On the general analytic solution in the baroclinic instability problem

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ABSTRACT

Analytical solutions of the baroclinic instability problem are found for Green's model including compressibility and for Hirota's model formulated in pressure coordinates. Some numerical results corresponding to Green's model are presented in order to study the influence of the compressibility parameter.

I. Introduction

The studies of the baroclinic instability problem have normally been made considering models with wind profiles that are linear with height. These models were studied in x, y, z coordinates and therefore involved the mathematical convenience of the supposition of restrictions in the compressibility.

Green (1960) found numerical solutions of a model of this type extending between two rigid walls. He considered the mass divergence effect of the compressibility, describing it by a factor $\varkappa$ that he assumed constant with height. Garcia & Norscini (1970) obtained the exact solutions of Green's model for the incompressible case ($\varkappa = 0$), expressed by confluent hypergeometric functions.

Hirota (1968) calculated numerically the solutions of a model formulated in pressure coordinates therefore including compressibility without restrictions. The assumption was made that the wind shear and static stability were constant throughout the whole atmosphere.

We shall deduce the exact solutions for Hirota's model. The combination of the vorticity and thermodynamic equations leads to a single differential equation derived by Wiin-Nielsen (1967)

$$(U-C)\,(U-C-C_*)\frac{d^2\Omega}{dp_*^2}-[2(U-C)-C_*]\frac{dU}{dp_*}\frac{d\Omega}{dp_*} - \frac{k^2\bar{s}p_0^2}{f^2}(U-C-C_*)^2\,\Omega=0 \qquad (1)$$

where $p_* = p/p_0\,(p_0 = 1\,000\,\text{mb})$, $C_* = \beta/k^2$ and $\omega' = \Omega\exp[ik(x-Ct)]$

is the perturbation in the vertical velocity $\omega = dp/dt$. An x, y, p coordinate system is used; $U = U(p)$ is the basic flow; θ is the potential temperature;

$$\bar{s} = -\frac{1}{\varrho\theta}\frac{\partial\theta}{\partial p}$$

is a measure of the static stability; f is the Coriolis parameter with a latitudinal variation $\beta = df/dy$; k is the wave number; $C = C_R + iC_i$ is the velocity of the wave.

Let us define a basic state by $U = U_T(1-p_*)$ and $\bar{s}$ constant, where U_T is the velocity at the top of the atmosphere ($p_* = 0$).

Equation (1) can be transformed into a confluent hypergeometric equation. Substitution according to transformation

$$\begin{aligned} &\xi = 2\alpha(U-C) && \text{where } \alpha^2 = k^2\bar{s}p_0^2/f^2U_T^2 \\ &\Omega = \eta - \xi\eta' && \text{where } \eta = \alpha^{-1}\xi e^{-\xi/2}\Phi(\xi) \end{aligned} \qquad (2)$$

where the primes indicate differentiation with respect to ξ, and taking ξ as the vertical coordinate, leads to

$$\xi\Phi'' + (c-\xi)\Phi' - a\Phi = 0 \qquad (3)$$

with parameters $c = 2$ and $a = 1 - \frac{1}{2}\alpha C_*$.

To define the problem we set the boundary conditions $\Omega = 0$ at the top and on the surface, that means $p_* = 0$ and $p_* = 1$. With the same transformation (2)

$$\Phi - 2\Phi' = 0 \quad \text{for} \begin{cases} p_* = 0 \\ p_* = 1 \end{cases} \tag{4}$$

Although the equation (3) and the boundary conditions (4) that correspond to Hirota's model have the same mathematical formulation as those obtained by Garcia and Norscini corresponding to Green's model with $\varkappa = 0$, the solutions are not the same because of the differences in the definitions of the variable ξ and the parameter a.

In the next section we shall deduce the general analytic solutions for Green's model ($\varkappa \neq 0$), that are confluent hypergeometric functions as well.

2. Analytic solutions for Green's model ($\varkappa \neq 0$)

The equations for large scale motions may be given with a frictionless, adiabatic and quasi-geostrophic approximation. The latitudinal variation of the Coriolis parameter is regarded as constant.

$$\frac{\partial v}{\partial z} - \frac{1}{f}\frac{\partial \theta^*}{\partial x} = 0; \; \theta^* = g \ln \theta \tag{5}$$

$$\frac{D\zeta}{Dt} + v\beta + f\frac{\partial u}{\partial x} = 0 \tag{6}$$

$$\frac{D\theta^*}{Dt} - fv\frac{dU}{dz} + \sigma g w = 0 \tag{7}$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} - w\gamma = 0 \tag{8}$$

Equation (5) is the meridional component of the thermal wind equation; (6) the vertical component of the vorticity equation; (7) the energy equation and (8) the continuity equation, in x, y, z coordinates with u, v, w as zonal, meridional and vertical components of the velocity vector $\mathbf{v}$; ζ is the vertical component of the relative vorticity;

$$\gamma = -\frac{1}{\varrho}\frac{\partial \varrho}{\partial z} \quad \text{and} \quad \sigma = \frac{1}{\theta}\frac{\partial \theta}{\partial z}$$

are assumed constant in the model.

The system admits a basic zonal flow $U(z)$ with constant shear (β-plane approximation is considered).

Combining the linearized perturbation equations into a single differential equation for the meridional velocity

$$(U-C)\frac{\partial^2 v}{\partial z^2} - (U-C)\gamma\frac{\partial v}{\partial z} + \left\{\left[\frac{\beta}{k^2} - (U-C)\right]\frac{k^2}{f^2}\sigma g + \gamma\frac{dU}{dz}\right\} v = 0 \tag{9}$$

where we have neglected the dependence on the y-coordinate. The differential equation (9) is similar to the non-dimensional form derived by Green, and can be integrated in terms of confluent hypergeometric functions.

Considering that the motion takes place between two rigid horizontal surfaces placed at $z = 0$ and $z = 2h$, the basic zonal flow may be written $U = (\bar{U}/h)z$, where $\bar{U} = U(h)$ is the vertical mean value of the wind.

The boundary conditions (vanishing vertical velocities at $z = 0$ and $z = 2h$) lead to two equations that must be satisfied by the solutions of equation (9). Using the linearized perturbation equation deduced from (7)

$$(U-C)f\frac{\partial v}{\partial z} - f\frac{dU}{dz}v + \sigma g w = 0 \tag{10}$$

the boundary conditions can be written

$$\left[(U-C) - \frac{\beta}{k^2}\right] v = 0 \quad \text{for} \begin{cases} z = 0 \\ z = 2h \end{cases} \tag{11}$$

The following non-dimensional parameters may be defined

$$\lambda = \frac{f}{(\sigma g)^{1/2} kh} \quad \text{wave length}$$

$$\Lambda = \frac{f^2}{\sigma g h \beta}\frac{\bar{U}}{h} \quad \text{vertical shear}$$

$$\varkappa = 2h\gamma \quad \text{mass divergence}$$

$$c_r = \frac{C_r}{\bar{U}} \quad \text{phase velocity}$$

$$N = \left(\frac{f}{2\mu\beta\bar{U}}\right) kC_i \quad \text{growth rate}$$

where $\mu^2 = R_i = \sigma g h^2/\bar{U}^2 \gg 1$ is the Richardson number for the model.

The transformation

$$ve^{b\xi} = \Phi\xi \tag{12}$$

where $\xi = \frac{2}{\lambda}\sqrt{\frac{\varkappa^2\lambda^2}{16}+1}\left(\frac{U-C}{\bar{U}}\right)$

and $b = \frac{1}{2}\left(1-\gamma\Big/\frac{\partial\xi}{\partial z}\right)$ leads to the

confluent hypergeometric equation

$$\xi\Phi'' + (c-\xi)\,\Phi' - a\Phi = 0 \tag{13}$$

where $c = 2$ and $a = 1-\frac{1}{2}\alpha$;

$$\alpha = \left(\frac{\mu\beta}{kf}+\frac{\varkappa\lambda}{2}\right)\Big/\sqrt{\frac{\varkappa^2\lambda^2}{16}+1} \tag{14}$$

The general solution of equation (13) is

$$\Phi = A\Phi_1 + B\Phi_2 \tag{15}$$

where Φ_1 and Φ_2 are confluent hypergeometric functions. If a is not a negative integer, Φ_1 and Φ_2 are the series

$$\Phi_1 = \varphi(a, c, \xi) = \sum_{r=0}^{\infty}\frac{(a)_r}{(c)_r}\frac{\xi^r}{r!}$$

$$\Phi_2\Psi(a,2,\xi) = \frac{1}{\Gamma(a-1)}\left\{\varphi(a,2,\xi)\ln\xi + \sum_{r=0}^{\infty}\frac{(a)_r}{(c)_r}\times[\psi(a+r)-\psi(1+r)-\psi(2+r)]\frac{\xi^r}{r!}\right\} + \frac{1}{\xi}\frac{1}{\Gamma(a)} \tag{16}$$

where $(a)_r = a(a+1)\,(a+2)\,\ldots\,(a+r-n)$ and $\psi(x) = d\ln\Gamma(x)/dx$.

We have again obtained solutions that are confluent hypergeometric functions, but in this case the variable ξ and the parameter a depend on the mass divergence parameter $\varkappa$.

The general solution (15) has to satisfy the boundary conditions (11). The corresponding expression may be derived using (12)

$$\Phi - 2m\,\Phi' = 0 \quad \text{for}\begin{cases} z=0 \\ z=2h \end{cases} \tag{17}$$

where

$$m = \sqrt{\frac{\varkappa^2\lambda^2}{16}+1}\Big/\left(-\frac{\varkappa\lambda}{4}+\sqrt{\frac{\varkappa^2\lambda^2}{16}+1}\right)$$

This expression is similar to that found in the first section, with the exception of the factor m that depends on the compressibility.

It can be seen that the baroclinic instability problem admits confluent hypergeometric solutions in the compressible and incompressible cases. This result could be expected as the reason for the similar numerical solutions found by Green for both cases.

3. Numerical results

Numerical methods were used to adjust the general solution (15) to the boundary condition (17), in order to study with better accuracy the influence of the compressibility factor $\varkappa$ on the solutions of the baroclinic instability problem. This point has been studied by Green using purely numerical methods.

In order to calculate the solutions, we assigned values to α, non-dimensional parameter that was defined (14) as a combination of f, β, σ, h, $\bar{U}$, g and k, which therefore were left unfixed. Considering values of α between 0.6 and 35.0, and $\varkappa = 2$ we computed the series (16) for complex values of ξ, in order to find the values that satisfied equation (17). The corresponding values of $\Gamma(a)$ and $\psi(a)$ where taken from Davis (1963).

The results we obtained are shown on Figs. 1 and 2.

(*a*) *Growth rate*. Fig. 1 shows the variation of the growth rate as a function of wave length, for constant values of the shear. The full lines correspond to the case $\varkappa = 2$, the dashed ones represent the same for $\varkappa = 0$ (after Garcia and Norscini).

It is found that the general configuration shows only slight changes for different values of $\varkappa$. There is a general displacement to shorter wave lengths when $\varkappa$ increases. The maximum value of the growth rate is larger for increasing values of $\varkappa$; in particular, for each value of the shear it is larger for $\varkappa = 2$ than for $\varkappa = 0$.

The increase in the growth rate with increasing values of $\varkappa$ becomes more significant towards the short waves. These results agree completely with those obtained by Green.

On Fig. 1 we can observe that in the region where the growth rate takes minimum values, it decreases as $\varkappa$ increases, but this is only a consequence of the general shifting we have mentioned.

The growth rate of the dominant wave deter-

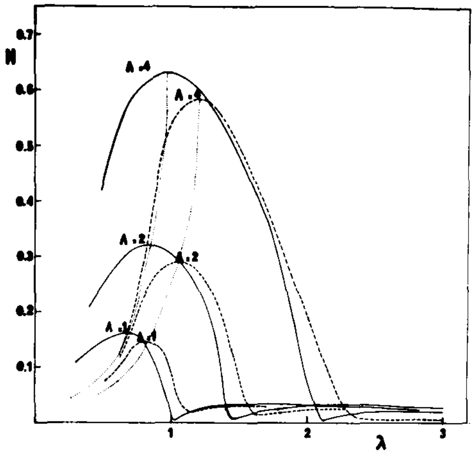

Fig. 1. Growth rate as a function of wave length, for several values of the shear, with $\varkappa = 2$ (full lines) and $\varkappa = 0$ (dashed lines, after Garcia & Norscini).

mined for $\varkappa = 2$ differs from the growth rate for the dominant wave in the case $\varkappa = 0$ by about 10%, while their wave lengths differ by about 20%.

(*b*) *Phase velocity.* In Fig. 2 the phase velocity of the unstable waves is represented for several values of the shear, with full lines in the case $\varkappa = 2$ and dashed lines for $\varkappa = 0$. The value of the phase velocity diminishes considerably when $\varkappa$ increases. The lowering effect of compressibility on the steering level was pointed out by Eady (1949) and found in Green's calculations.

The values of the phase velocity of the dominant waves in the cases $\varkappa = 0$ and $\varkappa = 2$ differ by about 30%.

(*c*) *Realistic values of* $\varkappa$. The value $\varkappa = 2$ was chosen for simplicity reasons, although it probably overestimates the mean value of the mass divergence parameter in the troposphere in middle-latitudes.

For comparison consider the values of γ calculated by Gates (1961), on the basis of observations of 40 stations in the USA during 10 years. Considering $H = 2h = 10$ km, the values of $\varkappa$ vary between 0.6 and 1.0 depending on the altitude. It has to be pointed out that the region under consideration included stations from 25° N to 49° N, and that stations with too low latitude have a lowering effect on the mean value.

Other values of $\varkappa$ can be calculated from the U.S. Standard Atmosphere Supplements (1966). Using the tables of γ for 30° N to 60° N, for the same value of H the values of $\varkappa$ vary between 0.90 and 1.56, depending on latitude, altitude and season.

The phase velocity and growth rate of a wave next to the dominant one was calculated for $\varkappa$ between 0 and 2.

$\varkappa$	α	λ	Λ	c_r	ε_{c_r}	N	ε_N
0.00	0.3	1.0574	3.5247	0.7420	—	0.5244	—
1.00	0.8	1.0558	3.5252	0.5871	21	0.5553	6
1.46	1.0	1.0575	3.5256	0.5243	29	0.5598	7
2.00	1.2	1.0574	3.5244	0.4576	38	0.5563	6

where $\varepsilon_{c_r} = [c_r(\varkappa = 0) - c_r(\varkappa)]/c_r(\varkappa = 0)$ and $\varepsilon_N = [N(\varkappa = 0) - N(\varkappa)]/N(\varkappa = 0)$.

These results show that the supposition $\varkappa = 2$ probably leads to too small values of c_r, but a correct estimation of N.

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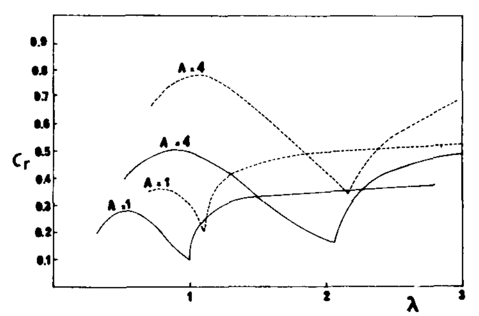

Fig. 2. Phase velocity of the unstable solutions, for some constant values of the shear, as a function of wave length, where full lines correspond to the case $\varkappa = 2$ and dashed lines to $\varkappa = 0$ (after Garcia & Norscini).

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ОБ ОБЩЕМ АНАЛИТИЧЕСКОМ РЕШЕНИИ ЗАДАЧИ О БАРОКЛИННОЙ НЕУСТОЙЧИВОСТИ

Найдены аналитические решения задачи о бароклинной неустойчивости для модели Грина с учетом сжимаемости и для модели Хирота, сформулированной в координатах давления. Приводятся некоторые численные результаты в случае модели Грина с целью изучения влияния параметра сжимаемости.