Geometry from simultaneous satellite ranging

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ABSTRACT

The geometric approach to satellite geodesy through simultaneous observations is discussed in general and as regards pure simultaneous ranging to a satellite in some detail. Closed expressions are derived for the geometric conditions imposed upon a network of satellite tracking stations joining in simultaneous ranging. The mathematical procedure suggested is symmetric in all stations and only involves the unknown distances between them and the measured ranges to a satellite. A comparison is made with similar 1- and 2-dimensional problems.

1. Introduction

Satellite geodesy is concerned with the acquisition and processing of geodetic information from orbital observations of artificial earth satellites, the observations being made from earth-based tracking stations. This loosely termed and somewhat restrictive definition serves the present purpose. Geometric connections between the tracking sites resulting from the observations from these sites are certainly pieces of geodetic information. In a way such connections sample the size and shape of the earth's surface the determination of which should be the ultimate objective of geodetic activity.

Geometric ties could be obtained from the observations relatively easily if the orbits of the satellites were known; but they are not, because the forces acting on the satellite are not. On the other hand the measured orbits yield valuable information on these forces, notably on the geopotential, the latter to be used geodetically in another approach. Because of the presence of non-geopotential forces it can be difficult to extract the geodetic data from the satellite observations, but even if such forces could be eliminated there remains through the observations a complex interplay between the geometry of the network of tracking stations and the geopotential.

These problems can be by-passed using the artifice of making the observations simultaneously from a number of stations and this approach leads to what is known as geometric satellite geodesy. Geometric satellite geodesy is a degenerate case in that the orbits disintegrate into some discrete points the locations of which are supposed to be unknown and to be not related to each other through the orbit. The simultaneity of the observations imposes geometric conditions upon the participating tracking sites. This approach is simple, straightforward and clear-cut. The price paid for this simplicity is the obvious inability to produce information concerning the geopotential and an immediate consequence of this is that the results obtained are in no way related to the centre of mass of the earth.

There is certainly no intention of giving an exposition of observation techniques but a few brief remarks do not seem to be out of place. The observations could be of different type, both optical and non-optical, usually referred to as radio-observations. Several kinds of radio-observation have been suggested or used, but insofar as simultaneous observations are concerned the SECOR (Sequential Collation of Range) system is probably the most typical. Photographic determination of direction avoids the problems of astronomic refraction to a great extent and can nowadays be made with standard deviations of about one second of arc. Optical ranging to satellites was introduced around 1965 when sufficiently powerful lasers became available. Such ranging is practically limited to special satellites equipped with corner cube retro-reflectors. Accuracies of about 1 meter standard deviation are being claimed already and it is likely that within the next

5 years this figure will be brought down to some decimeters or even below that.

The required degree of simultaneity is through the velocity of the satellite intimately connected with the accuracy of the observations. Consequently the present timing requirement will be as high as 1 000 or 100 μsec with respect to an arbitrary time reference, common however to all stations participating in a simultaneous observation event. Such requirements are rather demanding and though at present worldwide time distribution with an accuracy of 100 μsec is quite feasible, it is something else to produce observations with that degree of simultaneity. Therefore it has become practice to construct so-called synthetic simultaneous observations by interpolation in sequences of nearly simultaneous observations. Any distinction between strictly and synthetic simultaneous observations is irrelevant in the present context.

2. About space-triangulation

Although we should be primarily interested here in the treatment of simultaneous rangings it is worth while to briefly consider simultaneous directions as well. In the past 6 or 7 years many simultaneous directions have been obtained photographically, most of them pairwise but also many in triple, quadruple and higher multiple events. Once these directions have been properly corrected and reduced to an earth-fixed system of reference their geometrical treatment is not so difficult.

n simultaneous directions yield $2n-3$ conditions to be independently satisfied by the $\frac{1}{2}n(n-1)$ unit vectors $\mathbf{e}_{ij} = \|e^1_{ij}, e^2_{ij}, e^3_{ij}\|$ which stand for the chord-line directions between stations P_i and P_j ($i, j = 1 \dots n$; $i \neq j$). The conditions are of type

$$|E| \equiv \begin{vmatrix} e^1_{is} & e^2_{is} & e^3_{is} \\ e^1_{js} & e^2_{js} & e^3_{js} \\ e^1_{ij} & e^2_{ij} & e^3_{ij} \end{vmatrix} = 0 \qquad (1)$$

where $\|e^1_{is}, e^2_{is}, e^3_{is}\|$ and $\|e^1_{js}, e^2_{js}, e^3_{js}\|$ are the directions to the satellite P_s as measured from P_i and P_j respectively. (1) simply states that the three vectors should be linearly dependent and hence coplanar.

Another n-tuple of simultaneous directions from the same stations completes a sufficient set of conditions to calculate directions $\mathbf{e}_{ij}$ uniquely as intersections of pairs of planes. These planes are basic to the approach which was introduced by Väisälä (1946) as an astronomical method of triangulation using balloon or rocket-borne flashes as simultaneous targets.

It should be mentioned that additional conditions of type (1) enter when $n > 2$ expressing the requirement that any three vectors $\mathbf{e}_{ij}$, $\mathbf{e}_{jk}$, $\mathbf{e}_{ki}$ between ground-stations must be linearly dependent. Therefore the system of conditions resulting from two n-tuple events is overdetermined when $n > 2$. Like most problems this fairly simple geometric one is quite involved statistically, since vector $\mathbf{e}_{is}$ possibly makes an element of several conditions and moreover $\mathbf{e}_{is}$ and $\mathbf{e}_{js}$ could in general be correlated, but these complications are far beyond the scope of this paper.

From (1) it follows that

$$|E \cdot E^*| = |E| \cdot |E^*| = \begin{vmatrix} \mathbf{e}_{is}\cdot\mathbf{e}_{is} & \mathbf{e}_{is}\cdot\mathbf{e}_{js} & \mathbf{e}_{is}\cdot\mathbf{e}_{ij} \\ \mathbf{e}_{js}\cdot\mathbf{e}_{is} & \mathbf{e}_{js}\cdot\mathbf{e}_{js} & \mathbf{e}_{js}\cdot\mathbf{e}_{ij} \\ \mathbf{e}_{ij}\cdot\mathbf{e}_{is} & \mathbf{e}_{ij}\cdot\mathbf{e}_{js} & \mathbf{e}_{ij}\cdot\mathbf{e}_{ij} \end{vmatrix} = $$
$$= \begin{vmatrix} 1 & \cos\varphi_s & \cos\varphi_i \\ \cos\varphi_s & 1 & \cos\varphi_j \\ \cos\varphi_i & \cos\varphi_j & 1 \end{vmatrix} = 0 \qquad (2)$$

E^* is meant to be the transpose of E and φ_s stands for the angle between vectors $\mathbf{e}_{is}$ and $\mathbf{e}_{js}$ etc.

The determinant in the third member of (2) is a Gram-determinant the vanishing of which here is a necessary and sufficient condition for the linear dependence of vectors $\mathbf{e}_{is}$, $\mathbf{e}_{js}$, $\mathbf{e}_{ij}$ (Courant & Hilbert, 1953). Hence (2) appears as another, awkward, way of writing (1), but (2) is of interest since its generalizations to orders higher than 3 have a useful application in another field of geometrical satellite geodesy. They yield a compact and general mode to express in closed form the geometric relations imposed upon the network of observing stations participating in simultaneous ranging in much the same way as conditions (1) do upon a network participating in simultaneous direction measurements.

3. Four-station ranging

Four-station ranging has been dealt with repeatedly and the process has been called 'trispheration' in connection with SECOR, although in general (spatial) 'trilateration' is used.

Trispheration suggests a stepwise process where three of the four stations are in known locations, the fourth is unknown and to be determined through ranging to satellites simultaneously with the three privileged stations. Each quadruple event instantaneously fixes the satellite in space and this in turn yields a spherical locus for the unknown station. Three of such loci, and thus three quadruple ranging events are in general necessary and sufficient to determine the fourth station by intersection of three spheres. Whatever reasonable locations are chosen for the three 'known' stations there will be a solution for the fourth, as long as the three locating spheres intersect. In fact there will be two solutions but one should be able to intelligently discard the one which is not applicable.

This procedure is illustrative and if the linear uncertainties in the locations of the known stations are at most of the order of those in the range-measurements it can be applied without harm, but with laser rangings, accurate to say 1 meter, this can hardly be so. No distances on earth will be known to $1:10^6$ or $1:10^7$ over a few thousands of kilometers as would be required here and the trispheration technique must lead to serious discrepancies in the network that would show up at a later stage.

Because of this asymmetric way of treatment the trispheration approach is not so attractive and in practice variation-of-coordinates methods are preferred. Such methods work conveniently in conjunction with modern high speed digital computers and are quite general in terms of the number of stations but tend to obscure the relevant geometric relations a clear understanding of which is desirable when dealing with network optimization and related problems. The present treatment of the problem preserves the symmetry and only involves the relevant geometric quantities.

The smallest number of geodetically meaningful simultaneous rangings to a satellite is four. It is obvious that two or three simultaneous rangings do not, in general, constrain the geometry of the tracking network, unless such rangings are supplemented by additional information, for example directions measured at the same instant of time. Although this goes with a limitation in generality of application

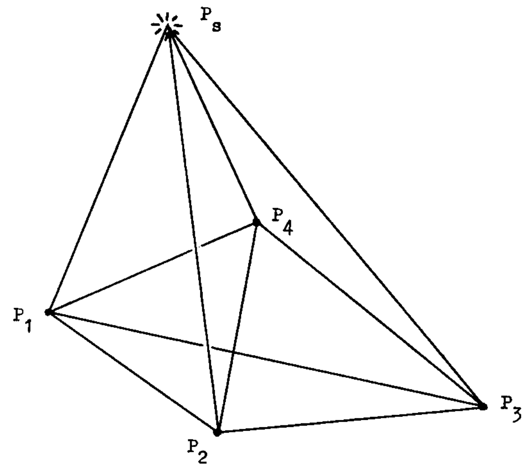

Fig. 1.

the present work confines itself to pure ranging.

The elementary configuration (see Fig. 1) can be described by four tracking sites P_i ($i = 1 \ldots 4$), one satellite location P_s, six distances l_{ij} ($i, j = 1 \ldots 4$; $i \neq j$) mutually between the observation sites and four distances l_{is} ($i = 1 \ldots 4$) between the observation sites and the satellite. It should be understood that distances l_{ij} are unknown and the problem before us is to find the relation mathematically connecting the unknown distances l_{ij} with the measured l_{is}. Since the configuration is determined unambiguously by altogether nine out of ten distances l, such a relation is expected to be unique.

Distances l are of course connected through

$$\frac{l_{is}^2 + l_{js}^2 - l_{ij}^2}{2 l_{is} l_{js}} = \cos \varphi_{ij}^s \tag{3}$$

where however the six angles φ_{ij}^s are unknown. They should be the angles subtended at P_s by P_i and P_j and consequently their cosines must be expressible as the conventional inner products of four unit vectors $\mathbf{e}_{is}$ taken two at a time:

$$\cos \varphi_{ij}^s = \mathbf{e}_{is} \cdot \mathbf{e}_{js} \tag{4}$$

Because four 3-dimensional vectors are linearly dependent in 3-dimensional space we must have

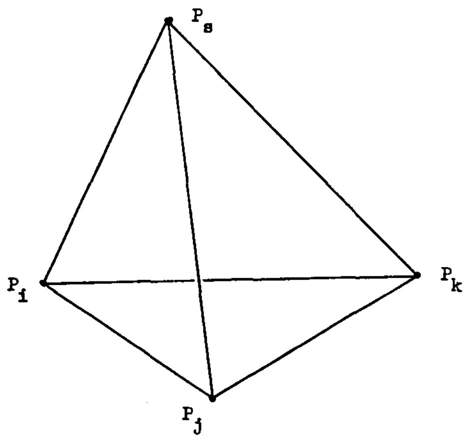

Fig. 2.

$$|G^s| \equiv \begin{vmatrix} 1 & \cos\varphi_{12}^s & \cos\varphi_{13}^s & \cos\varphi_{14}^s \\ \cos\varphi_{21}^s & 1 & \cos\varphi_{23}^s & \cos\varphi_{24}^s \\ \cos\varphi_{31}^s & \cos\varphi_{32}^s & 1 & \cos\varphi_{34}^s \\ \cos\varphi_{41}^s & \cos\varphi_{42}^s & \cos\varphi_{43}^s & 1 \end{vmatrix} = 0 \qquad (5)$$

The determinant here is again of Gram-type and its vanishing is a necessary and sufficient condition for the linear dependence of vectors $\mathbf{e}_i$. It should be noted that these vectors are only an intermediary through which (5) is derived but do not play an essential role, as they should not.

Also the similarity between (5) and (2) should be noted. As it stands (5) may be interpreted as a relation between the arc-lengths φ_{ij}^s in the spherical quadrangle obtained as the instantaneous central projection through P_s of the network P_i on the celestial sphere.

Inserting (3) into (5), then noting that both rows and columns have common factors l_{is}^{-1} and that l_{is} must be taken $\neq 0$ for all i, (5) reduces to the equivalent expression:

$$\begin{vmatrix} l_{1s}^2+l_{1s}^2 \ldots\ldots\ldots l_{1s}^2+l_{4s}^2-l_{14}^2 \\ \vdots \qquad\qquad \vdots \\ l_{4s}^2+l_{1s}^2-l_{41}^2 \ldots l_{4s}^2+l_{4s}^2 \end{vmatrix} = 0 \qquad (6)$$

Introduce quantities $t_{ij}^s \equiv l_{is}^2 + l_{js}^2 - l_{ij}^2$ (7)

If we define $l_{ij}=0$ for $i=j$, then t_{ij}^s is symmetric in i and j because l_{ij} is so and (6) becomes

$$|T^s| = \begin{vmatrix} t_{11}^s & t_{12}^s & t_{13}^s & t_{14}^s \\ t_{21}^s & t_{22}^s & t_{23}^s & t_{24}^s \\ t_{31}^s & t_{32}^s & t_{33}^s & t_{34}^s \\ t_{41}^s & t_{42}^s & t_{43}^s & t_{44}^s \end{vmatrix} = 0 \qquad (8)$$

This, or rather (6) is the condition we were to find.

There is no reason why one, unaware of the number of conditions, should not add the similar conditions $|T^i| = 0$ ($i = 1 \ldots 4$) obtained by permutation of indices, but they are really satisfied when (8) is, as could be expected, a result that however does not seem so trivial as it appears. Observe the following identity which clearly holds in tetrahedron $P_iP_jP_kP_s$ (see Fig. 2):

$$t_{ij}^s - t_{ik}^s - t_{kj}^s + t_{kk}^s = t_{ij}^k \quad (i, j \neq k \neq s) \qquad (9)$$

This is so because:

$$\begin{aligned} t_{ij}^s &= l_{is}^2 + l_{js}^2 - l_{ij}^2 \\ -t_{ik}^s &= -l_{is}^2 - l_{ks}^2 + l_{ik}^2 \\ -t_{kj}^s &= -l_{js}^2 - l_{ks}^2 + l_{kj}^2 \\ t_{kk}^s &= 2l_{ks}^2 \end{aligned}$$

Adding these identities and making use of the fact that by (7) $l_{ik}^2 + l_{kj}^2 - l_{ij}^2 = t_{ij}^k$ we find (9). Putting in (9) $i=k, j=k, i=j=k$ in turn we find $t_{kj}^k = t_{ik}^k = t_{kk}^k = 0$. Taking these special values of t_{ij}^k as being zero by definition the inequalities $i, j \neq k, s$ may be waived and we see that (9) equally well holds if $k = s$.

Denote matrix

$$\begin{vmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \text{ by } R_{s,1},$$

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \text{ by } R_{s,2} \text{ etc.}$$

Then it follows from (8), (9) and the properties of t_{ij}^k that

$$\begin{aligned} |T^1| &= |R_{s,1}^*| \cdot |T^s| \cdot |R_{s,1}| \\ |T^2| &= |R_{s,2}^*| \cdot |T^s| \cdot |R_{s,2}| \ldots \end{aligned}$$

and since

$|R_{s,1}| \cdot |R_{s,2}|, \ldots \neq 0$, that $|T^1| = |T^2| = |T^3| = |T^4| = 0$

This proves that $|T^s| = 0$ really entails $|T^i| = 0$ ($i = 1 \ldots 4$) and that $|G^s| = 0$ entails $|G^i| = 0$.

The operation on $|T^s|$ amounts to the subtraction of row (column) i of $|T^s|$ from all other rows (columns) and the subsequent subtraction of column (row) i from all other columns (rows) as can be readily verified.

The "equivalence-property" of determinants $|T|$ or rather $|G|$ as just derived will be quite useful. Using a similar argument it can be immediately proved to hold for any finite order of vanishing determinants $|G|$ of the type considered here. The transformation of $|G|$ to $|T|$ was convenient to prove the "equivalence-property", but in general the $|G|$-form of the determinant as expressed by (5) is more handy.

We notice that the four principal minors of order 3 in (5) are of type (2). Usually they will not vanish but at least one of them would if three of the e-vectors were linearly dependent and thus the satellite P_s were coplanar with three of the ground-stations P_i. All four would vanish if the satellite were coplanar with all four stations. Even this highly exceptional situation one could think of in satellite geodesy. It will never occur exactly, but practical arrangements of ground-stations could be so that it is approximated rather closely. If so, the 3-dimensional ranging problem degenerates towards one of two dimensions and conditions (5) can be replaced by two independently vanishing determinants of 3rd order f.i.:

$$\begin{vmatrix} 1 & \cos\varphi_{12}^s & \cos\varphi_{13}^s \\ \cos\varphi_{21}^s & 1 & \cos\varphi_{23}^s \\ \cos\varphi_{31}^s & \cos\varphi_s^{32} & 1 \end{vmatrix} = \begin{vmatrix} 1 & \cos\varphi_{12}^s & \cos\varphi_{14}^s \\ \cos\varphi_{21}^s & 1 & \cos\varphi_{24}^s \\ \cos\varphi_{41}^s & \cos\varphi_{42}^s & 1 \end{vmatrix} = 0$$

Geometrically one can easily "see" that the vanishing of these two 3rd order principal minors is sufficient to cancel both the original 4th order determinant and its two remaining principal minors of order 3. Similar situations of non-independence will arise where more than four stations join in simultaneous ranging. The two "singular" cases mentioned here will not be treated any further because they are so exceptional, although the latter is standard in trilateration problems of plane, or almost-plane land-surveying. There even the vanishing or almost-vanishing of 2nd order determinants, thus the linear dependence of two e-vectors, could be considered but not so in practical satellite geodesy.

All 2nd order determinants vanish in 1-dimensional problems of ranging, which frequently occur in elementary surveying. Then f.i. the determinant-condition

$$\begin{vmatrix} 1 & \cos\varphi_{12}^s \\ \cos\varphi_{21}^s & 1 \end{vmatrix} = 0$$

'collapses' to give $l_{12} = |l_{1s} \pm l_{2s}|$, a result that is obviously correct, but which in practical work is obtained by simpler means. This sophisticated treatment however clearly demonstrates how our problem of satellite ranging is mathematically a straightforward generalization of this elementary problem of surveying.

In passing the usual non-vanishing of the 3rd order determinants in 3-dimensional space pointed to the well known fact that three simultaneous rangings to a satellite generally do not yield direct geometric information.

Noting the symmetry of the determinants we will from now on make it practice to write them in a convenient diagonal-notation, so that (5) appears as

$$|G^s(1, 2, 3, 4) \equiv g^s(1, 2, 3, 4) = 0 \qquad (5\text{a})$$

This notation should be self-explanatory, the number of index-arguments indicating the order of the determinant. The quantities $\cos\varphi_{ij}^s$ appearing in all determinants g^s should be understood to be merely shorthand notations for the left-hand member of (3).

Any 'reasonably' chosen set of values for the six ground (chord-line)-distances l_{ij} between four points P_i on the earth's surface is compatible, i.e. that there can exist four points with these distances between. The word 'reasonably' requires f.i. $l_{ik} \leq l_{ij} + l_{jk}$. But when simultaneous rangings to a satellite have been made, so that the distances l_{is} to an unknown point P_s are given the six distances have to satisfy a non-linear condition like (5) or (5a).

Six of these four-station rangings to six suitably distributed instantaneous satellite positions in space will lead to six non-linear equations that can in general be solved for the six unknown distances l_{ij}. These in turn determine the figure between the four P_i. The orientation of this figure and its location with respect to the earth's centre of mass remain unknown.

Quantities related to orientation and absolute location are absent both from the result

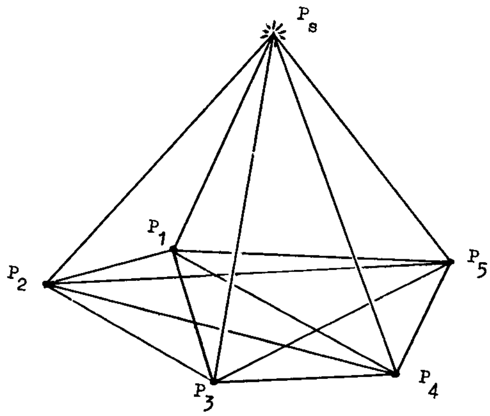

Fig. 3.

and from the mathematics leading to it, as they should be preferably. Moreover the approach is entirely symmetric in quantities of the same class and the conditions come in closed form. These qualities make perhaps the only charm of this method. Only the really relevant quantities are involved: the distances between the stations and the measured ranges to the satellite(s). Elegant closed expressions appeal from a purely mathematical point of view but their practical value could be doubted since standard least squares adjustment procedures require their linearization after all. But the other qualities clearly reveal the real nature of the problem and therefore the present work might be more than an attempt to obtain mathematical elegance and rigour.

It is tempting to investigate how stable the solutions for figures determined in this way really are and to try and optimize both the geometry of the network and the observation process, but these problems will not be taken up here.

4. Multiple-station ranging

With five or more stations joining in simultaneous ranging the idea is the same, but there are some complications. Confining ourselves for the present to the five-station case (see Fig. 3), we should realize that the ten ground-distances l_{ij} have to be constrained because we have just found that the distances l between five points in space satisfy one condition. Moreover we find that the l_{ij} have to satisfy five 4th order determinant-conditions of type (5), but in addition one of 5th order to be derived along the same lines, because five vectors are linearly dependent in 3-dimensional space. We expect only three conditions to be satisfied independently and therefore not all seven as quoted above may be considered. The constraining-condition will have to be respected, because it will be even there if no satellite observations are made at all.

If it is properly understood what (5a) means, it should be clear that the vanishing of two of the 4th order determinants (having three points P_i in common) entails the vanishing of all others, including that of 5th order. So in all we have three conditions on ten distances l_{ij}. Two of these conditions are obtained by satellite ranging. It is not true therefore that four 5-station ranging events could determine (or rather over-determine) the ten distances because in this process the constraining-condition should be counted only once and each additional event adds only two new conditions. So the total number of conditions from m 5-station events will be $3+(m-1)2=2m+1$ and therefore m must be at least 5.

It should be remarked that the constraining-condition we met is again of type (5), wherein one of the stations takes over the role of the satellite. It could be written as

$$g^1(2, 3, 4, 5) = 0$$

or in any of its four alternative forms. So together we find as one possible complete set of conditions:

$$\begin{aligned} g^1(2, 3, 4, 5) &= 0 \\ g^s(1, 2, 3, 4) &= 0 \\ g^s(2, 3, 4, 5) &= 0 \end{aligned} \tag{10}$$

As anticipated already this set is by no means unique. Applying the 'equivalence-property' we may write as one alternative:

$$\begin{aligned} g^2(1, 3, 4, 5) &= 0 \\ g^2(1, 3, 4, s) &= 0 \\ g^2(3, 4, 5, s) &= 0 \end{aligned} \tag{11}$$

It was stated that such a set is sufficient to cancel all other 4th order determinants together with that of 5th order. We will show that this is so. A few preliminary remarks will facilitate the argument substantially.

The g-determinants have the following characteristic properties: (a) they are symmetric;

(*b*) all diagonal elements are equal to 1; (*c*) all off-diagonal elements are of absolute value at most 1.

To prove that a g-determinant of order $n > 3$ vanishes it is sufficient to show that its matrix G can be written as the product $E \cdot E^*$, where the matrix E is composed of unit vectors $\mathbf{e}_i$ as follows:

$$E = \begin{vmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{e}_n \end{vmatrix}$$

Because then G is a pure Gram-matrix and its determinant g must vanish since $n > 3$.

We know that the vanishing of one of the five 4th order g-determinants is sufficient to make the ten distances between five points in space compatible because (6) must be unique. This means that its matrix G can be expressed in the form $E \cdot E^*$ if $g = 0$. This can be so because $E \cdot E^*$ involves eight independent parameters, two for each unit vector, which have to satisfy only 7 conditions.

For orders higher than 4 this converse is in general not true. To make it possible $2n$ parameters would have to satisfy $\frac{1}{2}n(n-1) + 1$ conditions. This requires

$$\tfrac{1}{2}\, n(n-1) + 1 \leqslant 2n$$
$$\text{or:} \quad n^2 - 5n + 2 \leqslant 0$$

and this cannot be satisfied by positive integers $n > 4$. If it were so that $G = E \cdot E^*$ if $g = 0$ for any order > 4, $g = 0$ would imply that all determinants of order down to 4 would automatically vanish and there could be only one independent condition even for very large values of n, which is obviously not true.

Because of the validity of the converse statement only for $n = 4$ the 4th order determinants will play a central role. It can be proved that

$$\begin{aligned} g^s(1, 2, 3, 4, 5) = 0 \text{ if } g^s(1, 2, 3, 4) &= 0 \\ g^s(2, 3, 4, 5) &= 0 \\ g^s(1, 3, 4, 5) &= 0 \\ g^s(1, 2, 4, 5) &= 0 \\ g^s(1, 2, 3, 5) &= 0 \end{aligned}$$

i.e. if all its 4th order principal minors vanish.

In full we may write:

$$G^s(1, 2, 3, 4) = \begin{vmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \end{vmatrix} \cdot \left| \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4 \right|$$

$$G^s(2, 3, 4, 5) = \begin{vmatrix} \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \\ \mathbf{e}_5 \end{vmatrix} \cdot \left| \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5 \right|$$

etc.

But then also,

$$G^s(1, 2, 3, 4, 0) = \begin{vmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \\ 0 \end{vmatrix} \cdot \left| \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, 0 \right| = E_5 \cdot E_5^*$$

$$G^s(0, 2, 3, 4, 5) = \begin{vmatrix} 0 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \\ \mathbf{e}_5 \end{vmatrix} \cdot \left| 0, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5 \right| = E_1 \cdot E_1^*$$

etc.

where the '0' in $G^s(1, 2, 3, 4, 0)$ means that $G^s(1, 2, 3, 4)$ is extended by both a 5th row and a 5th column of all zeros. The notation E_i is self-explanatory.

Now write $\quad D = \sum_i E_i$, then:

$$D = 4 \cdot \begin{vmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \\ \mathbf{e}_5 \end{vmatrix}$$

Expand $D \cdot D^*$, noting that $D^* = \sum_i E_i^*$ and that $E_j \cdot E_i^* = (E_i \cdot E_j^*)^*$.

Combining we find f.i.:

$$E_1 \cdot E_2^* + (E_1 \cdot E_2^*)^* =$$

$$= \begin{vmatrix} 0 & \cos\varphi_{12}^s & \cos\varphi_{13}^s & \cos\varphi_{14}^s & \cos\varphi_{15}^s \\ \cos\varphi_{21}^s & 0 & \cos\varphi_{23}^s & \cos\varphi_{24}^s & \cos\varphi_{25}^s \\ \cos\varphi_{31}^s & \cos\varphi_{32}^s & 2 & 2\cos\varphi_{34}^s & 2\cos\varphi_{35}^s \\ \cos\varphi_{41}^s & \cos\varphi_{42}^s & 2\cos\varphi_{43}^s & 2 & 2\cos\varphi_{45}^s \\ \cos\varphi_{51}^s & \cos\varphi_{52}^s & 2\cos\varphi_{53}^s & 2\cos\varphi_{54}^s & 2 \end{vmatrix},$$

hence a matrix similar to $G^s(1, 2, 3, 4, 5)$ with however all elements not in any of rows or columns 1 and 2, multiplied by 2 and the diagonal elements in positions 1 and 2 replaced by 0. Analogous results hold for the other four $E_i \cdot E_j^* + (E_i \cdot E_j^*)^*$.

Inserting everything we find

$$D \cdot D^* = 4^2 \cdot G^s(1, 2, 3, 4, 5),$$

from which it follows that we may write

$$G^s(1,2,3,4,5) = \begin{vmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \\ \mathbf{e}_5 \end{vmatrix} \cdot \begin{vmatrix} \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \mathbf{e}_5 \end{vmatrix} = E \cdot E^*$$

and consequently $g^s(1,2,3,4,5) = 0$.

The original statement can be proved along the same lines. From (11):

$$\begin{aligned} G^2(1,3,4,5,0) &= E_s \cdot E_s^* \\ G^2(1,3,4,0,s) &= E_5 \cdot E_5^* \\ G^2(0,3,4,5,s) &= E_1 \cdot E_1^* \end{aligned} \tag{12}$$

Now write $S = E_s + E_5 + E_1$ and expand to get, making use of (12):

$$S \cdot S^* = \begin{vmatrix} 4 & 6\cos\varphi_{13}^2 & 6\cos\varphi_{14}^2 & 4\cos\varphi_{15}^2 & 4\cos\varphi_{1s}^2 \\ 6\cos\varphi_{31}^2 & 9 & 9\cos\varphi_{34}^2 & 6\cos\varphi_{35}^2 & 6\cos\varphi_{3s}^2 \\ 6\cos\varphi_{41}^2 & 9\cos\varphi_{43}^2 & 9 & 6\cos\varphi_{45}^2 & 6\cos\varphi_{4s}^2 \\ 4\cos\varphi_{51}^2 & 6\cos\varphi_{53}^2 & 6\cos\varphi_{54}^2 & 4 & 4\cos\varphi_{5s}^2 \\ 4\cos\varphi_{s1}^2 & 6\cos\varphi_{s3}^2 & 6\cos\varphi_{s4}^2 & 4\cos\varphi_{s5}^2 & 4 \end{vmatrix} \tag{13}$$

On the other hand

$$S = \begin{vmatrix} 2\mathbf{e}_1 \\ 3\mathbf{e}_3 \\ 3\mathbf{e}_4 \\ 2\mathbf{e}_5 \\ 2\mathbf{e}_s \end{vmatrix} \tag{14}$$

Then on pre- and post-multiplying both (13) and (14) with the diagonal matrix

$$\begin{vmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix}$$

and dividing through the common factor $6^2 = 36$, we see

$$G^2(1,3,4,5,s) = F \cdot F^* \tag{15}$$

where

$$F = \begin{vmatrix} \mathbf{e}_1 \\ \mathbf{e}_3 \\ \mathbf{e}_4 \\ \mathbf{e}_5 \\ \mathbf{e}_s \end{vmatrix}$$

and therefore

$$g^2(1,3,4,5,s) = 0 \tag{16}$$

Also by simply pre- and post-multiplying both members of (15) with the diagonal matrices

$$\begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} \text{ and } \begin{vmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

respectively:

$$\begin{aligned} G^2(1, 0, 4, 5, s) &= F_2 \cdot F_2^* \\ \text{and } G^2(1, 3, 0, 5, s) &= F_4 \cdot F_4^* \end{aligned}$$

and thus:

$$g^2(1, 4, 5, s) = g^2(1, 3, 5, s) = 0$$

This completes the proof of the statement under (11).

Applying the "equivalence-property" to (16) we arrive at the alternative condition

$$g^s(1, 2, 3, 4, 5) = 0$$

but from only this no direct conclusion as to the vanishing of its 4th order principal minors may be drawn, as was pointed out before.

This kind of proof is general for g-determinants of finite order $n > 3$, starting out from vanishing 4th order principal minors together —and this is essential—at least once containing all distances l not involving the vertex (here P_2). The argument must be symmetric as regards the vertex chosen, but in satellite geodetic applications the set of conditions chosen should contain all constraining-conditions, because the compatability of the distances l_{ij} must be guaranteed.

Through generalizations of the statement the validity of which was just proved the idea can be easily extended to multiple-station ranging problems, where more than five stations participate simultaneously. Here too an nth order g-determinant must vanish together with all its principal minors down to order 4. This is necessary, but again it is sufficient that a limited number of 4th order principal minors vanish. This general case will not be dealt with because it does not seem to have essentially new features.

Instead we will consider the mere number of independent conditions in simultaneous n-station ranging. For the total number $N(n)$ of independent conditions in one single event we have (for $n > 4$) the recursive formula

$$N(n) = N(n-1) + n - 3 \text{ with } N(4) = 1.$$

$N(n-1)$ is the number of constraining-conditions and $n-3$ is the number of 4th order determinant-conditions imposed upon the network by the simultaneous satellite observation. $N(n)$ appears as the partial sum of an arithmetic progression of second order and thus

$$N(n) = \frac{(n-2)(n-3)}{2} \qquad (n>4) \tag{17}$$

After m simultaneous n-station rangings we will be able to sum up

$$\frac{(n-2)(n-3)}{2} + (m-1)(n-3)$$

independent non-linear 4th order determinant-conditions to be satisfied by the $\frac{1}{2}n(n-1)$ distances l_{ij}. We may not expect to solve this system as long as

$$\frac{(n-2)(n-3)}{2} + (m-1)(n-3) < \frac{n(n-1)}{2}$$

or:

$$m < 3\frac{n-2}{n-3}$$

This suggests that for any $n > 5$ at least four events are required to determine the network.

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ГЕОМЕТРИЯ ДЛЯ ОДНОВРЕМЕННЫХ ИЗМЕРЕНИЙ РАССТОЯНИЙ С ПОМОЩЬЮ СПУТНИКОВ

Обсуждается общий геометрический подход к спутниковой геодезии, использующий одновременные наблюдения. Вводятся замкнутые выражения для геометрических условий, накладываемых на сеть станций, следящих за спутниками и производящих одновременные измерения расстояний до них. Предлагаемая математическая процедура симметрична по отношению ко всем станциям и содержит лишь неизвестные расстояния между станциями и измеряемые расстояния до спутников. Делается сравнение аналогичных одномерной и двумерной задач.