Two examples of penetrative convection

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ABSTRACT

Critical Rayleigh numbers are computed for a two-layer model of fluid heated from below in two cases; namely, the upper layer possesses (*a*) an infinite coefficient of heat conduction, or (*b*) a large negative Rayleigh number. Case (*b*) has already been studied by Gribov & Gurevich (1957), but because of an incorrect numerical evaluation, it is reconsidered here.

1. Introduction

In a horizontal layer of fluid heated from below, convection sets in when the temperature gradient exceeds a certain critical value, which depends on the special boundary conditions of the problem. Most of the former work, see for instance Chandrasekhar (1961), contains the following three cases: (*a*) the layer is confined between two "free" planes, (*b*) the layer is confined between two "rigid" planes, and (*c*) one surface is free, the other rigid. No slip occurs on a rigid surface, and no tangential stresses act on a free surface. For both kinds of bounding surfaces, which are fixed in position, the normal component of the flow and the temperature perturbation must vanish. In all three cases the principle of exchange of stabilities is valid and instability sets in as stationary convection.

However, in some cases, convection arising in an unstable layer may penetrate into a neighbouring stable layer. Examples of this situation are the convection zones of stars, unstable layers in the earth's atmosphere or the thin layer at the surface of the oceans, which is cooled by evaporation. Correspondingly, there is a series of theoretical studies, which are concerned with models of several layers of fluid with different properties of stability. Lick (1965) describes a two-layer model with a time-dependent temperature gradient. Gribov & Gurevich (1957) investigate the influence of the bordering stable layers on the considered layer, and handle the special cases, in which the bordering layers (*a*) are near the stability limit, or (*b*) possess considerable stability. Of special interest are those cases in which the coefficients of volume expansion, heat conduction or viscosity in the different layers have different values. Welander (1964) has shown that in such a situation under certain conditions even instability may occur although the temperature gradient of the different layers is positive, corresponding to a stable stratification.

In the present analysis I consider first a two-layer model with one finite layer ($0 \leqslant z \leqslant d$) and one infinite layer ($z > d$) possessing an infinite coefficient of heat conduction. Secondly, I compute once again the critical Rayleigh number of the Gribov & Gurevich case (*b*), because these authors made a mistake in the numerical evaluation of the eigenvalue equation.

2. Basic equations

Using the Boussinesq approximation one can derive the following equations for the perturbations π, ϑ, u_i of pressure, temperature, and the three components of velocity:

$$\frac{\partial u_i}{\partial t} = -\frac{1}{\varrho}\frac{\partial \pi}{\partial x_i} + \lambda_i g \alpha \vartheta + \nu \Delta u_i \tag{1}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{2}$$

$$\frac{\partial \vartheta}{\partial t} = \beta \lambda_j u_j + \varkappa \Delta \vartheta \tag{3}$$

where $\varkappa$ denotes the coefficient of thermometric conductivity, ν the kinematic viscosity, α the coefficient of volume expansion, and g

the acceleration of gravity; λ_i is the unit vector (0, 0,1). The temperature gradient of the unperturbed state is

$$-\beta = \frac{\partial T_0}{\partial z} \quad (4)$$

One can analyse an arbitrary disturbance into a set of normal modes,

$$\left.\begin{aligned} u_z &= W(z) \\ \vartheta &= \Theta(z) \end{aligned}\right\} \exp(ik_x x + ik_y y + \sigma t) \quad (5)$$

and correspondingly for the dependence of u_x, u_y and π. In the following, I consider only such solutions which describe stationary convection. Then, in the case of marginal stability, σ vanishes and from the basic equations one can derive the well-known eigenvalue equation

$$(D^2 - a^2)^3 W = -\mathrm{Ra}^2 W \quad (6)$$

where $a = d(k_x^2 + k_y^2)^{1/2}$ is a non-dimensional horizontal wave number, $R = \alpha\beta g d^4/\varkappa\nu$ the Rayleigh number (a non-dimensional temperature gradient), D denotes the Operator (d/dz) and d the thickness of the instable layer.

The solution of (6) is given by

$$W = \sum_{j=1}^{3} (A_j \sinh \mu_j z + C_j \cosh \mu_j z) \quad (7)$$

where

$$\mu_j^2 = a^2 - (\mathrm{Ra}^2)^{1/3} \exp(2\pi i(j-1)/3) \quad (8)$$

3. Boundary conditions

It is convenient to use d as a unit of length. The unstable layer is then located between $z = 0$ and $z = 1$. Let the quantities belonging to this layer be unprimed, whereas the quantities belonging to the upper stable region are marked by a prime.

Let, for simplicity, the plane $z = 0$ be a free surface. Then (see e.g. Chandrasekhar, 1961)

$$W = 0, \quad D^2 W = 0 \quad \text{and} \quad D^4 W = 0 \quad \text{for} \quad z = 0 \quad (9)$$

It follows that $C_j = 0$ for $j = 1, 2, 3$. At $z = \infty$, all perturbations have to vanish; that is, W' must only contain terms decaying exponentially with increasing z.

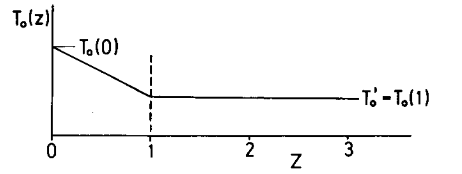

Fig. 1. Unperturbed temperature profile in the case $\varkappa' = \infty$.

Provided that the density and the viscosity have the same values in the two layers, one has from the equations (1) and (2) the following conditions of continuity

$$\left.\begin{aligned} W &= W' \\ DW &= DW' \\ D^2 W &= D^2 W' \\ D^3 W &= D^3 W' \end{aligned}\right\} \text{for } z = 1 \quad (10\,\mathrm{a})$$

From the continuity of the temperature perturbation ensues the continuity of $(D^2 - a^2)^2 W$, and with respect to (10 a)

$$D^4 W = D^4 W' \quad \text{for } z = 1 \quad (10\,\mathrm{b})$$

where $\alpha = \alpha'$ is assumed. The condition of continuous heat flow gives

$$\varkappa D(D^2 - a^2)^2 W = \varkappa' D(D^2 - a^2)^2 W' \text{ for } z = 1 \quad (10\,\mathrm{c})$$

4. The case $\varkappa' = \infty$

This case describes the penetration of convection into a semi-infinite isothermal layer. For, since $\varkappa' = \infty$ one has $T_0' =$ const. (Fig. 1) The basic equation for W' is reduced to

$$(D^2 - a^2)^3 W' = 0$$

The following solution fulfils the boundary condition at $z = \infty$

$$W' = e^{-az}(B_0 + zB_1 + z^2 B_2) \quad (11)$$

However, from (10 c) we have $D(D^2 - a^2)^2 W'/_{z=1} = 0$ and therefore $B_2 = 0$. The remaining coefficients B_j and A_j of the solutions (7) and (11) can be determined up to a common factor from the conditions (10 a) and (10 b). These conditions give a homogeneous linear system of equations whose determinant must be zero:

$$\begin{vmatrix} \sinh\mu_1 & \sinh\mu_2 & \sinh\mu_3 & 1 & 1 \\ \mu_1\cosh\mu_1 & . & . & -a & -a+1 \\ \mu_1^2\sinh\mu_1 & . & . & a^2 & a^2-2a \\ \mu_1^3\cosh\mu_1 & . & . & -a^3 & -a^3+3a^2 \\ \mu_1^4\sinh\mu_1 & . & . & a^4 & a^4-4a^3 \end{vmatrix} = 0 \qquad (12)$$

Equation (12) gives the Rayleigh number R as an implicit function of the wave number a. The lowest possible value of R determines the critical temperature gradient for the onset of convection. The result of a numerical evaluation of (12) is

$$R_{\min} = 225$$

and

$$a_{\min} = 1.25$$

Comparing this result with the free-free case ($R_{\min} = 27\pi^4/4 \approx 657.5$ and $a_{\min} = \pi/\sqrt{2} \approx 2.22$) one sees that instability in the two-layer model with $\varkappa' = \infty$ may occur at considerably lower Rayleigh numbers. The value $a_{\min} = 1.25$ also gives an estimate of the penetration depth of W' (Eq. 11).

Letting $\mu_1 = i\omega$, $\mu_2 = \mu_3^* = \delta - i\gamma$ one gets from the critical values of R and a: $\omega = 2.345$, $\delta = 2.55$ and $\gamma = 1.19$. I set arbitrarily $A_1 = -i$. Then evaluation of the continuity conditions yield numerical values of A_2, A_3, B_0 and B_1 and the complete solution is given by

$$W = \sin\omega z + 1.22\sinh\delta z\cos\gamma z + 0.0868\cosh\delta z\sin\gamma z$$

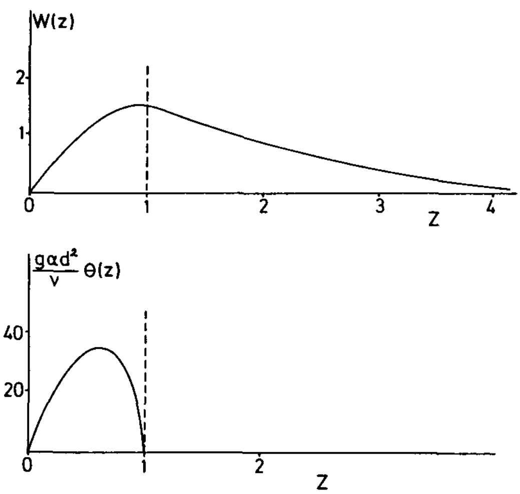

Fig. 2. Velocity $W(z)$ and temperature perturbation $\Theta(z)$ in the case $\varkappa' = \infty$.

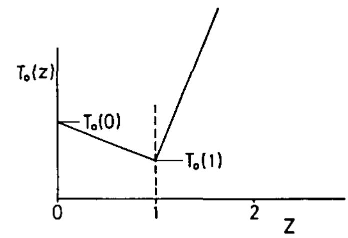

Fig. 3. Unperturbed temperature profile in the case $-R' \gg 1$.

$$\Theta = \frac{\nu}{g\alpha d^2}(49.7\sin\omega z + 0.73\sinh\delta z\cos\gamma z - 7.30\cosh\delta z\sin\gamma z)$$

$$W' = -1.33\, e^{-az} + 6.58\, z e^{-az}$$

$$\Theta' = 0$$

The solution shows quite appreciable penetration (Fig. 2).

5. The case $-R' \gg 1$

Gribov & Gurevich (1957) study besides other special cases a two-layer model with a large negative Rayleigh number R' prescribed in the upper region $z > 1$, which therefore possesses considerable stability. There is a heat sink at $z = 1$ in this model and the unperturbed temperature profile is qualitatively given by Fig. 3. The authors derive the following characteristic equation

$$(\mu_2^2 - \mu_3^2)\mu_1\coth\mu_1 + (\mu_3^2 - \mu_1^2)\mu_2\coth\mu_2 + (\mu_1^2 - \mu_2^2)\mu_3\coth\mu_3 = 0 \qquad (13)$$

where the μ_j are given by (8) again. Equation (13) represents a function $R(a)$, the minimum of which according to Gribov & Gurevich occurs at $a_{\min} = 1.9$ and has the value $R_{\min} = 370$. However, a correct numerical evaluation of equation (13) yields

$$R_{\min} = 2\,434.65$$

and

$$a_{\min} = 3.629$$

The critical Rayleigh number is considerably higher than even in the rigid-free case ($R_{\min} = 1\,100.65$ and $a_{\min} = 2.682$). In a sense, this is plausible since the two compared cases lead to two eigenvalue problems differing from each other in one boundary condition only. Due to $-R' \gg 1$, one has in the case treated here $D^2W = 0$ at $z = 1$, whereas in the rigid-free case the condition $(D^2 - a^2)^2 W = 0$ must be fulfilled.

The other boundary conditions are

$$W = 0, \quad DW = 0 \qquad \text{for } z = 1$$

and

$$W = 0, \ D^2W = 0, \ D^4W = 0 \text{ for } z = 0$$

for both problems. The vanishing of the second derivative is a stronger constraint on the flow than the vanishing of $(D^2 - a^2)^2 W$. That is, one expects higher stability in the former case than in the latter.

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ДВА ПРИМЕРА ПРОНИКАЮЩЕЙ КОНВЕКЦИИ

Вычислены критические числа Рэлея для двухслойной модели жидкости, подогреваемой снизу. Рассмотрены два случая: верхний слой обладает (а) бесконечным коэффициентом теплопроводности и (б) большим отрицательным числом Рэлея. Случай (б) уже был изучен Грибовым и Гуревичем (1957), однако, из-за ошибки в вычислениях он здесь рассмотрен заново.