SHORTER CONTRIBUTION

A note on inertial instability[1]

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Introduction

Blumen & Washington (1969) suggest that "pure" centrifugal instability should be a rare phenomenon in the atmosphere. This conclusion is based upon an analysis of a two-layer hydrostatic model and appears to contradict the concept of inertial instability set forth in text books on dynamic meteorology. In addition this result seems contrary to previous theoretical results concerning the stability of a balanced vortex to cylindrically symmetrical disturbances. See, e.g., Eliassen & Kleinschmidt (1957) for a listing of references.

Acceptance of this general conclusion of Blumen & Washington (henceforth referred to as BW) would eliminate inertial instability as an explanation for the upper bound on observed synoptic scale wind shears on the equatorward side of jet streams. See Reiter (1961). It is the purpose of this note to discuss certain limitations of the analysis in BW and to offer a simple model which provides a more general understanding of inertial instability as it might occur in the atmosphere.

The special nature of the BW model is easy to understand. We recall that their model consisted of two incompressible layers separated by an interface with zonally symmetrical heights variations. To facilitate analysis BW made the additional assumption that the upper layer was inert. This has the effect of limiting the basic geostrophic current to a special *baroclinic* form since the basic flow is not zero in the lower layer. The conclusions given in BW were thus inappropriate for general symmetric disturbances on a barotropic current for a two-layer model. On the other hand, the governing equations in BW coincide with those derived from a one-layer model using a reduced value of gravity. In this case the basic flow is indeed barotropic, but the disturbances are then constrained to be barotropic also.

It would seem that neither interpretation of the BW model corresponds to a model of general (i.e., baroclinic) symmetric disturbances on a barotropic base current. The consequences of these special features for current stability can be discussed in terms of the interface slope in the mass continuity equation. As mentioned in BW, this basic slope has the effect of stabilizing the disturbances even when the horizontal divergence is zero.

The analysis shown in the next section is an attempt to provide a link between the results of BW (where the response of the pressure field to the disturbance motion is very special) and other results where the response is lessened or even eliminated by the use of particle dynamics.

Mathematical analysis

The model considered here is hydrostatic, adiabatic and inviscid with no variations in the x-direction. A tangent plane is assumed. In all these respects this model is identical to that of BW. The primary difference is in the vertical structure. The BW model was essentially a one-layer model, and the dynamic interactions did not involve any vertical variation of the horizontal velocity within the lower layer. In the model considered here variations in the vertical are permitted.

The linear model is formulated in pressure coordinates with the mean state assumed to be barotropic, of a constant static stability, and with a locally constant horizontal wind shear.

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The linearized perturbation equations can be written:

$$\frac{\partial u_1}{\partial t} + v_1 \left(F \frac{dU_0}{dy} - 1 \right) = 0, \tag{1}$$

$$\frac{\partial v_1}{\partial t} + u_1 + \frac{\partial h}{\partial y} 1 = 0, \tag{2}$$

$$\frac{\partial \omega_1}{\partial p} + \frac{\partial v}{\partial y} 1 = 0, \tag{3}$$

$$\frac{\partial}{\partial t} \left(\frac{\partial h_1}{\partial p} \right) + \omega_1 S = 0. \tag{4}$$

These are the two horizontal equations of motion, the continuity equation and the first law of thermodynamics respectively. The notation is made as similar as possible to that used by BW:

t: time nondimensionalized by f, the constant Coriolis parameter.

y: The horizontal coordinate in the northward direction nondimensionalized by the radius of deformation, defined as c/f where c is a reference horizontal phase speed of long gravity waves.

p: the vertical coordinate of pressure nondimensionalized by a reference pressure, p_r.

u_1: perturbation x-component of velocity nondimensionalized by c.

v_1: perturbation y-component of velocity nondimensionalized by c.

U_0: Mean x-component of velocity nondimensionalized by c.

F: Froude number, the scaling magnitude of U_0.

h_1: perturbation geopotential height of a pressure surface nondimensionalized by the depth c^2/g.

ω_1: perturbation total pressure derivative nondimensionalized by p_r and f.

S: Static stability of the mean state equal to

$$-\frac{p_r}{\varrho_0 c^2} \frac{\partial \ln \theta_0}{\partial p}$$

where $\varrho_0(y, p)$ and $\theta_0(y, p)$ are the density and potential temperature of the mean state.

The coefficients of the governing equations are all assumed to be constant in space and time in order to facilitate the mathematical analysis. In order to evaluate local wave disturbances the following forms are assumed for the dependent variables:

$$u_1 = \text{Real}\,(\hat{u}\, e^{i(ly + mp - \sigma t)}), \tag{5 a}$$

$$v_1 = \text{Real}\,(\hat{v}\, e^{i(ly + mp - \sigma t)}), \tag{5 b}$$

$$\omega_1 = \text{Real}\,(\hat{\omega}\, e^{i(ly + mp - \sigma t)}), \tag{5 c}$$

$$h_1 = \text{Real}\,(\hat{h}\, e^{i(ly + mp - \sigma t)}). \tag{5 d}$$

Equations (1) to (4) are converted to equivalent equations for the Fourier coefficients, $\hat{u}$, $\hat{v}$, $\hat{\omega}$, $\hat{h}$. This set of equations is homogeneous and possesses non-trivial solutions only if the determinant of the coefficients is zero. This condition yields the following frequency equation (if we disregard the geostrophic root, $\sigma = 0$:

$$\sigma = \pm \sqrt{\frac{l^2 S}{m^2} - \left(F \frac{dU_0}{dy} - 1 \right)}. \tag{6}$$

This indicates that instability will occur for[1]

$$F \frac{dU_0}{dy} > \frac{l^2 S}{m^2} + 1. \tag{7}$$

Discussion

The two terms on the right hand side of the inequality (7) can be related to physical processes. If the first term is ignored, the condition for instability is

$$F \frac{dU_0}{dy} > 1 \tag{8}$$

which is the result obtained from particle dynamics considerations where the pressure field is assumed to be independent of fluid motions. The first term on the right contains the static stability parameter S and can be related to the baroclinic velocity-pressure interactions in the fluid. If S is negative, then this term always tends to destabilize the flow (see equation (7)]; i.e., the effects of static instability are incorporated in this analysis.

For stable stratification the term $l^2 S/m^2$ in (7) is proportional to the ratio of the radius of deformation to the horizontal length scale of the

[1] The authors' attention has recently been called to a paper by S. O. R. Wilson (*Quart. J. R. Met. Soc. 91*, 132–139) where a comparable expression is derived.

wave. This indicates that waves of small horizontal scale will be more stable than those of larger scale. The term also shows that shallow wave modes (large m) will be less stable than deep ones. The deeper wave modes correspond to the more barotropic wave motions, and in this respect the results are consistent with the result of the BW model where the barotropic disturbances are unconditionally stable.

When discussing instability in the real atmosphere, one must consider all possible vertical wave modes as well as the special barotropic one. The instability criterion (7) indicates that for large m the criterion approaches the classical criterion, $F\,dU_0/dy > 1$, and we can conclude that inertial instability is a relevant concept for baroclinic disturbances in the atmosphere. The analysis does show that the instability would tend to proceed with motions with vertical scales much less than the "total depth" of the atmosphere.

In closing it should be noted that for a fluid with a baroclinic mean state, the tendency for instability may be even greater than indicated by (7). (See e.g., Eliassen & Kleinschmidt, 1957.) This supports our contention that inertial instability is relevant to atmospheric motions and sets the results of BW into an even more special category. These conclusions also suggest that caution must be exercised in studying primitive-equation models with a one-layer representation.

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