A new method for the computation of seiches in oblong basins

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ABSTRACT

An iterative numerical method is presented for the computation of seiches in oblong lakes, bays and channels. The method is based on numerical forward integration of the governing equations and on an original iterative technique. Results from the application of the method to various basins show that the method is reliable, very stable and efficient. The application of the method to similar eigenvalue problems, occurring frequently in many branches of continuum mechanics, is discussed briefly. There is no apparent reason why the method would not be applicable to such problems.

1. Introduction

Seiches are the natural modes of sloshing oscillations in large, shallow water basins such as lakes and bays. The computation of seiches is a linear problem involving, in general, the solution of a set of partial differential equations with certain boundary conditions. Most lakes and bays are elongated in one direction (oblong basins), and the task of computing their seiches reduces from a two-dimensional to a much simpler one-dimensional problem. For marginal cases where the basin is not very much elongated in one direction, the lower seiche modes of the basin can still be computed from the one-dimensional theory with acceptable accuracy.

The computation of the seiches in oblong basins involves the solution of two first order ordinary linear differential equations with variable coefficients and homogeneous boundary conditions; i.e., it is a second order eigenvalue problem. The form of the boundary conditions depends on whether the basin is a lake, bay, or channel.

In the period 1904–1936, a number of methods were developed for computing the seiches in oblong lakes. The methods of Chrystal (1904, 1905*a*, 1905*b*), Honda et al. (1908), Proudman (1914), Defant (1918), Ertel (1933), and Hidaka (1936) are well known. Defant (1961) gives a concise review of these methods as well as an extensive bibliography of other pertinent literature. Although these methods were developed for computing the seiches in oblong lakes, most of them can be extended to apply to the computation of seiches in oblong bays and channels. With the exception of Defant's method, these methods are based on various analytical techniques. This decided preference for analytical techniques versus numerical techniques is probably due to the fact that a few decades ago numerical techniques and high speed computing machines were nowhere near their present level of development and application. Defant's method and the method presented here are based on iterative numerical procedures.

2. Statement of the problem

The differential equations governing the seiches of oblong basins can be written in the following normalized (dimensionless) form:

$$\frac{d}{dX}(S\Xi) = -\Omega BH \tag{1}$$

$$\frac{d}{dX}H = \Omega\Xi \tag{2}$$

The normalized variables in the above equations are related to the physical variables of the problem through the following equations:

$$X = \frac{x}{l} \tag{3}$$

$$\Xi = \frac{\omega\xi}{\eta_0}\sqrt{\frac{\bar{h}}{g}} \tag{4}$$

$$H = \frac{\eta}{\eta_0} \tag{5}$$

$$S = \frac{s}{\bar{s}} \tag{6}$$

$$B = \frac{b}{\bar{b}} \tag{7}$$

$$\Omega = \frac{l\omega}{\sqrt{g\bar{h}}} \tag{8}$$

The independent variable x is a linear coordinate measured along the talweg of the basin, which is defined as the line connecting the lowest points of the bottom of an oblong basin. The length of the talweg is l. The dependent variables ξ and η are the displacement amplitudes of the seiche in the horizontal and vertical directions, respectively. The functions s and b are the distributions along the talweg of the cross-sectional area and of the width of the free surface of the basin, respectively. The constant g is the acceleration of gravity, and ω is the angular frequency of the seiche. $\bar{s}$ and $\bar{b}$ are the averages of s and b over the talweg, and $\bar{h}$ is an average depth defined by $\bar{s} = \bar{b}\bar{h}$. η_0 is a reference value for η.

Just as ξ, η, s and b are functions of x, the normalized variables Ξ, H, S and B are functions of X.

The solution of equations (1) and (2) for Ξ and H must satisfy two boundary conditions. The type of the boundary conditions depends on whether the basin is closed at both ends (lake), or closed at one end only (bay), or open at both ends (channel). At closed ends Ξ must be equal to zero, while at open ends H must be equal to zero.[1] Thus, the solution of equations (1) and (2) must satisfy the following homogeneous boundary conditions:

Lakes
$$\Xi(0) = 0 \tag{9}$$
$$\Xi(1) = 0 \tag{10}$$

Bays (open end at $X = 1$)
$$\Xi(0) = 0 \tag{11}$$
$$\mathrm{H}(1) = 0 \tag{12}$$

Channels
$$\mathrm{H}(0) = 0 \tag{13}$$
$$\mathrm{H}(1) = 0 \tag{14}$$

The homogeneity of the boundary conditions makes this problem an eigenvalue problem. For given S and B, equations (1) and (2) have non-trivial solutions satisfying the boundary conditions for only certain, unknown but unique, values of the normalized angular frequency Ω. The solution of the problem yields both the eigenvalues Ω and the corresponding eigenfunctions Ξ and H. The eigenfunctions describe the shape of the various seiche modes. The period T of a particular mode is related to the corresponding eigenvalue Ω through the following equation:

$$T = \frac{2\pi l}{\Omega\sqrt{g\bar{h}}} \tag{15}$$

3. Description of the method

Since the coefficients S and B are variable, closed form analytic solutions of equations (1) and (2) cannot be obtained in general. A numerical technique must be used to obtain solutions for arbitrary variations of S and B.

First, a numerical method used commonly to solve eigenvalue problems such as the present will be described. The method is based on forward integration from one boundary to the other. To carry out such an integration in the seiche problem, the initial values (at $X = 0$) of the dependent variables Ξ and H as well as the value of the unknown parameter Ω are required. The boundary conditions, equations (9) to (14), provide the initial value for one of the two dependent variables. A nonzero initial value for the other dependent variable can be selected arbitrarily since the boundary conditions are homogeneous—the solution of eigenvalue problems has an arbitrary scaling factor. Since the equations are in normalized form, it is expedient to select the initial value of 1 for the other dependent variable.

With the initial values of Ξ and H thus known, an approximate value of Ω is selected, and equations (1) and (2) are integrated forward from $X = 0$ to $X = 1$. Merian's formula for rec-

[1] This condition for open ends is an idealized condition, since it does not take into consideration the way in which a partially open basin is actually connected to the ocean or to another much larger basin. After computing the seiches of partially open basins with this idealized condition, opening corrections to the seiche periods can be made with formulae developed by Rayleigh (1894) and Proudman (1925).

tangular basins can be useful in the selection of the first approximate value of Ω. If the obtained solution satisfies the boundary condition at $X = 1$, the problem is solved and the assumed Ω is the correct eigenvalue of the obtained seiche mode. If not, a second value of Ω is selected somehow and the forward integration is repeated. For the third and following forward integrations, the required values of Ω are obtained from linear extrapolation of two preceding values. The iterations are terminated when a desired accuracy is obtained.

The method described above has been used by Defant in computing the seiches of oblong lakes, and is used commonly in other similar problems of continuum mechanics. The iterations usually converge to the solution, although the speed of convergence may be quite slow—depending on the problem. An original technique for obtaining the approximate eigenvalues for the successive iterations will now be described.

Instead of carrying out the forward integration to exactly the point $X = 1$, carry out the integration to a point X' at which the second homogeneous boundary condition is satisfied taking into consideration the number of nodes that a particular seiche mode should have. X' can be smaller or larger than 1 depending on the relation of the approximate value of Ω used in this integration to the correct value. Since this forward integration may have to be carried to points beyond the end of the basin ($X = 1$), the question arises as to what values of S and B should be used for such points. Considering the periodicity that the exact solution must have beyond the point $X = 1$, it seems that the best choice for rapid convergence is as follows: for points beyond $X = 1$, use the values of S and B at the reflections of these points about $X = 1$.

Now, let Ω_i be the value of Ω used in the i forward integration, and X'_i the resulting value of X'. For the next forward integration, use the value Ω_{i+1} for Ω given by the following formula:

$$\Omega_{i+1} = \Omega_i X'_i \qquad (16)$$

and so on for the following integrations. This formula was derived from similarity principles, which show that the normalized angular frequency of a seiche must be proportional to the talweg length l (see equation (8)). For rectangular basins (constant B and S), this formula will give the correct value of Ω for a particular seiche mode in exactly one iteration, regardless of the initial value of Ω used in the forward integration. Results from the application of this procedure to various basins with variable S and B show that the successive values of Ω obtained from equation (16) approach the correct value very rapidly, but the convergence is not always uniform. What usually happens is that the obtained values of Ω first approach the correct value very rapidly from one side, depending on the relation of the initial value of Ω to the correct value, and then oscillate about the correct value with very small but everlasting deviations. This, however, is not a serious handicap of the method. If and when the oscillation starts, use the following linear interpolation formula instead of equation (16) to compute the values of Ω for the remaining one or two forward integrations:

$$\Omega = \frac{(X'_+ - 1)\,\Omega_+ + (1 - X'_-)\Omega_-}{X'_+ - X'_-} \qquad (17)$$

X'_+ is the smallest of all values of X' larger than 1 computed from preceding integrations. X'_- is the largest of all values of X' smaller than 1 computed from preceding integrations. Ω_+ and Ω_- are the values of Ω which produced X'_- and X'_+, respectively.

Although the author has not tried the method described above on other similar eigenvalue problems of continuum mechanics, he sees no reason why the method would not be applicable to such problems. One important aspect of the method must be pointed out. The iterative formula of equation (16) must reflect the exact dependence of the eigenvalues Ω on the maximum value l of the independent variable x. For the seiche problem, this dependence is linear (see equation (8)) and, hence, the dependence of Ω_{i+1} on X'_i should also be linear as in equation (16). This may not be the case for other problems, and equation (16) must be modified accordingly, which can be done readily once the dependence of Ω on l has been established through the appropriate normalization of the physical variables.

Finally, it should be noted that, unlike other methods commonly used, this method requires only one initial approximate value of the eigenvalue. For the seiche problem, this value can be obtained from Merian's formula or from a more judicious selection. A similar selection can be made for other problems.

Table 1. *Shape and dimensions of bay*

Distance Along Talweg – x (10^3 ft)	Cross sectional area – s (10^6 ft^2)	Free surface width – b (10^3 ft)
0.0	0.2	5.0
2.5	0.9	7.5
5.0	1.5	10.0
7.5	2.0	12.0
10.5	2.5	13.7
13.5	2.9	14.7
16.0	3.1	14.8
18.5	3.2	14.33
22.5	3.3	13.71
25.0	3.4	13.33
28.0	3.5	12.83
31.5	3.7	12.3
35.0	4.0	11.83
38.0	4.3	11.43
42.0	4.8	10.9
46.0	5.4	10.4
50.0	6.0	10.0

4. Application of the method to various basins

The method has been tested with a digital computer program using Gill's procedure of the fourth order Runge-Kutta integrating technique for the forward integration. A variety of basins and modes have been considered, and the method proved very successful in all cases. Results from the application of the method to three basins (a lake, a bay, and a channel) are presented and discussed below.

The lake is a symmetric basin with linear variation of depth and free surface width along the talweg. The cross-sectional area and the width of the free surface are given by the following normalized expressions:

For $0 \leqslant X \leqslant 0.5$
$$S = (0.01 + 1.98\,X)^2/0.3367 \tag{18}$$

$$B = (0.01 + 1.98\,X)/0.505 \tag{19}$$

and for $0.5 < X \leqslant 1$
$$S = (1.99 - 1.98\,X)^2/0.3367 \tag{20}$$

$$B = (1.99 - 1.98\,X)/0.505 \tag{21}$$

It should be noted that this basin is quite adverse to computation by a numerical method since S and B approach almost zero at the ends of the basin. Because of the way in which S enters in equation (1), zero values of S cannot be allowed in a numerical method for solving this equation.

The bay is an irregular basin whose shape and dimensions are described in Table 1. The opening of the bay is at the far end of the basin (x = 50 000 feet). The normalization of the characteristics of this basin was done by the computer program.

The channel is a basin of uniform depth. The normalized free surface width B is given by

$$B = 1 - \frac{2}{3}X + X^2 \tag{22}$$

Since the depth is constant, the same function describes the normalized cross-sectional area S.

The iterations were terminated when X' was close to 1 with a tolerance of 10^{-5}. This means that the results are accurate in four to five figures. A sufficiently small integrating step was selected so that the integration errors were insignificant.

For a given desired accuracy, the speed of the method (number of iterations) depends on the initial value of Ω. The normalized angular frequencies for the modes of rectangular basins, as given by Merian's formula, were used as initial values of Ω. They are π, 2π, 3π, etc., for lakes and channels, and $\pi/2$, $3\pi/2$, $5\pi/2$, etc., for bays.

Some significant results for the first four modes are summarized in Table 2. Figs. 1 and 2

Table 2. *Summary of results*

Basin, mode	Normalized angular frequency (Ω) Initial	Final	Number of iterations	Period (min)
Lake, 1	π	4.6482	19	8.4255
Lake, 2	2π	6.235	6	6.282
Lake, 3	3π	8.530	6	4.591
Lake, 4	4π	10.253	6	3.8195
Bay, 1	$\pi/2$	1.7117	4	31.931
Bay, 2	$3\pi/2$	4.8931	4	11.170
Bay, 3	$5\pi/2$	7.689	4	7.108
Bay, 4	$7\pi/2$	10.539	4	5.1859
Channel, 1	π	3.2969	3	6.8581
Channel, 2	2π	6.357	3	3.557
Channel, 3	3π	9.474	3	2.387
Channel, 4	4π	12.602	2	1.7941

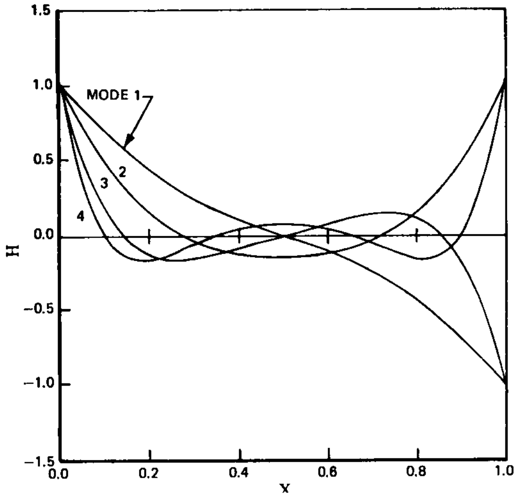

Fig. 1. Distribution of the normalized vertical displacement (H) of the lake for modes 1 to 4.

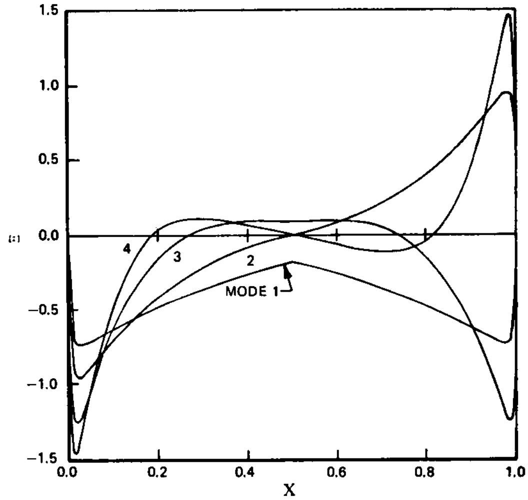

Fig. 2. Distribution of the normalized horizontal displacement (Ξ) of the lake for modes 1 to 4.

show the shapes of the modes for the lake as plotted by the computer program. Once the normalized angular frequency of a seiche mode is known, all that is needed to compute the seiche period are the talweg length l and the average depth $\bar{h}$ of the basin (see equation 15)). The periods given as an illustration in the last column of Table 2 were obtained with the following values of l and $\bar{h}$: 30 000 and 200 feet for the lake, and 15 000 and 150 feet for the channel. The values of l and $\bar{h}$ for the bay can be deduced from the data given in Table 1. They are 50 000 and 285.2 feet.

Table 2 shows that, except for the first mode of the lake, a very small number of iterations was required to obtain solutions accurate in at least four figures. Nineteen iterations were required for the first mode of the lake primarily because the initial value of Ω is 33 percent below the correct value. These results, as well as other results not presented here, show that the method is efficient. The method is also reliable and very stable. No difficulties whatsoever were encountered in any computation; convergence to the solution was always fast and uniform.

5. Conclusions

A new iterative numerical method has been developed for the computation of seiches in oblong lakes, bays and channels.

Results from the application of the method to various basins with a digital computer program show that the method is efficient, reliable and very stable.

There is no apparent reason why the method would not be applicable to similar eigenvalue problems occurring frequently in many branches of continuum mechanics. It appears very worthwhile to try this method on such problems.

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НОВЫЙ МЕТОД ВЫЧИСЛЕНИЙ СЕЙШ В УДЛИНЕННЫХ БАССЕЙНАХ

Дается интерационный численный метод для вычисления сейш в удлиненных озерах, заливах и проливах. Метод основывается на численном интегрировании уравнений шагами по времени вперед и на оригинальной технике итераций. Результаты применений метода к различным бассейнам показывают, что метод надежен, очень устойчив и эффективен. Кратко обсуждается применение метода к аналогичным задачам на собственные значения, часто встречающимся во многих разделах механики сплошных сред. Нет очевидных причин, почему бы этот метод не мог бы применяться для решения таких задач.