which is similar to the solution obtained if one adopts a model of a homogeneous fluid with a free surface. It seems, therefore, that one may be able to analyze the dynamics of the modes corresponding to all eigenvalues except the smallest by applying Burger's assumption and the simplified boundary condition $\omega = 0$, $p = p_0$.

Such a study, in which the basic current will have both vertical and horizontal shear, will be published later.

The analysis described here has been limited to a beta plane approximation with simple lateral boundary conditions corresponding to rigid walls at certain latitudes. It would naturally be possible, but laborious, to extend the analysis to a spherical geometry and apply methods similar to those used by Diky & Golitsyn (1968) or Eliasen & Machenhauer (1969) for the divergent barotropic mode.

APPENDIX

The transcendental equation to be solved in order to find the eigenvalues is equation (3.8). In the general case it involves the Bessel functions of order n and $n+1$. Available tables list only the functions J_0 and J_1. It is therefore necessary to rewrite (3.8) in such a way that these functions only appear in the equation. This goal can be accomplished by using the relation

$$\mu\, J_{n-1}(\mu) + \mu\, J_{n+1}(\mu) = 2n\, J_n(\mu) \qquad \text{(A.1)}$$

from which a given Bessel function can be expressed in terms of the two functions having the order of 1 and 2 below the given function.

In this investigation we have only solved (3.8) for the two cases $n = 3$ and $n = 4$ which result in reasonable atmospheric lapse rates. These cases are (3.9) and (3.10). By repeated use of (A.1) we can write (3.9) in the form

$$\frac{J_1(\mu)}{J_0(\mu)} = \frac{4r^2 - 3 + \frac{1}{8}\mu^2}{8r^2 - 6 + (1 - r^2)\,\mu^2}\,\mu = H(\mu); \quad n = 3 \qquad \text{(A.2)}$$

r^2 is a constant given in (3.9). Using (3.1) and (3.2) we find

$$r^2 = \frac{\sigma_0 p_0^2}{RT_0} = \frac{R}{C_p} - \frac{R\gamma}{g} = \frac{R}{C_p} - \frac{1}{n+1} \qquad \text{(A.3)}$$

For $n = 3$, we have therefore $r^2 = 0.036$. Equation (A.2) was solved by tabulating $J_1(\mu)/J_0(\mu)$ from available tables and computing $H(\mu)$ according to the defining expression in (A.2). The intersections of the curves $J_1(\mu)/J_0(\mu)$ and $H(\mu)$ determine the eigenvalues μ.

Using (A.1) we can transform (3.10) into the equation

$$\frac{J_1(\mu)}{J_0(\mu)} = \frac{24r^2 - 19.2 + (1.2 - r^2)\,\mu^2}{48r^2 - 38.4 + 2(1.2 - r^2)\,\mu^2 - (6r^2 - 4.8 + 0.1\mu^2)\,\mu^2} \times \mu = G(\mu) \qquad \text{(A.4)}$$

where r^2 according to (A.3) has the value 0.086. Equation (A.4) was solved in a way analogous to (A.2).

The procedures described above work well in practice for large values of μ. However, for small values of μ a very high accuracy is needed to determine the intersections of the curves. In order to determine the small eigenvalues it is, therefore, preferable to use a different procedure.

The functions $\Lambda_n(\mu)$ defined by the expression

$$J_n(\mu) = \frac{1}{n!}\left(\frac{\mu}{2}\right)^n \Lambda_n(\mu) \qquad \text{(A.5)}$$

are tabulated for $1 \leqslant n \leqslant 8$. Using the definition $\mu = 2(n+1)S$ in agreement with (3.9) and (3.10), we find by substituting (A.5) in (3.8) that

$$\mu^2 \frac{\Lambda_{n+1}(\mu)}{\Lambda_n(\mu)} = 4(n+1)\left[\frac{R}{C_p}(n+1)-1\right] \quad \text{(A.6)}$$

From (A.6) we find for $n=3$ and $n=4$ the equations

$$\mu^2 \frac{\Lambda_4(\mu)}{\Lambda_3(\mu)} = 2.304; \quad n=3 \quad \text{(A.7)}$$

$$\mu^2 \frac{\Lambda_4(\mu)}{\Lambda_4(\mu)} = 8.6; \quad n=4 \quad \text{(A.8)}$$

These equations are easy to solve using available tables for small values of μ.

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О ДВИЖЕНИИ РАЗЛИЧНЫХ ВЕРТИКАЛЬНЫХ МОД НЕСТАЦИОНАРНЫХ УЛЬТРАДЛИННЫХ ВОЛН

Исследуется движение и структура очень длинных нестационарных волн путем решения линеаризированных уравнений. В основном состоянии движение отсутствует, вертикальный градиент температуры постоянен. Движение предполагается адиабатическим и невязким, используется геометрия β-плоскости.

С помощью разделения переменных показано, что решение образует набор функций с различной вертикальной структурой, каждая из которых характеризуется эквивалентной глубиной, убывающей по мере возрастания собственного числа. Существование наименьшего собственного числа зависит от использования граничного условия $w \neq 0$ на поверхности земли. В то же время собственные числа более высокого порядка лишь слегка возмущаются при использовании граничного условия $w=0$. Фазовая скорость, соответствующая наименьшему собственному числу, хорошо согласуется с результатами наблюдений 500-миллибаровой поверхности или первой вертикальной моды. Волна, соответствующая этому собственному значению, ведет себя подобно волне в однородной жидкости со свободной поверхностью. Для больших собственных чисел мы получаем фазовые скорости, которые согласуются с наблюдаемым движением высоких вертикальных мод. В конце показано, что уравнения, основанные на допущении Бюргера, дают хорошие результаты для высоких вертикальных мод, но не для моды, соответствующей наименьшему собственному значению.