The coastal boundary layers of a lake when horizontal and vertical Ekman numbers are of different orders of magnitudes

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ABSTRACT

We investigate the linear hydrostatic boundary layers which bring the horizontal motion to rest at the shore of a lake of horizontal extent L and depth H containing a homogeneous fluid with constant, though vastly differing, horizontal and vertical eddy viscosities, A_H and A_V. The lake rotates with angular speed $f/2$ about a vertical axis and the motion is induced by a steady, nonuniform wind stress. Our results are applicable when

$$(H/L)^2 \leqslant (A_V/fH^2)^{\frac{1}{2}}\,(A_H/fL^2) \leqslant (A_V/fH^2)^3 < < 1.$$

It is found that under these conditions the flow near the shore is determined by the local wind stress and the local value of the interior velocity parallel to the coast. It is found that under certain wind stress conditions there is a region of flow, adjacent to the shore, in which the fluid is effectively confined to the coast.

Introduction

Over the past decade, the boundary layers adjacent to the vertical walls of rotating containers have come under intensive theoretical investigation (see Greenspan (1968) for a summary). These analyses have generally assumed that the depth and horizontal extent of the containers are comparable or that the horizontal and vertical Ekman numbers are of comparable magnitude. There are cases, namely flows in shallow lakes, where both these assumptions are invalid. We shall construct a theory capable of dealing with situations when

$$(H/L)^2 \leqslant (A_V/fH^2)^{\frac{1}{2}}\,(A_H/fL^2) \leqslant (A_V/fH^2)^3 < < 1$$

where H, L, A_H, A_V and $f/2$ are the depth, width, horizontal and vertical eddy viscosities, and angular rotation rate for the flow under consideration.

The model

Consider the flow of a homogeneous fluid of density ϱ contained in a basin of uniform depth H, horizontal extent L, rotating about a vertical axis with angular speed $f/2$, acted upon by a steady nonuniform wind stress at the free surface, and undergoing a turbulent motion with constant horizontal and vertical eddy viscosities A_H and A_V. The motion is viewed from a cartesian reference frame with the origin at some point on the shore, at the free surface, the z' axis taken vertically upwards, the x' axis is normal to the coast directed into the fluid, and the y' axis is parallel to the coast and counterclockwise from the x' axis (Fig. 1). The dimensional (mean) velocities in the (x', y', z') directions are (u', v', w'). The flow is caused by a wind stress (τ_x', τ_y') applied at the free surface. We non-dimensionalize the variables as follows:

$$L(x, y) = (x', y'),\; Hz = z',\; U(u, v) = (u', v')$$

$$U\,\frac{H}{L}\,\omega = \omega',\; \varrho fULp - \varrho' gz' + \frac{1}{8}\,\varrho f^2 r'^2 = p' \qquad (1)$$

$$\tau_M \equiv MAX[\tau_x'(x', y'), \tau_y'(x', y')],$$

$$U \equiv \frac{\tau_M}{\varrho\sqrt{A_V f}}, \quad \varepsilon = \frac{U}{fL}$$

$$\tau_M(\tau_x, \tau_y) = (\tau_x', \tau_y'),\; E_H = A_H/fL^2,\; E_V = A_V/fH^2$$

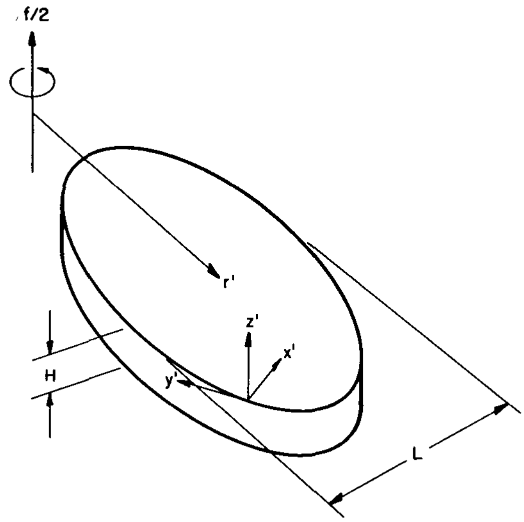

Fig. 1. Geometry of the basin.

where r' is the horizontal distance from the axis of rotation. The dimensionless governing equations are:

$$\varepsilon \frac{du}{dt} - v = -\frac{\partial p}{\partial x} + E_H \nabla_H^2 u + E_V \frac{\partial^2 u}{\partial z^2} \tag{2a}$$

$$\varepsilon \frac{dv}{dt} + u = -\frac{\partial p}{\partial y} + E_H \nabla_H^2 v + E_V \frac{\partial^2 v}{\partial z^2} \tag{2b}$$

$$\varepsilon \left(\frac{H}{L}\right)^2 \frac{dw}{d\tau} = -\frac{\partial p}{\partial z} + \left(\frac{H}{L}\right)^2 \left(E_H \nabla_H^2 \omega + E_V \frac{\partial^2 \omega}{\partial z^2}\right) \tag{2c'}$$

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial z} \tag{2d}$$

where

$$\frac{d}{dt} \equiv u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + \omega \frac{\partial}{\partial z}, \nabla_H^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y}$$

We first require that ε is sufficiently small so that the inertial terms are, everywhere, negligible. We further assume that

$$(H/L)^2 \leqslant E_H E_V^{\frac{1}{2}} < < 1$$

We can later verify that this allows us to drop the "viscous" terms in (2c′) and replace this equation with

$$0 = -\frac{\partial p}{\partial z} \tag{2c}$$

By doing this we surrender the possibility of setting the vertical velocity to rest at the shore. There will be a thin boundary layer adjacent to the coast, in which the pressure is non-hydrostatic, which brings the vertical motion to rest, and which will not be discussed here. The horizontal velocities supported by this layer, which can be shown to be of width H, are of a smaller order than those considered in this paper and hence do not affect our analysis. Thus, the solutions obtained in this paper will be valid at all points in the flow, except for the pressure and vertical velocity which will be in error within distance H from the shore.

The boundary conditions for the linear problem we have formulated are as follows:

$$\text{At } z = 0, E_v^{\frac{1}{2}} \frac{\partial u}{\partial z} = \tau_x, E_V^{\frac{1}{2}} \frac{\partial v}{\partial z} = \tau_y, w = 0 \tag{3a}$$

$$\text{At } z = -1, u = 0, v = 0, w = 0 \tag{3b}$$

$$\text{At the coast, } u = v = 0 \tag{3c}$$

We now proceed along well established lines. We assume that the flow field can be decomposed into two solutions. First is the interior solution for which $\partial/\partial x$, $\partial/\partial y = O(1)$ and which is defined for all (x, y, z). Second is the coastal solution, which decays away from the coast and has large gradients normal to the coast. The two solutions overlap near the coast and together satisfy (3c).

The interior solution

The interior solution has $\partial/\partial x$, $\partial/\partial y = O(1)$. Assuming first that $E_H = O(E_V^{\frac{5}{2}})$, we can, through order E_V^2, neglect horizontal friction, and the governing equations become:

$$-v = -\frac{\partial p}{dx} + E_V \frac{\partial^2 u}{\partial z^2} \tag{4a}$$

$$u = -\frac{\partial p}{\partial y} + E_V \frac{\partial^2 v}{\partial z^2} \tag{4b}$$

$$0 = -\frac{\partial p}{\partial z} \tag{4c}$$

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{4d}$$

The solution to these equations must satisfy (3 a, 3 b). The interior solution is itself composed of three parts, i.e., the top Ekman layer (T) solution which has $\partial/\partial z = O(E_V^{-\frac{1}{2}})$ and vanishes away from the top, the inviscid (I) solution has $\partial/\partial z = O(1)$ and exists for $-1 \leqslant z \leqslant 0$, and the bottom Ekman layer (B) solution which has $\partial/\partial z = O(E_V^{-\frac{1}{2}})$ and vanishes away from the bottom. We first consider the inviscid solution. We expand the dependent variables as follows:

$$(u_I, v_I, \omega_I, p_I) = \sum_{K=0}^{\infty} E_V^{K/2} (u_{IK}, v_{IK}, \omega_{IK}, p_{IK})$$

Hence, the zero, first and second order solutions satisfy:

$$-v_{IK} = -\frac{\partial p_{IK}}{\partial x} \tag{6 a}$$

$$u_{IK} = -\frac{\partial p_{IK}}{\partial y} \tag{6 b}$$

$$0 = -\frac{\partial p_{IK}}{\partial z} \tag{6 c}$$

$$0 = \frac{\partial \omega_{IK}}{\partial z} + \frac{\partial u_{IK}}{\partial x} + \frac{\partial v_{IK}}{\partial y}, \tag{6 d}$$

where $k = 0, 1, 2$. Eliminating the pressure through cross-differentiating (6 a) and (6 b) and differentiating these equations with respect to z leads to:

$$\frac{\partial w_{IK}}{\partial z} = \frac{\partial u_{IK}}{\partial z} = \frac{\partial v_{IK}}{\partial z} = 0$$

Clearly the shear stress condition of (3 a) must be satisfied by the top solution alone. This solution, which must vanish away from $z = 0$, satisfies:

$$-v_T = \frac{\partial^2 u_T}{\partial \bar{z}^2} \tag{7 a}$$

$$u_T = \frac{\partial^2 v_T}{\partial \bar{z}^2} \tag{7 b}$$

$$p_T = 0 \tag{7 c}$$

$$\frac{\partial w_T}{\partial \bar{z}} = -E_V^{\frac{1}{2}} \left(\frac{\partial u_T}{\partial x} + \frac{\partial v_T}{\partial y} \right), \tag{7 d}$$

where $\bar{z} = z E_V^{-\frac{1}{2}} (-\infty < \bar{z} \leqslant 0)$. The solution to this set is well known and to order E_V^2 is

$$u_T + i v_T = (\tau_x + i \tau_y) \exp\left(\sqrt{i}\, \bar{z} - i\pi/4\right) \tag{8 a}$$

$$\omega_T \big|_{z=0} = -E_V^{\frac{1}{2}} \left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y} \right) \tag{8 b}$$

and the volume flux carried by the top Ekman layer, M_T, is

$$M_T = M_{Tx} + i M_{Ty} \equiv E_V^{\frac{1}{2}} \int_{-\infty}^{0} (u_T + i v_T)\, d\bar{z}$$

$$= E_V^{\frac{1}{2}} (\tau_y - i \tau_x) \tag{8 c}$$

To satisfy the last part of (3 a), i.e.,

$$\omega_T + \omega_I \big|_{z=0} = 0,$$

requires:

$$\omega_{I0} \big|_{z=0} = 0;\ \omega_{I1} \big|_{z=0} = \frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y};\ \omega_{I2} \big|_{z=0} = 0 \tag{9}$$

Since $\dfrac{\partial \omega_{IK}}{\partial z} = 0$, this implies:

$$\omega_{I0} = \omega_{I2} \equiv 0,\ \omega_{I1} = \frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}$$

Thus, we have satisfied (3 a). The bottom solution satisfies equations (7) with the subscript B replacing the T and $\bar{z}$ redefined as $\bar{z} \equiv (z+1) E_V^{-\frac{1}{2}} (0 \leqslant \bar{z} < \infty)$. The solution of this set of equations, with satisfies the first two parts of (3 b), is

$$u_B + i v_B = -(u_I + i v_I) \exp\left(-\sqrt{i}\, \bar{z}\right) \tag{10 a}$$

$$\omega_B \big|_{z=-1} = -\frac{E_V^{\frac{1}{2}}}{\sqrt{2}} \left(\frac{\partial v_I}{\partial x} - \frac{\partial u_I}{\partial y} \right) \tag{10 b}$$

and the volume flux carried by the bottom Ekman layer, M_B, is

$$M_B = M_{Bx} + i M_{By} \equiv E_V^{\frac{1}{2}} \int_0^{\infty} (u_B + i v_B)\, d\bar{z} \tag{10 c}$$

$$= -\frac{E_V^{\frac{1}{2}}}{\sqrt{2}} \left((v_{I0} + u_{I0}) + i (v_{I0} - u_{I0}) \right) + 0(E_V)$$

To satisfy the last part of (3 b), i.e.,

$$\omega_B + \omega_I \big|_{z=-1} = 0$$

requires, from (10b),

$$\omega_{I0}\big|_{z=-1} = 0, \ \omega_{I1}\big|_{z=-1} = \frac{1}{\sqrt{2}}\left(\frac{\partial v_{I0}}{\partial x} - \frac{\partial u_{I0}}{\partial y}\right), \qquad (11)$$

$$\omega_{I2}\big|_{z=-1} = \frac{1}{\sqrt{2}}\left(\frac{\partial v_{I1}}{\partial x} - \frac{\partial u_{I1}}{\partial y}\right)$$

Equating the results of (9) and (11) yields:

$$\frac{1}{\sqrt{2}}\left(\frac{\partial v_{I0}}{\partial x} - \frac{\partial u_{I0}}{\partial y}\right) = \frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}$$

or

$$\frac{1}{\sqrt{2}} \nabla_H^2 p_{I0} = \frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{dy} \qquad (12)$$

and

$$\frac{1}{\sqrt{2}}\left(\frac{\partial v_{I1}}{\partial x} - \frac{\partial u_{I1}}{\partial y}\right) = 0$$

or

$$\nabla_H^2 p_{I1} = 0 \qquad (13)$$

The analysis leading to (12) and (13) is fairly standard, see Welander (1957) for a different approach. The coastal solutions must satisfy null conditions at $z = 0, -1$, and together with the interior solution satisfy (3c). We shall end our analysis at order $E_V^{\frac{1}{2}}$ as to proceed to a higher order in the coastal solution appears to be difficult. The vanishing of horizontal motion at the coast places boundary conditions on (12) and (13). We require as our first boundary condition that the horizontal component of the zeroth order interior velocity normal to the coast must vanish, i.e.,

$$\underset{\sim}{n} \times \nabla p_{I0} = 0, \qquad (14)$$

where $\underset{\sim}{n}$ is a vector normal to the coast. Equations (12) and (14) specify the zeroth order interior velocity. If the right hand side of (12) does not vanish identically there will, in general, be a zeroth order velocity parallel to the shore. The coastal boundary layers must (I) bring this zeroth order tangential motion to rest. Further, the layer must (II) cause the volume flux of the top and bottom Ekman layers to vanish at the coast. In accomplishing (I) and (II) first order velocities are induced at the coast. The layers must (III) bring this first order motion to rest. In doing this we determine the value of the first order normal interior velocity at each point of the coast which sets the boundary condition to equation (13) which completes the problem through order $E_V^{\frac{1}{2}}$. We shall proceed by satisfying part of the condition (II) first.

The coastal boundary layers

Let us examine the flow in the region near the origin as this is a typical coastal area. We assume that the boundary layer solution

$$(\bar{u}, \bar{v}, \bar{\omega}, \bar{p}) \text{ has } \frac{\partial}{\partial x} >> \frac{\partial}{\partial y},$$

and so satisfies

$$-\bar{v} = -\frac{\partial \bar{p}}{\partial x} + E_H \frac{\partial^2 \bar{u}}{\partial x^2} + E_V \frac{\partial^2 \bar{u}}{\partial z^2} \qquad (15\text{a})$$

$$\bar{u} = -\frac{\partial \bar{p}}{\partial y} + E_H \frac{\partial^2 \bar{v}}{\partial x^2} + E_V \frac{\partial^2 \bar{u}}{\partial z^2} \qquad (15\text{b})$$

$$0 = -\frac{\partial \bar{p}}{\partial z} \qquad (15\text{c})$$

$$0 = \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{\omega}}{\partial z} \qquad (15\text{d})$$

These equations can be shown to have a more general application. Let (x, y) and $(\bar{u}, \bar{v})$ be the local normal and tangential coordinates and horizontal velocities. The horizontal mixing terms will then contain first derivatives with respect to x and y as well as terms involving the curvature (assumed to be on the order of L^{-1}). By examining the full equations in these coordinates we can show the following. If we have a layer of thickness $E_H^{\frac{1}{2}}$ with $(\bar{u}, \bar{v}) = O(E_V^{\frac{1}{2}})$, then the errors obtained by using equations (15) instead of the full equations will be at most of order E_V if $E_V \geq E_H$. Further, if we have a layer of thickness $E_H^{\frac{1}{2}}/E_V^{\frac{1}{4}}$ with $\bar{u} = O(E_V^{\frac{1}{4}})$ and $\bar{v} = O(1)$, then the errors obtained by using equations (15) instead of the full equations, will be of order $E_V^{\frac{3}{4}}$ in $\bar{u}$ and of order E_V in $\bar{v}$, if

$$E_H^{\frac{1}{2}} \leq E_V^{\frac{3}{4}} << 1$$

Therefore the solutions of equations (15) will be valid through order $E_V^{\frac{1}{2}}$ if we require the above inequality to hold, as we now do. We now turn to solving equations (15) through order $E_V^{\frac{1}{2}}$ subject to conditions (I)–(III). We assume that boundary layer solutions are composed of three parts. First we have the modified (by horizontal mixing) top and bottom (T, B) Ekman layer solutions and a solution (I) non-vanishing over $-1 \leqslant z \leqslant 0$ in which vertical mixing is unimportant.

The Ekman solutions satisfy

$$-\bar{v}_i = E_H \frac{\partial^2 \bar{u}_i}{\partial x^2} + \frac{\partial^2 \bar{u}_i}{\partial \bar{z}^2} \tag{16a}$$

$$\bar{u}_i = E_H \frac{\partial^2 \bar{v}_i}{\partial x^2} + \frac{\partial^2 \bar{v}_i}{\partial \bar{z}^2} \tag{16b}$$

$$\frac{\partial \bar{\omega}_i}{\partial \bar{z}} = -E_V^{\frac{1}{2}} \left(\frac{\partial \bar{u}_i}{\partial x} + \frac{\partial \bar{v}_i}{\partial y} \right) \tag{16c}$$

$$i = T, B,$$

where for the top layer $\bar{z} = z/E_V^{\frac{1}{2}}$, and for the bottom layer $\bar{z} = (z+1)/E_V^{\frac{1}{2}}$. Consider the top layer first. Define

$$\bar{M} \equiv \bar{M}_x + i\bar{M}_y \equiv E_V^{\frac{1}{2}} \int_{-\infty}^{0} (\bar{u}_T + i\bar{v}_T)\, dz$$

Integrating (16a) and (16b) with respect to $\bar{z}$, and noting the zero stress condition at $z = 0$, we obtain

$$-\bar{M}_y = E_H \frac{\partial^2 \bar{M}_x}{\partial x^2}$$

$$\bar{M}_x = E_H \frac{\partial^2 \bar{M}_y}{\partial x^2}$$

Solving this we find

$$\bar{M} = E_V^{\frac{1}{2}} A \exp\,(-\sqrt{i}\,x/E_H^{\frac{1}{2}})$$

Thus, we have a top Ekman layer of thickness $E_H^{\frac{1}{2}}$. We use this solution to satisfy (II) at the top by choosing

$$A = -(\tau_y - i\tau_x),$$

where $\quad \tau_x, \tau_y \equiv \tau_x \big|_{z=0}, \tau_y \big|_{z=0}$

Therefore,

$$\bar{M} = -E_V^{\frac{1}{2}}(\tau_y - i\tau_x) \exp\,(-\sqrt{i}\,x/E_H^{\frac{1}{2}}) \tag{17}$$

We now obtain the I solution for this boundary layer. We require that $\bar{\omega}_T + \bar{\omega}_I \big|_{z=0} = 0$. Integrating (16c) with respect to $\bar{z}$ leads to:

$$\bar{\omega}_T \big|_{z=0} = -\left\{ \frac{\partial \bar{M}_x}{\partial x} + \frac{\partial \bar{M}_y}{\partial y} \right\}$$

$$= -\frac{E_V^{\frac{1}{2}}}{E_H^{\frac{1}{2}}} \tau e^{-\bar{x}} \sin\,(\theta + \pi/4 - \bar{x}), \tag{18}$$

where

$$\bar{x} \equiv x/\sqrt{2E_H}, \quad \tau \equiv \sqrt{\tau_x^2 + \tau_y^2}, \quad \tan\theta = \frac{\tau_y}{\tau_x}$$

Hence

$$\bar{\omega}_I \big|_{z=0} = +\frac{E_V^{\frac{1}{2}}}{E_H^{\frac{1}{2}}} \tau e^{-\bar{x}} \sin\left(\theta + \frac{\pi}{4} - \bar{x}\right) \tag{19}$$

Now let us consider the induced horizontal velocities $\bar{u}_I$ and $\bar{v}_I$. With $\partial/\partial x = O(E_H^{-\frac{1}{2}})$, (15a) and (15b) imply that $\bar{u}_I = O(\bar{v}_I)$ and $\bar{p}_I = O(E_H^{\frac{1}{2}} \bar{v}_I)$. Equation (15d) implies that $\bar{u}_I = O(E_H^{\frac{1}{2}} \bar{\omega}_I) = o(E_V^{\frac{1}{2}})$. Therefore, to lowest order, the equations to be satisfied by the (I) solution are

$$-\bar{v}_I = -\frac{\partial \bar{p}_I}{\partial x} + E_H \frac{\partial^2 \bar{u}_I}{\partial x^2} \tag{20a}$$

$$\bar{u}_I = E_H \frac{\partial^2 \bar{v}_I}{\partial x^2} \tag{20b}$$

$$\frac{\partial \bar{u}_I}{\partial x} + \frac{\partial \bar{\omega}_I}{\partial z} = 0 \tag{20c}$$

$$\frac{\partial \bar{p}_I}{\partial z} = 0 \tag{20d}$$

Now the bottom Ekman layer exists solely to bring the horizontal motion to rest at the bottom. The horizontal velocity in this layer is of order $E_V^{\frac{1}{2}}$ and the vertical velocity

$$\bar{\omega}_B = O\left(E_V^{\frac{1}{2}} \frac{\partial \bar{u}_T}{\partial x}\right), \text{ or } \bar{\omega}_B = O\left(\frac{E_V}{E_H^{\frac{1}{2}}}\right)$$

Hence, the vertical velocity in the bottom layer is of order $E_V^{\frac{1}{2}}$ smaller than $\bar{\omega}_I$, and we must require that $\bar{\omega}_I \big|_{z=-1} = 0$. Requiring for matching purposes that $\frac{\partial \bar{u}_I}{\partial z} = 0$, we have, integrating (20c)

$$\bar{\omega}_I = -(z+1) \frac{\partial \bar{u}_I}{\partial x} \tag{21}$$

Setting z to zero in this expression, equating the result to the right-hand side of (19), and integrating once with respect to x yields $\tilde{u}_I$. Then integrating (20b) we obtain $\tilde{v}_I$. To summarize

$$\tilde{u}_I = E_V^{\frac{1}{2}} \tau e^{-\tilde{x}} \sin(\theta - \tilde{x}) \tag{22a}$$

$$\tilde{v}_I = -E_V^{\frac{1}{2}} \tau e^{-\tilde{x}} \cos(\theta - \tilde{x}) \tag{22b}$$

$$\bar{\omega}_I = (z+1) \frac{E_V^{\frac{1}{2}}}{E_H^{\frac{1}{2}}} \tau e^{-\tilde{x}} \sin\left(\theta + \frac{\pi}{4} - \tilde{x}\right) \tag{22c}$$

To carry the analysis to a higher order would require the explicit solution of the bottom Ekman layer, hence we stop at this order. This analysis is quite similar to one performed by Pedlosky (1968) for an oceanic circulation problem with $E_H \equiv E_V$.

We have now satisfied part of condition (II). To satisfy (I) and (III) requires another boundary layer solution ($\tilde{u}_i$, $\tilde{v}_i$, $\tilde{\omega}_i$, $i = T, B, I$). We therefore seek a solution of (15) for which $\tilde{v}_I = O(1)$, $\tilde{u}_I = o(\tilde{v}_I)$, and which has $\frac{\partial \tilde{u}_I}{\partial z} = \frac{\partial \tilde{v}_I}{\partial z} = 0$. We tentatively assume this layer has a thickness, say L_x, greater than $E_H^{\frac{1}{2}}$. If this last assumption is valid, then the top Ekman solution is identically zero since there is no non-trivial solution of (15a) and (15b), with the pressure and horizontal mixing terms dropped, which satisfies the zero shear stress condition at $z = 0$, and vanishes away from the top. Therefore $\tilde{\omega}_I|_{z=0} = 0$. Since $\frac{\partial \tilde{u}_I}{\partial z} = \frac{\partial \tilde{v}_I}{\partial z} = 0$, $\tilde{\omega}_I$ is proportional to z and must be of the same order as $\tilde{\omega}_B$ where $\tilde{\omega}_B = O\left(E_V^{\frac{1}{2}} \frac{(1)}{L_x}\right)$. Differentiating (15a) and (15b) with respect to x and y respectively, subtracting the results, and neglecting $\frac{\partial \tilde{u}_I}{\partial y}$ with respect to $\frac{\partial \tilde{v}_I}{\partial x}$, leads to

$$-\frac{\partial \tilde{\omega}_I}{\partial z} = E_H \frac{\partial^3 \tilde{v}_I}{\partial x^3} = O\left(\frac{E_H}{L_x^3}\right)$$

Since $\tilde{\omega}_I = O(\tilde{\omega}_B) = O\left(\frac{E_V^{\frac{1}{2}}}{L_x}\right)$, we find $L_x = O\left(\frac{E_H^{\frac{1}{2}}}{E_V^{\frac{1}{4}}}\right)$. Since we now know the length scale of this boundary layer, we define x by $\tilde{x}(E_H^{\frac{1}{2}}/(E_V/2)^{\frac{1}{4}}) = x$. Rewriting equations (15) in terms of this independent variable, we have:

$$-\tilde{v}_i = -\frac{(E_V/2)^{\frac{1}{4}}}{E_H^{\frac{1}{2}}} \frac{\partial \tilde{p}_i}{\partial \tilde{x}} + \left(\frac{E_V}{2}\right)^{\frac{1}{2}} \frac{\partial^2 \tilde{u}_i}{\partial \tilde{x}^2} + E_V \frac{\partial^2 \tilde{u}_i}{\partial z^2} \tag{23a}$$

$$\tilde{u}_i = -\frac{\partial \tilde{p}_i}{\partial y} + \left(\frac{E_V}{2}\right)^{\frac{1}{2}} \frac{\partial^2 \tilde{v}_i}{\partial \tilde{x}^2} + E_V \frac{\partial^2 \tilde{v}_i}{\partial z^2} \tag{23b}$$

$$0 = -\frac{\partial \tilde{p}_i}{\partial z} \tag{23c}$$

$$\frac{\partial \tilde{u}_i}{\partial \tilde{x}} + \frac{E_H^{\frac{1}{2}}}{(E_V/2)^{\frac{1}{4}}} \frac{\partial \tilde{\omega}_i}{\partial z} + \frac{E_H^{\frac{1}{2}}}{(E_V/2)^{\frac{1}{4}}} \frac{\partial \tilde{v}_i}{\partial y} = 0, \tag{23d}$$

where $i = I, B$.

We now expand the dependent variables as follows:

$$(\tilde{v}_I, \tilde{u}_I, \tilde{\omega}_I, \tilde{p}_I) = \sum_{K=0}^{\infty} (E_V/2)^K \left[\tilde{v}_{IK}, \left(\frac{E_V}{2}\right)^{\frac{1}{2}} \tilde{u}_{IK}, \frac{(E_V/2)^{\frac{3}{4}}}{E_H^{\frac{1}{2}}} \tilde{\omega}_{IK}, \frac{E_H^{\frac{1}{2}}}{(E_V/2)^{\frac{1}{4}}} \tilde{p}_{IK}\right] \tag{24}$$

Substituting these expansions into equations (23) leads to the following two sets of equations:

$$-\tilde{v}_{I0} = -\frac{\partial \tilde{p}_{I0}}{\partial \tilde{x}} \tag{25a}$$

$$\tilde{u}_{I0} = \frac{\partial^2 \tilde{v}_{I0}}{\partial \tilde{x}^2} \tag{25b}$$

$$0 = -\frac{\partial \tilde{p}_{I0}}{\partial z} \tag{25c}$$

$$\frac{\partial \tilde{u}_{I0}}{\partial \tilde{x}} + \frac{\partial \tilde{\omega}_{I0}}{\partial z} = 0 \tag{25d}$$

and

$$-\tilde{v}_{I1} = -\frac{\partial \tilde{p}_{I1}}{\partial \tilde{x}} \tag{26a}$$

$$\tilde{u}_{I1} = -K \frac{\partial \tilde{p}_{I0}}{\partial y} + \frac{\partial^2 \tilde{v}_{I1}}{\partial \tilde{x}^2} \tag{26b}$$

$$0 = -\frac{\partial \tilde{p}_{I1}}{\partial z} \tag{26c}$$

$$\frac{\partial \tilde{u}_{I1}}{\partial \tilde{x}} + \frac{\partial \tilde{\omega}_{I1}}{\partial z} + K \frac{\partial \tilde{v}_{I0}}{\partial y} = 0, \tag{26d}$$

where

$$1 \geq K \equiv \frac{E_H^{\frac{1}{2}}}{(E_V/2)^{\frac{3}{4}}}$$

The boundary conditions at $z=0$ are

$$\tilde{\omega}_{I0}=\tilde{\omega}_{I1}=0$$

Differentiating (25 b) with respect to $\tilde{x}$, substituting the result into (25 d), noting $\frac{\partial \tilde{v}_I}{\partial z}=0$, and then integrating with respect to z yields

$$\tilde{\omega}_{I0}=-z\frac{\partial^3 \tilde{v}_{I0}}{\partial \tilde{x}^3} \tag{28 a}$$

Differentiating (26 b) with respect to $\tilde{x}$, and substituting the result into (26 d) leads to

$$\frac{\partial^3 \tilde{v}_{I1}}{\partial \tilde{x}^3}-K\frac{\partial^2 \tilde{p}_{I0}}{\partial \tilde{x}\partial y}+K\frac{\partial \tilde{v}_{I0}}{\partial y}+\frac{\partial \tilde{\omega}_{I1}}{\partial z}=0$$

From (25 a), the middle terms cancel and we obtain

$$\tilde{\omega}_{I1}=-z\frac{\partial^3 \tilde{v}_{I1}}{\partial \tilde{x}^3} \tag{28 b}$$

We note that $\tilde{u}_{I1}$ is the only dependent variable affected by the value of K. Since $\tilde{u}_{I1}=O(E_V)$, and we shall end our analysis at order $E_V^{\frac{1}{2}}$, K does not affect our results. We now analyze the bottom solution to obtain a second relation between $\tilde{\omega}_I$ and $\tilde{v}_I$. Let $\bar{z}=(z+1)/E_V^{\frac{1}{2}}$. Expanding the dependent variables as follows:

$$(\tilde{u}_B, \tilde{v}_B, \tilde{\omega}_B, \tilde{p})=\sum_{K=0}^{\infty}(E_V/2)^K\left[\tilde{u}_{BK}, \tilde{v}_{BK}, \frac{(E_V/2)^{\frac{1}{2}}}{E_H^{\frac{1}{2}}}\tilde{\omega}_{BK}, 0\right] \tag{29}$$

and substituting these expansion into (23) leads to

$$-\tilde{v}_{B0}=\frac{\partial^2 \tilde{u}_{B0}}{\partial \bar{z}^2} \tag{30 a}$$

$$\tilde{u}_{B0}=\frac{\partial^2 \tilde{v}_{B0}}{\partial \bar{z}^2} \tag{30 b}$$

$$\frac{\partial \tilde{u}_{B0}}{\partial \tilde{x}}+\frac{1}{\sqrt{2}}\frac{\partial \tilde{\omega}_{B0}}{\partial \bar{z}}=0 \tag{30 c}$$

and

$$-\tilde{v}_{B1}=\frac{\partial^2 \tilde{u}_{B1}}{\partial \bar{z}^2}+\frac{\partial^2 \tilde{u}_{B0}}{\partial \tilde{x}^2} \tag{31 a}$$

$$\tilde{u}_{B1}=\frac{\partial^2 \tilde{v}_{B1}}{\partial \bar{z}^2}+\frac{\partial^2 \tilde{v}_{B0}}{\partial \tilde{x}^2} \tag{31 b}$$

$$\frac{\partial \tilde{u}_{B1}}{\partial \tilde{x}}+\frac{1}{\sqrt{2}}\frac{\partial \tilde{\omega}_{B1}}{\partial \bar{z}}=0 \tag{31 c}$$

The boundary conditions at $z=-1$ ($\bar{z}=0$) are

$$\tilde{u}_{B0}=0 \tag{32 a}$$

$$\tilde{v}_{B0}=-\tilde{v}_{I0} \tag{32 b}$$

$$\tilde{\omega}_{B0}=-\tilde{\omega}_{I0} \tag{32 c}$$

and

$$\tilde{u}_{B1}=-\tilde{u}_{I0} \tag{33 a}$$

$$\tilde{v}_{B1}=-\tilde{v}_{I1} \tag{33 b}$$

$$\tilde{\omega}_{B1}=-\tilde{\omega}_{I1} \tag{33 c}$$

The solution to equations (30) satisfying (32 a, b) is

$$(\tilde{u}_{B0}+i\tilde{v}_{B0})=-i\tilde{v}_{I0}\exp(-\sqrt{i}\bar{z}) \tag{34}$$

Integrating (30 c) with respect to $\bar{z}$ leads to

$$\tilde{\omega}_{B0}\big|_{\bar{z}=0}=\sqrt{2}\frac{\partial}{\partial \tilde{x}}\int_0^\infty \tilde{u}_{B0}\,d\bar{z}$$

or, using (34),

$$\tilde{\omega}_{B0}\big|_{\bar{z}=0}=-\frac{\partial \tilde{v}_{I0}}{\partial \tilde{x}}$$

Satisfying (32 c), and using (28 a) evaluated at $z=-1$, leads to

$$\frac{\partial^3 \tilde{v}_{I0}}{\partial \tilde{x}^3}=\frac{\partial \tilde{v}_{I0}}{\partial \tilde{x}}$$

or

$$\tilde{v}_{I0}=A\bar{e}^{\tilde{x}} \tag{34 a}$$

$$\tilde{u}_{I0}=A\bar{e}^{\tilde{x}} \tag{34 b}$$

$$\tilde{\omega}_{I0}=zA\bar{e}^{\tilde{x}} \tag{34 c}$$

We also note that the total mass flux carried by the bottom Ekman layer, $\tilde{M}_B$, is, to lowest order

$$\tilde{M}_B=\tilde{M}_{Bx}+i\tilde{M}_{By}=E_V^{\frac{1}{2}}\int_0^\infty(\tilde{u}_B+i\tilde{v}_B)\,d\bar{z}$$

$$=-\frac{E_V^{\frac{1}{2}}}{\sqrt{2}}(1+i)A\bar{e}^{\tilde{x}} \tag{34 d}$$

To solve equations (31), we rewrite (31 a, b) as

$$\frac{\partial^2}{\partial \bar{z}^2}\tilde{S}_{B1} - i\tilde{S}_{B1} = -\frac{\partial^2 \tilde{S}_{B0}}{\partial \tilde{x}^2}$$

where $\tilde{S}_{Bi} = \tilde{u}_{Bi} + i\tilde{v}_{Bi}$ $i = 0, 1$. Using equations (34)

$$\frac{\partial^2 \tilde{S}_{B1}}{\partial \bar{z}^2} - i\tilde{S}_{B1} = iA\,\bar{e}^{\tilde{x}} \exp\,(-\sqrt{i}\,\bar{z}) \qquad (35)$$

The solution to (35) which satisfies (33a, b) is

$$\tilde{S}_{B1} = -\left[(A\,\bar{e}^{\tilde{x}} + i\tilde{v}_{I1}) + \frac{\sqrt{i}}{2}\,A\,\bar{z}\,\bar{e}^{\tilde{x}}\right] \exp\,(-\sqrt{i}\bar{z}) \qquad (36)$$

Substituting (36) into (31) and integrating with respect to $\bar{z}$ leads to

$$\tilde{\omega}_{B1}\big|_{\bar{z}=0} = -\frac{\partial \tilde{v}_{I1}}{\partial \tilde{x}} + \frac{3}{2}A\,\bar{e}^{\tilde{x}}$$

Satisfying the final condition, (33c), using (28b) evaluated at $z = -1$, yields:

$$\frac{\partial^3 \tilde{v}_{I1}}{\partial \tilde{x}^3} = \frac{\partial \tilde{v}_{I1}}{\partial \tilde{x}} - \frac{3}{2}A\,\bar{e}^{\tilde{x}}$$

We solve this to obtain:

$$\tilde{v}_{I1} = \sqrt{2}\,B\bar{e}^{\tilde{x}} - \tfrac{3}{4}A\,\tilde{x}\,\bar{e}^{\tilde{x}} \qquad (37)$$

To summarize our coastal boundary layer solutions through order $E_V^{\frac{1}{2}}$, and now letting

$$(\tilde{\bar{u}}_I, \tilde{\bar{v}}_I, \tilde{\bar{\omega}}_I)$$

denote the total coastal boundary layer correction to the interior solution, u_I, v_I, ω_I, we have:

$$\tilde{\bar{v}}_I = A\,\bar{e}^{\alpha\bar{x}} + E_V^{\frac{1}{2}}\left[B\,\bar{e}^{\alpha\bar{x}} - \frac{3}{4}\frac{A}{\sqrt{2}}\alpha\,\bar{x}\,\bar{e}^{\alpha\bar{x}} - \tau\,\bar{e}^{\bar{x}}\cos\,(\theta - \bar{x})\right] + 0(E_V) \quad (38\text{a})$$

$$\tilde{\bar{u}}_I = E_V^{\frac{1}{2}}\left[\frac{A}{\sqrt{2}}\,\bar{e}^{\alpha\bar{x}} + \tau\,\bar{e}^{\bar{x}}\sin\,(\theta - \bar{x})\right] + O(E_V) \quad (38\text{b})$$

$$\tilde{\bar{\omega}}_I = \frac{E_V^{\frac{1}{2}}}{E_H^{\frac{1}{2}}}\left[(z+1)\,\tau\,\bar{e}^{\bar{x}}\sin\left(\theta + \frac{\pi}{4} - \bar{x}\right) + \frac{E_V^{\frac{1}{4}}}{2^{\frac{3}{4}}}\,zAe^{-\alpha\bar{x}}\right], \quad (38\text{c})$$

where $\alpha \equiv (2E_V)^{\frac{1}{4}}$, $\bar{x} = x/\sqrt{2E_H}$. The volume flux carried by the modified top and bottom layers, to order, $E_V^{\frac{1}{2}}$, are

$$\tilde{\bar{M}}_T = -E_V^{\frac{1}{2}}(\tau_y - i\tau_x)\exp\,(-\sqrt{2i}\,\bar{x}) \quad (38\text{d})$$

$$\tilde{\bar{M}}_B = -\frac{E_V^{\frac{1}{2}}}{\sqrt{2}}(1+i)\,A\,\bar{e}^{\alpha\bar{x}} \quad (38\text{e})$$

The interior solution near $x = 0$ is, noting that we reqired that $u_{I0} = 0$ at $x = 0$,

$$u_I = E_V^{\frac{1}{2}}u_{I1}\big|_{x=0} + O(E_V) \quad (39\text{a})$$

$$v_I = v_{I0}\big|_{x=0} + E_V^{\frac{1}{2}}v_{I1}\big|_{x=0} + O(E_V) \quad (39\text{b})$$

where $v_{I0}\big|_{x=0}$ has already been determined from (12), (14) and (6a). The volume flux carried by the *interior* top and bottom Ekman layers are, near $x = 0$,

$$M_T = E_v^{\frac{1}{2}}(\tau_y - i\tau_x) \quad (39\text{c})$$

$$M_B = -\frac{E_V^{\frac{1}{2}}}{\sqrt{2}}(1+i)\,v_{I0}\big|_{x=0}, \quad (39\text{d})$$

To satisfy condition (I), we choose

$$A = -v_{I0}\big|_{x=0}$$

This choice of A also brings the volume flux in the bottom Ekman layers to zero at $x = 0$; this completes condition (II). To satisfy condition (III), at $x = 0$, we must choose

$$u_{I1}\big|_{x=0} = -\left[\frac{A}{\sqrt{2}} + \tau\sin\theta\right] = -\left[\tau\sin\theta - \frac{v_{I0}}{\sqrt{2}}\big|_{x=0}\right] \quad (41\text{a})$$

and $$v_{I1}\big|_{x=0} = -[B - \tau\cos\theta] \quad (41\text{b})$$

We note that the right-hand side of (41a) is known from the zeroth order solution and equals the volume flux normal to the coast carried shorewards by the interior Ekman layers. Therefore, with $u_{I1}\big|_{x=0}\left(= -\frac{\partial p_{I1}}{\partial y}\big|_{x=0}\right)$ specified at the coast, equation (13) can be solved yielding $v_{I1}\big|_{x=0}$.

With $v_{I1}\big|_{x=0}$ known the problem is completed by choosing

$$B = \tau\cos\theta - v_{I1}\big|_{x=0}$$

Thus, the total velocity near the coast and outside of the Ekman layers which we now denote as u_T, v_T and ω_T, is

$$v_T = v_{I0}\big|_{x=0}(1-\bar{e}^{\alpha\bar{x}}) + E_V^{\frac{1}{2}}\left[\frac{3\,v_{I0}}{4\sqrt{2}}\Big|_{x=0}\alpha\bar{x}\,\bar{e}^{\alpha\bar{x}} + \tau(\cos\theta\,\bar{e}^{\alpha\bar{x}} - \bar{e}^{\bar{x}}\cos(\theta-\bar{x})) + v_{I1}\big|_{x=0}(1-\bar{e}^{\alpha\bar{x}})\right] + O(E_V) \quad (42\,a)$$

$$u_T = E_V^{\frac{1}{2}}\left[\frac{v_{I0}}{\sqrt{2}}\Big|_{x=0}(1-\bar{e}^{\alpha\bar{x}}) + \tau\bar{e}^{\bar{x}}\sin(\theta-\bar{x}) - \tau\sin\theta\right] + O(E_V) \quad (42\,b)$$

$$\omega_T = \frac{E_V^{\frac{1}{2}}}{E_H^{\frac{1}{2}}}\left[(z+1)\,\tau\bar{e}^{\bar{x}}\sin\left(\theta+\frac{\pi}{4}-\bar{x}\right) - \frac{E_V^{\frac{1}{2}}}{2^{\frac{1}{2}}}\,z\,v_{I0}\big|_{x=0}\bar{e}^{\alpha\bar{x}} + O(E_V)\right] \quad (42\,c)$$

By substituting (42 c) into the right-hand side of (2 c), we can verify that the error produced by making the hydrostatic approximation is of order $E_V^{\frac{1}{2}}$ smaller than the smallest retained term if

$$(H/L)^2 \leqslant E_H E_V^{\frac{1}{2}} (\leqslant E_V^{\frac{3}{2}} < < 1) \quad (43)$$

We note that in (42 b) we omit terms of order E_V with respect to terms of order $E_V^{\frac{1}{2}}$. To be consistent with this, we should omit terms of order $E_V^{\frac{1}{2}}$ with respect to terms of order 1 in the rest of our solution. First consider the case when $v_{I0}|_{x=0} \neq 0$. The first and third terms inside the bracket on the right-hand side of (42 a) are order $E_V^{\frac{1}{2}}$ smaller than the term outside the bracket and must be omitted. The remaining term in the bracket is only order $E_V^{\frac{1}{2}}$ smaller than the term outside the bracket for $\bar{x}=O(1)$, and must be retained. Therefore for $v_{I0}|_{x=0}\neq 0$, we are consistent by taking

$$v_T = v_{I0}\big|_{x=0}(1-\bar{e}^{\alpha\bar{x}}) + E_V^{\frac{1}{2}}\tau\,[\cos\theta\bar{e}^{\alpha\bar{x}} - \bar{e}^{\bar{x}}\cos(\theta-\bar{x})] \quad (42\,a)$$

Equations (42 b, c) are consistent as they stand. If $v_{I0}|_{x=0}=0$ which may be caused by an irrotational wind stress field ($v_{I0}\equiv 0$), then dropping these terms in equations (42) leads to the consistent results:

$$v_T = E_V^{\frac{1}{2}}[v_{I1}|_{x=0}(1-\bar{e}^{\alpha\bar{x}}) + \tau(\cos\theta\,\bar{e}^{\alpha\bar{x}} - \bar{e}^{\bar{x}}\cos(\theta-\bar{x}))] \quad (44\,a)$$

$$u_T = E_V^{\frac{1}{2}}[\tau\bar{e}^{\bar{x}}\sin(\theta-\bar{x}) - \tau\sin\theta] \quad (44\,b)$$

$$\omega_T = \frac{E_V^{\frac{1}{2}}}{E_H^{\frac{1}{2}}}[(z+1)\,\tau\bar{e}^{\bar{x}}\sin(\theta+\pi/4-\bar{x})] \quad (44\,c)$$

In either case, we can define a stream function, ψ_T, such that

$$u_T = \frac{\partial\psi_T}{\partial z},\quad \omega_T = -\frac{\partial\psi_T}{\partial x} \quad (45)$$

This function is

$$\psi_T = E_V^{\frac{1}{2}}\left[\frac{v_{I0}}{\sqrt{2}}\Big|_{x=0}(1-\bar{e}^{\alpha\bar{x}})\,z - (1+z)\,\tau(\sin\theta - \bar{e}^{\bar{x}}\sin(\theta-\bar{x}))\right] \quad (46)$$

Lines of constant ψ_T then correspond to projections of streamlines in the (x, z) plane. Also $-\psi_T(0,\bar{x})$ equals the total volume flux carried by the top Ekman layers in the x direction, and $\psi_T(-1,\bar{x})$ equals the flux carried by the lower Ekman layers. If $v_{I0}|_{x=0}=0$, then (46) implies that it is possible for ψ_T to vanish at some value of $\bar{x}$, say $\bar{x}_c$, other than zero. In fact for $\bar{x}\neq 0$, ψ_T vanishes if

$$\tan\theta = -\frac{\bar{e}^{\bar{x}_c}\sin\bar{x}_c}{1-\bar{e}^{\bar{x}_c}\cos\bar{x}_c} \quad (47)$$

Except for the case $\theta\approx 0$, for each value of in the "confinement range" $(-(\pi/4), 0)$ and $(3\pi/4, \pi)$ there, will be one value of $\bar{x}$, say $\bar{x}_c$, where ψ_T, hence u_T vanishes. At this value of $\bar{x}$ there will be no net exchange of mass with the interior by way of the Ekman layer nor any exchange in the underlying region. The fluid in the range $(0, \bar{x}_c)$ is confined there and moves along the shore. This has practical significance in pollution problems. A plot of θ versus the confinement distance, $\bar{x}_c$, is presented in Fig. 2.

If $v_{I0}|_{x=0}\neq 0$, the flow field can become quite complicated and no generalizations seem possible. Therefore we illustrate a few interesting examples.

First we consider the case when $v_{I0}=0$, $v_{I1}=1.0$, $\tau=1.0$, $\theta=-0.25$ radians, and $E_V=0.01$. We plot the *total* volume flux carried in the $\bar{x}$

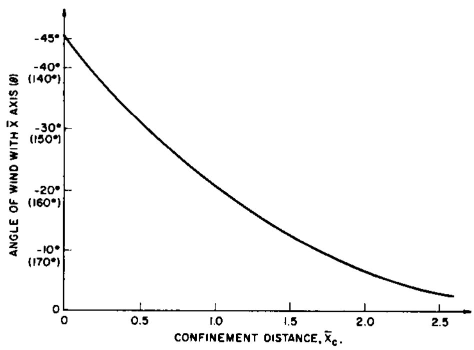

Fig. 2. A plot of the confinement distance, $\bar{x}_c$, versus the angle the wind makes with the $\bar{x}$ axis.

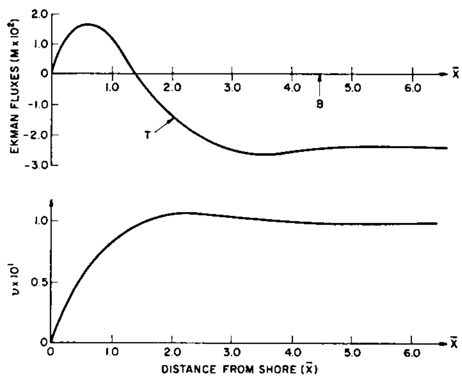

Fig. 3. A plot of the Ekman layer fluxes and the tangential velocity versus $\bar{x}$ for $v_{I0}|_{x=0} = 0$, $v_{I1}|_{x=0} = 1.0$, $\tau = 1.0$, $\theta = -0.25$, $E_V = 0.01$.

direction by the top and bottom Ekman layers, now denoted by M_T, M_B respectively, versus $\bar{x}$ and the tangential velocity versus $\bar{x}$ in Fig. 3, and we plot the streamlines in the $\bar{x}-z$ plane in Fig. 4. We note that, to lowest order, $M_{T\infty} = -0.0247$, $M_B \equiv 0$, and that this is a case of confinement. We see from Fig. 4 that the mass carried by the top layer towards the shore is ejected in the region $1.4 < \bar{x} < 3.0$. This mass flows outward filling the region $-1 \leqslant z \leqslant 0$. For $0 < \bar{x} < 1.4$ there is a region of confined flow.

We next consider the case when $v_{I_0} = 1.0$, $\tau = 1.0$, $\theta = -0.25$, and $E_V = 0.01$. The results are shown in Fig. 5 and 6. We note that $M_{T\infty} = -0.0247$ and $M_{B\infty} = -.0707$. The mass carried towards shore in the top layer is again ejected in the region $1.4 < \bar{x} < 3.0$. This mass now returns to the interior filling the region $-0.3 < z \leqslant 0$. The bottom layer also carries mass towards, shore. About three quarters of this mass is

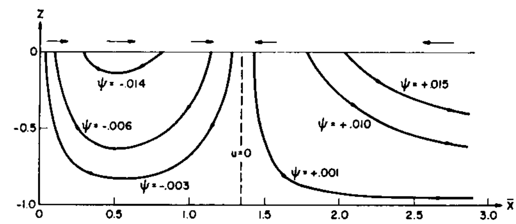

Fig. 4. Projection of the streamlines in the $\bar{x}-z$ plane for $v_{I0}|_{x=0} = 0$, $v_{I1}|_{x=0} = 1.0$, $\tau = 1.0$, $\theta = -0.25$, $E_V = 0.01$.

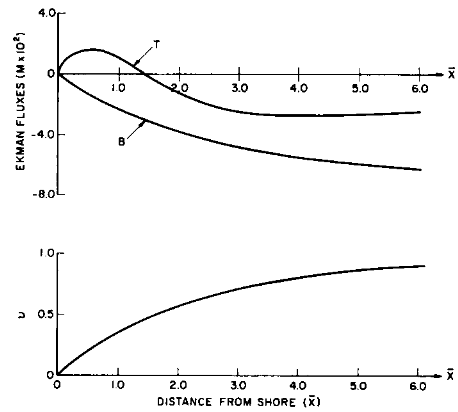

Fig. 5. A plot of the Ekman layer fluxes and the tangential velocity versus $\bar{x}$ for $v_{I0}|_{x=0} = 1.0$, $v_{I1}|_{x=0} = 0.0$, $\tau = 1.0$, $\theta = -0.25$, $E_V = 0.01$.

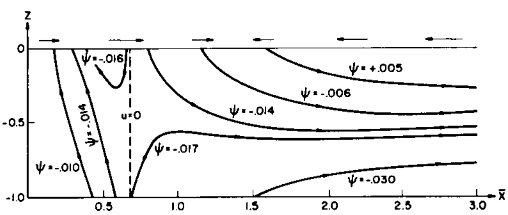

Fig. 6. Projection of the streamlines in the $\bar{x}-z$ plane for $v_{I0}|_{x=0} = 1.0$, $v_{I1}|_{x=0} = 0.0$, $\tau = 1.0$, $\theta = -0.25$, $E_V = 0.01$.

ejected in the region $\bar{x} > 0.7$ and returns directly to the interior. The remainder of the mass leaves the bottom layer in $0 < \bar{x} < 0.7$. It flows under a small recirculating region and enters the top layer in $0 < \bar{x} < 0.4$. The top layer carries this mass away from shore. It is ejected from the top layer in $0.7 < \bar{x} < 1.4$ and flows towards the interior. Hence the region $0 < \bar{x} < 1.4$ contains mostly waters originating in the bottom Ekman layer.

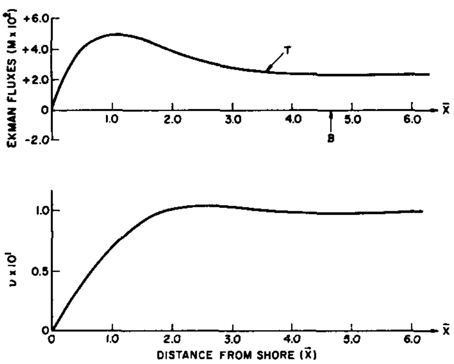

Fig. 7. A plot of the Ekman layer fluxes and the tangential velocity versus $\bar{x}$ for $v_{I0}|_{x=0}=0$, $v_{I1}|_{x=0}=1.0$, $\tau=1.0$, $\theta=+0.25$, $E_V=0.01$.

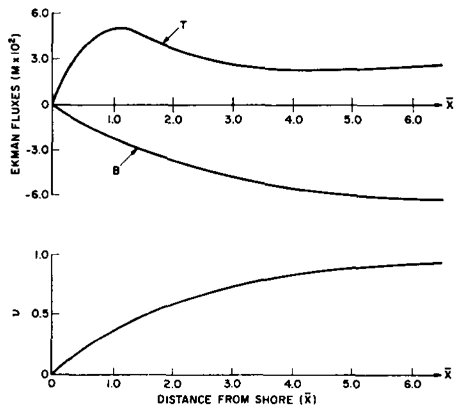

Fig. 9. A plot of the Ekman layer fluxes and the tangential velocity versus $\bar{x}$ for $v_{I0}|_{x=0}=1.0$, $v_{I1}|_{x=0}=0$, $\tau=1.0$, $\theta=+0.25$, $E_V+0.01$.

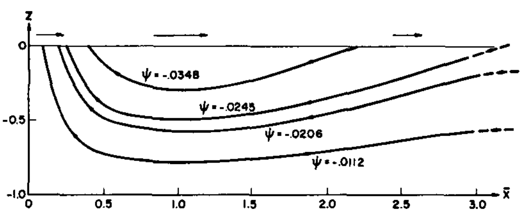

Fig. 8. Projection of the streamlines in the $\bar{x}-z$ plane for $v_{I0}|_{x=0}=0$, $v_{I1}|_{x=0}=1.0$, $\tau=1.0$, $\theta=+0.25$, $E_V=0.01$.

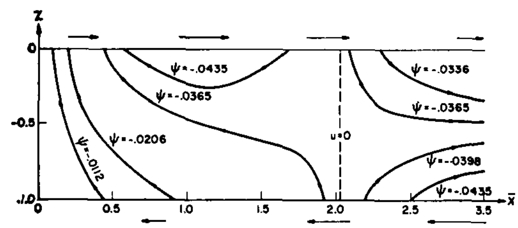

Fig. 10. Projection of the streamlines in the $\bar{x}-z$ plane for $v_{I0}|_{x=0}=1.0$, $v_{I1}|_{x=0}=0$, $\tau=1.0$, $\theta=+0.25$, $E_V-0.01$.

The third case considered is when $v_{I_0}=0$, $v_{I1}=1.0$, $\tau=1.0$, $\theta=+0.25$, and $E_V=0.01$. Here $M_{T\infty}=+0.0247$ and $M_B\equiv 0$. The relevant information appears in Fig. 7 and 8. A flow of mass of flux -0.0247 approaches the shore under the top Ekman layer. It flows under a large recirculating region which occupies the region $0.2<\bar{x}<4.0$ with a maximum depth $|z_{\max}|\sim 0.5$, and enters the top Ekman layer in $0<\bar{x}<0.2$. This layer flows outwards and continues to receive mass from the recirculating region until, at $\bar{x}=1.0$, $M_T\sim 0.05$. It then gradually ejects the excess mass back into the recirculating region in $1.0<\bar{x}<4.0$ and the remainder ($M=0.0247$) flows into the interior in the top Ekman layer.

The last case considered here has $v_{I_0}=1.0$, $\tau=1.0$, $\theta=0.25$, and $E_V=0.01$. Here $M_{T\infty}=+0.0247$, $M_{B\infty}=-.0707$, and the relevant material appears in Fig. 9 and 10. The bottom layer carries mass towards shore. About one-half of this mass leaves the layer in $\bar{x}>2.0$ and flows outward occupying the region $-1.0<z<-0.3$. The remaining half of the mass is ejected from the bottom layer in the region $0<\bar{x}<2.0$. The top layer flows outwards over the recirculating region and about one-third of the mass it carries at $\bar{x}=2.0$ is ejected and flows outwards occupying the region $-0.3<z\leqslant 0$. The remaining mass flows away from the shore in the top layer.

Thus we see some of the great variety of flow patterns described in Equation (46). Apart from the case of confinement, no generalizations seem possible regarding the details of the transfer of mass from the Ekman layers to the interior. It is this method of transfer which may be of significance to life at the edge of the basin.

Some Final Comments

We note that the parameter range under consideration

$$\left(\frac{H}{L}\right)^2 \leqslant \left(\frac{A_V}{fH^2}\right)^{\frac{1}{2}} \frac{A_H}{fL^2} \leqslant \left(\frac{A_V}{fH^2}\right)^3 < < 1$$

is of some physical interest: For example, reasonable values for (H, L, f, A_V, A_H) for Lake Erie, located in the northeastern United States, are respectively (2×10^3 cm., 10^7 cm, 10^{-4} rad/sec, 10 cm^2/sec, 10^4 cm^2/sec). The above inequality is satisfied by these values and becomes:

$$4 \times 10^{-8} \leqslant 10^{-7} \leqslant 10^{-5} < < 1$$

We also note that the results obtained here are based on the crude modeling of turbulent mixing by the use of constant values of the Austauch coefficients. How crude the results obtained by this modeling are, can be determined either by comparison with the results obtained by the use of a more realistic model of turbulent mixing or by comparison with observations. At present neither of these comparisons exists. If this theory predicts the general wind configuration under which confinement occurs or the general wind configuration under which mixing of top and bottom waters is enhanced or suppressed, it should be considered fairly successful.

This brings us to the effects of stratification which are currently under study for the above range of parameters. We may tentatively observe that since many lakes undergo fairly large annual variations in their stratification the results obtained here may apply over some period of the year. We must await the results of the current study for a more quantitative statement.

Lastly, we initially required that the radius of curvature of the coast be on the order of L. This condition is more stringent than it need be. We can simply require that ratio of boundary layer thickness to radius of curvature R be of order $E_V^{\frac{1}{2}}$. This leads to the condition that

$$\frac{R}{L} \geqslant E_V^{\frac{1}{2}}.$$

for our theory to be valid everywhere.

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ПРИБРЕЖНЫЙ ПОГРАНИЧНЫЙ СЛОЙ В ОЗЕРЕ, КОГДА ГОРИЗОНТАЛЬНОЕ И ВЕРТИКАЛЬНОЕ ЧИСЛА ЭКМАНА ИМЕЮТ РАЗЛИЧНЫЙ ПОРЯДОК ВЕЛИЧИН

Мы исследуем линейные гидростатические пограничные слои, которые обеспечивают затухание скорости у берега озера с горизонтальным размером L и глубиной H, содержащего однородную жидкость с постоянными, хотя и существенно различными горизонтальной и вертикальной вихревыми вязкостями A_H и A_V. Озеро вращается с угловой скоростью $f/2$ относительно вертикальной оси, и движение индуцируется стационарным неоднородным напряжением ветра. Наши результаты применимы, когда $(H/L^2 \leqslant (A_V/fH^2)^{1/2}(A_H/fL^2) \leqslant (A_V/fH^2)^3 \ll 1$.

Найдено, что при этих условиях течение вблизи берега определяется локальным напряжением ветра и локальным значением скорости, параллельной берегу. Найдено, что при некотором напряжении ветра существует течения, прилежащия к берегу, в которой жидкость эффективно ограничена у берега.