The flow of a stratified fluid over a vertical step

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ABSTRACT

Long's linearized model is used to study the two-dimensional flow of an incompressible, inviscid and stably stratified fluid over a step of finite height in a long channel bounded above by a rigid, horizontal lid. The height of this lid is varied in an attempt to determine its effect on the flow and the results, which are obtained for a range of values of the Richardson number, are compared with the observation of a glider pilot, Farley, taken in Switzerland in the lee of the Jura plateau.

1. Introduction

Farley (1965) has described in detail observations that he made when gliding in the lee of the Jura plateau. This plateau lies to the north-west of a line running roughly south-west to north-east in Switzerland, and rises gently to this line where it suddenly drops about 1 000 meters to a flat alluvial plain. The line is about 200 km long. Farley's flight path is shown in Fig. 1.

On that particular day, the Jura, as seen from above, was covered by a thin sheet of stratus which moved forward with the north-west wind in an unruffled layer. At the edge of the plateau this layer flowed sharply down the lee slope. Beyond this, 1–2 km from the drop and stretching parallel to the ridge for hundreds of km, was a line of cloud as indicated in Fig. 1. Parallel to this and about 10 km further downstream was a similar line of cloud.

Farley, towed by a powered aircraft, flew under the first-mentioned line of cloud as indicated in Fig. 1 and found that conditions were rather smooth. However, at the upstream lower edge of the cloud he encountered strong down currents. North-west of this point and above the ridge Farley encountered strong lift even though he could see the stratus moving down the lee-slope directly below him. Flying along the ridge, Farley also noticed that the strongest lift occurred when the lee-slope was steepest, even when it was a steep cliff. These observations led Farley to suggest the existence of a double vortex in the lee of the Jura. In this paper a mathematical investigation of Farley's phenomenon is presented.

Most theories of atmospheric flow have dealt with the flow of an inviscid, incompressible and stably stratified fluid over an obstacle, although Scorer (1949) also considered the two-dimensional flow over a step as in the case Farley has described. However, Scorer used a perturbation method to obtain his solutions so that no rotors or finite amplitude waves were predicted.

Corby (1954) gave a survey of the theoretical treatments together with a summary of the results of observations available at that time. All the treatments described by Corby are linear perturbation treatments, the authors having considered the atmosphere to be infinite in extent vertically so that the wave lengths of the perturbations are small compared with the depth of the atmosphere.

Long (1953) suggested that this may not be a good approximation since the waves set up by a mountain will have lengths of the order of the horizontal dimensions of the mountain, while the atmosphere may be considered to be about 12 km thick. He was able to develop a two-dimensional model of airflow over a mountain considered to be in a long channel, in which the non-linear inertial terms in the Eulerian equations of motion are represented exactly, by assuming a certain upstream relationship between the density and velocity. He considered the fluid to be inviscid, incompressible and stably

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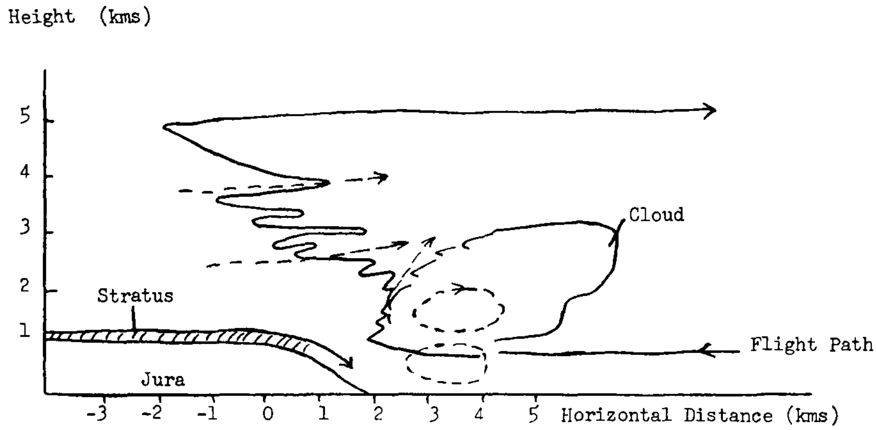

Fig. 1. Approximate scale drawing showing Jura plateau on left with stratus layer, vortex cloud and Farley's flight path. Taken from Farley (1965).

stratified. In particular, Long assumed that no disturbance propagates upstream and that the solution is bounded downstream. This last condition allows standing waves to exist in the lee of the obstacle.

Long and other workers (cf. Yih, 1965) used an inverse method, in which the shape of the obstacle depends on the Richardson number for the flow, to obtain their solution. Drazin & Moore (1967), Miles (1967) and Jones (1967) used Long's model to determine the flow past a vertical strip in a duct. Miles also obtained solutions describing the flow over a vertical strip in a half-plane.

The treatments of the problem of flow past a vertical strip make use of the fact that Long's linearized equation, which will be described in Section 2, takes the form of the Helmholtz wave equation so that the techniques of diffraction theory may be employed. These works are mathematical and the authors make no claim for any physical validity for large Richardson numbers. However, these treatments suggest a way of solving the two-dimensional problem of the flow of an inviscid, incompressible and stably stratified fluid over a step, which may be considered to be a good approximation to Farley's problem.

In Section 2, Long's model is briefly described. In Section 3 it is used to set up the mathematical model of flow over a vertical step; the method of solution, similar to that used by Drazin & Moore, is presented and the results are discussed. In Section 4, temperature readings given by Farley are used to determine the density gradient in the atmosphere. These readings were taken 25 km downstream of the Jura but it is assumed that they are fairly representative of the temperature distrubution upstream.

It was originally intended to use Farley's results to see if Long's assumed relationship between the velocity and density far upstream was satisfied on the day concerned, and if so, to determine the appropriate Richardson number to be used in the model. However, Long's condition was not satisfied and it is assumed that this is because the downstream velocity distribution is used. Hence the procedure was changed. Results are obtained for a series of values of the Richardson number and, for a certain range of these values, the predicted flows are found to be in good agreement with Farley's observations. The calculated potential density distribution and the appropriate Richardson numbers are then used to determine the upstream velocities which produce Farley's phenomenon. The results give quite realistic values for the velocities producing the phenomenon.

2. Long's model

Long (1953) showed that the general two-dimensional steady-state equations of motion and continuity for an inviscid, incompressible and stably stratified fluid can be reduced to a single vorticity equation. This result was first obtained by Dubreil-Jacotin (1937). Long further showed that this vorticity equation assumes a linear form if the relationship

$$\varrho'_{-\infty} U'^{2}_{-\infty} = \text{a constant} \tag{2.1}$$

is satisfied far upstream between the density ϱ' and the velocity U' of the fluid. The suffix $-\infty$ denotes that variables are considered far upstream at $x' = -\infty$, the flow being from left to right in the (x', y') plane. (Yih (1960) has obtained more distributions which yield a linear governing equation for the described fluid.)

The linear vorticity equation can be written in the dimensional form

$$\nabla'^2 \phi + k'^2 \phi = 0 \tag{2.2}$$

where

$$\nabla'^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{2.3}$$

and

$$k'^2 = \left(\frac{g}{\varrho' U'^2} \frac{\partial \varrho'}{\partial y'} \right)_{-\infty} \tag{2.4}$$

k being the Richardson number, g gravity and

$$\phi'(x', y') = y' - y'_{-\infty} \tag{2.5}$$

the variable $y'_{-\infty}(x', y')$ being the height at $x' = -\infty$ of the streamline through any point (x', y'). Thus $\phi'(x', y')$ represents the vertical displacement of any particle at a point (x', y') from its original height. The non-linear inertial terms of the Eulerian equations of motion are represented exactly in (2.2).

Long (1955) discussed the physical implications of assumption (2.1) and conluded that, though this condition might not be realised exactly in the atmosphere, such a model should exhibit the essential features of atmospheric flow. He obtained exact solutions to equation (2.2), for the flow over an obstacle $y' = h'(x')$ in a channel bounded above by a rigid horizontal plane $y' = T$, by assuming that no disturbance propagates upstream and that the solution is bounded downstream.

The equations of motion of a steady inviscid fluid do not distinguish between upstream and downstream so that some assumption is necessary to explain the absence of upstream waves. Without some additional restriction, solutions of the steady-state wave equation (2.2) are well known to be indeterminate. Rayleigh (1883) showed that this indeterminacy can be removed by introducing a fictitious viscous term—su into the analysis (Lamb, 1932). The limiting case as s tends to zero in the viscous solution obtained gives a determinate inviscid solution, the only remaining effect of friction being the absence of upstream waves. Alternatively, the transient problem can be solved starting the flow from rest at time $t = 0$, say. The solution as t tends to infinity describes a steady state free of upstream waves. Crapper (1959) has remarked that this procedure is formally identical to the use of Rayleigh's fictitious viscosity in the limit as s tends to zero. Hence Long's assumption that there is no disturbance upstream in the steady-state problem has considerable support and it is generally assumed that this condition is sufficient to ensure the uniqueness of the solution.

3. Flow over a vertical step

In view of the comments in Section 2 the mathematical model for the flow of the described fluid over a vertical step in a channel of height T km may be set up as follows. First introduce non-dimensional unprimed variables

$$x = \pi x'/T;\ y = \pi y'/T;\ \phi = \pi(y' - y'_{-\infty})/T \tag{3.1}$$

It is assumed that the channel has a rigid lid, the non-dimensional width of the channel being π downstream $(x \geqslant 0)$ and d upstream $(x \leqslant 0)$. Then the vertical displacement, $\phi(x, y)$, of any particle from its undisturbed upstream position satisfies the wave equation

$$\nabla^2 \phi + k^2 \phi = 0 \tag{3.2}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{3.3}$$

$$k^2 = \left(\frac{g}{\varrho' U'^2} \frac{\partial \varrho'}{\partial y} \frac{T^2}{\pi^2} \right)_{-\infty} \tag{3.4}$$

and

$$\phi = y - y_{-\infty} \tag{3.5}$$

everywhere in the region (see Fig. 2) $-\infty < x \leqslant 0$, $\pi - d \leqslant y \leqslant \pi$; $0 \leqslant x < \infty$, $0 \leqslant y \leqslant \pi$ together with the following boundary conditions.

$$\phi = 0 \text{ on } y = \pi - d,\ -\infty < x \leqslant 0 \tag{3.6}$$

$$\phi = 0 \text{ on } y = \pi, \ -\infty < x < \infty, \tag{3.7}$$

$$\phi = -(\pi - d) \text{ on } y = 0, \ 0 \leqslant x < \infty, \tag{3.8}$$

$$\phi = y - (\pi - d) \text{ on } x = 0, \ 0 \leqslant y \leqslant \pi - d, \tag{3.9}$$

$$\phi \to 0 \text{ as } x \to -\infty \tag{3.10}$$

$$\phi \text{ is bounded as } x \to +\infty \tag{3.11}$$

$$\phi \text{ and } \partial\phi/\partial x \text{ are continous } x = 0, \ \pi - d \leqslant y \leqslant \pi \tag{3.12}$$

In order to study Farley's phenomenon, solutions to the problem (3.2)–(3.12) are obtained for a range of values of the Richardson number k. The height of the upper boundary in the atmosphere T is varied in order to determine its effect on the flow while the height of the Jura plateau, $(\pi - d)T/\pi$, is taken to be 1 km.

The problem (3.2)–(3.12) may be solved by forming an intergal equation in the unknown function

$$g(y) = \sum_{r=L+1}^{\infty} D_r \sin(\pi r(\pi - y)/d) \quad (\pi - d \leqslant y \leqslant \pi)$$

and expanding this integral equation in the modes of the small channel. This procedure is identical to that used by Drazin & Moore (1967) and has been described by Jones (1964). It leads to an infinite set of linear algebraic equations which can be solved effectively by truncation. However, an equivalent set of equations can be obtained directly without the use of an integral equation and this may be accomplished as follows.

Following Drazin & Moore, write the solution to equation (3.2) as

$$\phi(x,y) = \sum_{r=L+1}^{\infty} D_r e^{-\mu_r |x|} \sin \pi r(\pi - y)/d \quad (x \leqslant 0, \pi - d \leqslant y \leqslant \pi)$$

$$\phi(x,y) = -\frac{(\pi - d)\sin k(\pi - y)}{\sin k\pi} + \sum_{r=1}^{K} (A_r \cos \lambda_r x + B_r \sin \lambda_r x) \sin ry + \sum_{r=K+1}^{\infty} A_r e^{-\lambda_r x} \sin ry \quad (x \geqslant 0, 0 \leqslant y \leqslant \pi), \tag{3.13}$$

which satisfies all the boundary conditions except those at $x = 0$. The parameters and variables introduced in (3.13) are as follows.

$$L = [dk/\pi] < K = [k], \tag{3.14}$$

$$\mu_r = ((r\pi/d)^2 - k^2)^{\frac{1}{2}}, \quad \lambda_r = \begin{cases} (k^2 - r^2)^{\frac{1}{2}} & (r < k) \\ (r^2 - k^2)^{\frac{1}{2}} & (r > k) \end{cases} \tag{3.15}$$

The problem has now been reduced to that of determining the unknown coefficents A_r, B_r and D_r and this is achieved as follows. Use conditions (3.9) and (3.12) to obtain the relationships

$$\frac{-(\pi - d)\sin k(\pi - y)}{\sin k\pi} + \sum_{r=1}^{\infty} A_r \sin ry = \begin{cases} y - (\pi - d) & (0 \leqslant y \leqslant \pi - d) \\ \sum_{r=L+1}^{\infty} D_r \sin \pi r(\pi - y)/d & (\pi - d \leqslant y \leqslant \pi) \end{cases} \tag{3.16}$$

$$\sum_{r=L+1}^{\infty} \mu_r D_r \sin \pi r(\pi - y)/d = \sum_{r=1}^{K} \lambda_r B_r \sin ry - \sum_{r=K+1}^{\infty} \lambda_r A_r \sin ry \quad (\pi - d \leqslant y \leqslant \pi) \tag{3.17}$$

between the coefficients. From (3.16) it is possible to write A_r as

$$A_r = \sum_{s=L+1}^{\infty} A_{rs} D_s + C_r \quad (r = 1, 2, \ldots) \tag{3.18}$$

where

$$A_{rs} = \frac{2}{\pi} \int_{\pi - d}^{\pi} \sin(\pi s(\pi - y)/d) \cdot \sin ry dy \tag{3.19}$$

and

$$C_r = \frac{2}{\pi} \int_0^{\pi - d} (y - (\pi - d)) \sin ry dy + \frac{2(\pi - d)}{\pi \sin k\pi} \int_0^{\pi} \times \sin k(\pi - y) \cdot \sin ry dy \tag{3.20}$$

Substitute the expression for A_r defined in (3.18) into (3.17). It follows that

$$\sum_{r=1}^{K} \lambda_r B_r \sin ry = \sum_{r=K+1}^{\infty} \lambda_r \left(\sum_{s=L+1}^{\infty} A_{rs} D_s + C_r \right) \times \sin ry + \sum_{r=L+1}^{\infty} \mu_r D_r \sin \pi r(\pi - y)/d \tag{3.21}$$

holds within the range $\pi - d \leqslant y \leqslant \pi$.

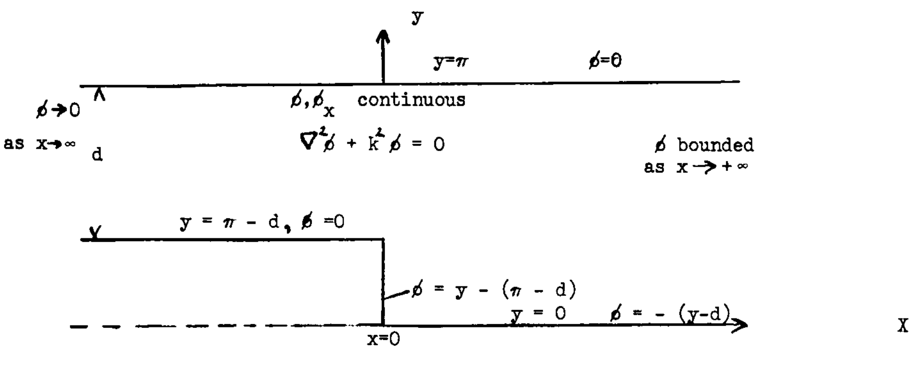

Fig. 2. Mathematical model of the Jura plateau problem.

Expand $\sin ry$ (cf. Jones, 1964, page 266) in terms of the complete set of orthonormal functions over the range $\pi-d \leqslant y \leqslant \pi$. Hence

$$\sin ry = \sum_{t=1}^{\infty} A'_{rt} \sin \pi t(\pi - y)/d \qquad (3.22)$$

where

$$A'_{rt} = \pi A_{rt}/d \qquad (3.23)$$

Substitute from (3.22) into (3.21), multiply (3.21) by $\sin \pi t(\pi-y)/d$ and integrate the resulting expression from $\pi - d$ to π. This procedure yields the following infinite set of equations

$$\sum_{r=1}^{K} \lambda_r B_r A_{rt} = \sum_{r=K+1}^{\infty} \lambda_r \left(\sum_{s=L+1}^{\infty} A_{rs} D_s + C_r \right) A_{rt}$$

$$(t=1,2,\ldots,L) \quad \sum_{r=1}^{K} \lambda_r B_r A_{rt}$$

$$= \sum_{r=K+1}^{\infty} \left(\sum_{s=L+1}^{\infty} (\lambda_r A_{rs} A_{rt} + d\mu_s \delta_{st}/\pi) D_s + \lambda_r C_r A_{rt} \right)$$

$$(t=L+1, L+2, \ldots) \qquad (3.24)$$

In this infinite set the unknowns are the coefficients B_r and D_s. From (3.19) it can easily be shown that

$$A_{rs} = \begin{cases} \dfrac{(-1)^{r+s}\, 2sd \sin rd}{(\pi s)^2 - (rd)^2} & rd \neq \pi s \\ (-1)^{r+1}(d - \sin 2rd/2r)/\pi & rd = \pi s \end{cases} \qquad (3.25)$$

so that

$$\lambda_r A_{rs} A_{rt} = 0(r^{-3}) \qquad (3.26)$$

and similarly, from (3.20), that

$$C_r = \frac{2}{\pi}\left(\frac{\sin r(\pi-d)}{r^2} + (\pi - d)\left(\frac{r}{r^2-k^2} - \frac{1}{r} \right) \right) = 0(r^{-2})$$

so that

$$\lambda_r C_r A_{rt} = 0(r^{-3}) \qquad (3.27)$$

Hence the infinite series in r in (3.24) converges rapidly for each s. Even so, the infinite system (3.24), which corresponds to that obtained by Drazin & Moore, cannot be solved exactly. An appropriate numerical solution can, however, be obtained by a truncation method, i.e. assume that $D_s = 0$ for all $s > M$, say, where M is a finite integer. (3.24) then becomes

$$\sum_{r=1}^{K} \lambda_r B_r A_{rt} = \sum_{r=K+1}^{N} \lambda_r \left(\sum_{s=L+1}^{M} A_{rs} D_s + C_r \right) A_{rt}$$

$$(t=1,2,\ldots,L) \quad \sum_{r=1}^{K} \lambda_r B_r A_{rt}$$

$$= \sum_{r=L+1}^{N} \left(\sum_{s=L+1}^{M} (\lambda_r A_{rs} A_{rt} + d\mu_s \delta_{st}/\pi) D_s + \lambda_r C_r A_{rt} \right)$$

$$(t=L+1, L+2, \ldots) \qquad (3.28)$$

where N is sufficiently large to ensure the convergence of the r series for each s.

(3.28) contains $K+M-L$ unknowns and so the first $K+M-L$ equations of (3.28) are taken to solve for B_r $(r=1,2,\ldots,K)$ and D_s $(s = L+1, L+2, \ldots, M)$. A_r $(r=1,2,\ldots)$ is then calculated from (3.18).

The method of solution was, of course, numerical. The computer used was an Elliott 503 and it was possible to invert a 35×35 matrix on the machine. Hence $K+M-L$ was taken as 35 in order to solve for B_r and D_s. The accuracy of the method was tested by substituting the values of the coefficients in (3.13) and matching the solutions and their derivatives for $x \gtrsim 0$ at $x=0, \pi-d \leqslant y \leqslant \pi$, and by calculating $y-\infty$ along the step face $x=0$, $0 \leqslant y \leqslant \pi - d$. The error was at

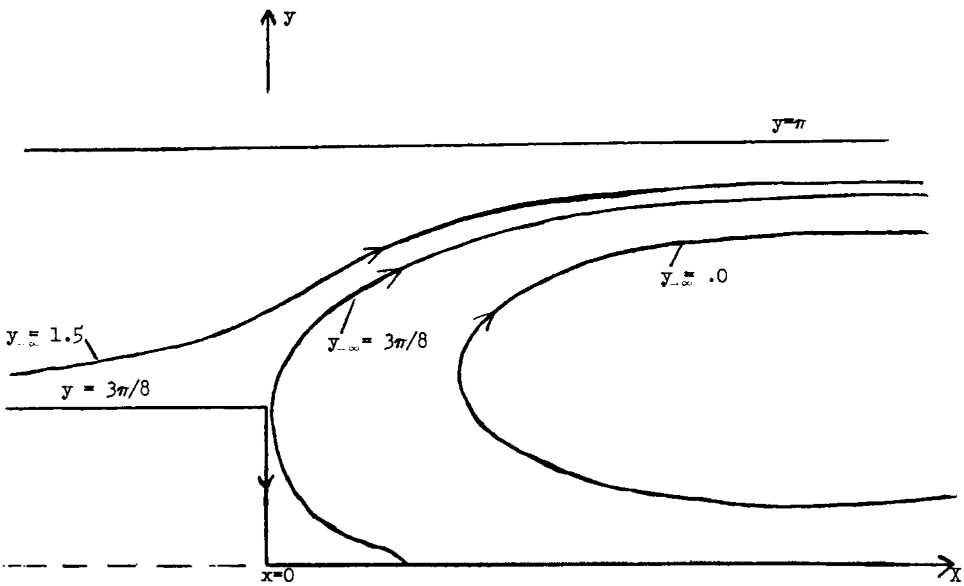

Fig. 3. Flow over a vertical step. $T = 3$ km, $d = 5\pi/8$, $k = 1.2$.

most 1% and was typically of order 10^{-4}. This is not surprising since D_{L+1} was of order 1 or 10^{-1} in most cases while D_M was of order 10^{-7} or 10^{-8}. Hence the truncation method of solution is very effective.

Solutions were obtained for various values of k and various step heights, that is for various upper bounds in the atmosphere since 2 represents a scale model of the atmospheric flow. These flows are shown in Fig. 3–9. Also, since the relationship between the primed dimensional variables and the unprimed non-dimensional variables is of the form

$$y = y'\pi/T \tag{3.29}$$

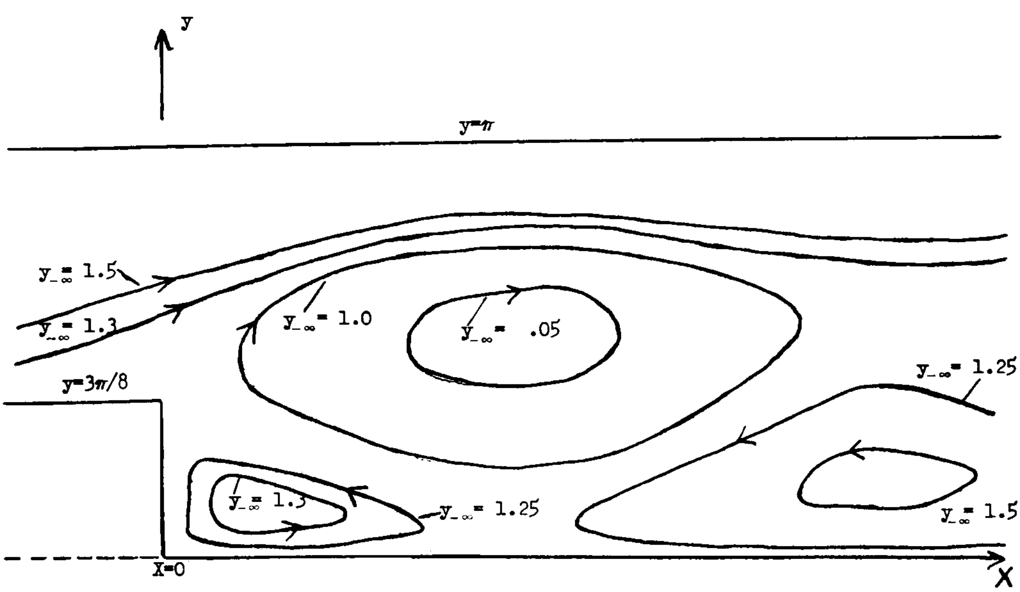

Fig. 4. Flow over a vertical step. $T = 3$ km, $d = 5\pi/8$, $k = 1.5$.

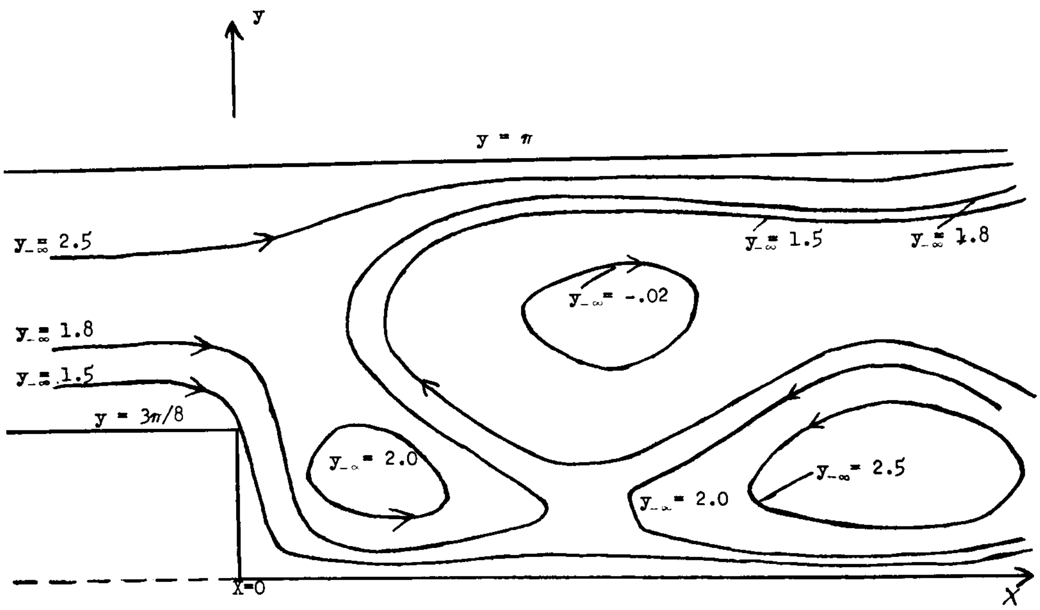

Fig. 5. Flow over a vertical step. $T = 3$ km, $d = 5\pi/8$, $k = 1.8$.

it is seen that if T is taken to be 3 km, the upper bound of Fig. 1, the step height is approximately $3\pi/8$ in Fig. 2. That is,

$y' = 1.125$ km in Fig. 1 when $y = 3\pi/8$ in Fig. 2

$y = \pi$ in Fig. 2 when $y' = 3$ km in Fig. 1

Although 3 km might seem low for the upper bound of the atmosphere, in Farley's diagram (Fig. 1) the stream lines are rising only slightly

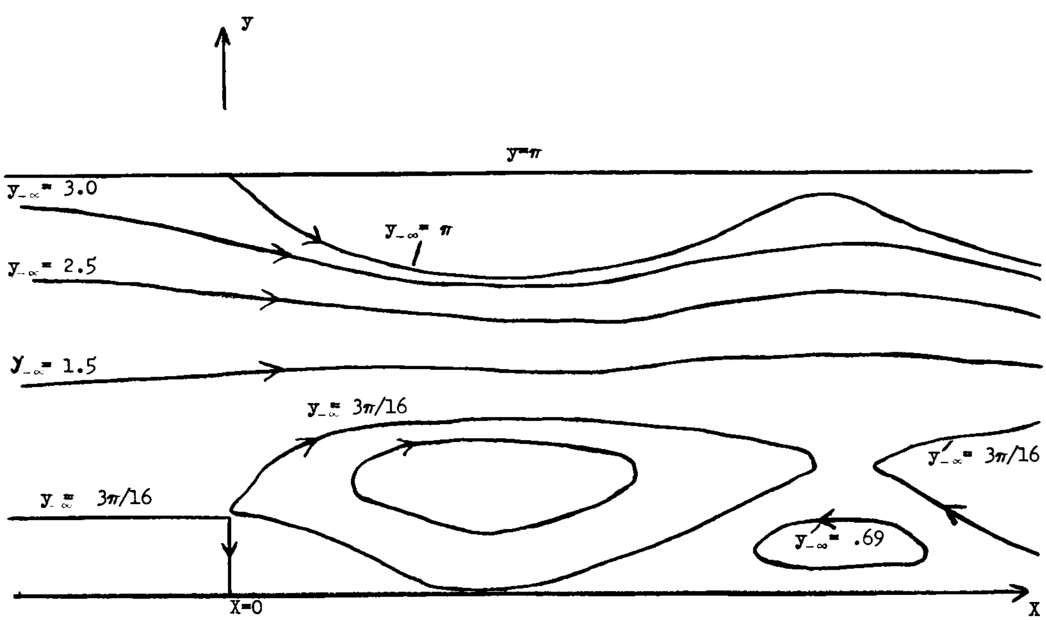

Fig. 6. Flow over a vertical step. $T = 6$ km, $d = 13\pi/16$, $k = 2.4$.

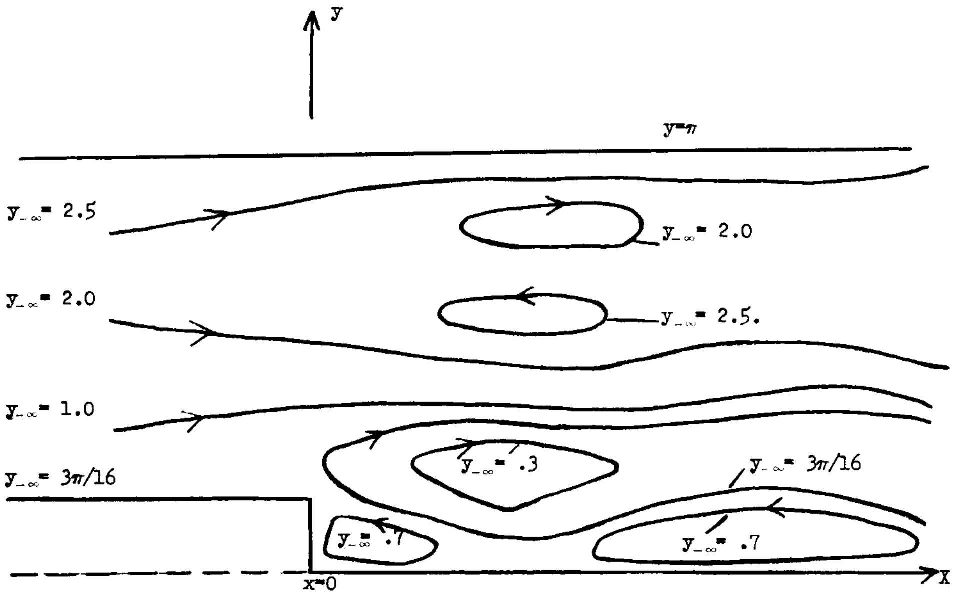

Fig. 7. Flow over a vertical step. $T = 6$ km, $d = 13\pi/16$, $k = 3.6$.

and quite uniformly at this level. Also, the height of the upper bound is varied in order to investigate its effect on the flow.

If T is doubled it follows from (3.4) that k must be doubled in order to represent the same physical conditions as before.

In Fig. 3, 4 and 5 the flow is represented for $T = 3$ km and $k = 1.2$, 1.5 and 1.8 respectively. T is doubled and the flow, corresponding to that shown in Fig. 3, is represented in Fig. 6. Here $T = 6$ km and $k = 2.4$ It can be seen that the flow below 3 km in both scale models is the same which suggests that in this case the flow at 3 km is approximately horizontal and the presence of an upper bound at this height does not affect the flow adversely below this level.

The case shown in Fig. 4 cannot be represented for $T = 6$ km because k then becomes an integer and the method of solution breaks down.

The flow corresponding to that shown in Fig. 5 is represented in Fig. 7 with $T = 6$ km and $k = 3.6$. Again the flow below 3 km is very similar in both figures. What is more, the stream lines at 3 km are now rising slightly above the step as observed by Farley.

T was next taken to be 12 km but here the results are disappointing. All cases considered exhibit the same features to a certain degree—a line of long rotors one above the other in the lee of the step. Little agreement with the corresponding cases for smaller T is attained. In order to reproduce the same physical conditions for various values of T is hat been assumed that k depends only on T. However, k also depends on $U'\infty$ and if anything can be inferred from Farley's results (that is, wind velocities recorded downstream) $U_{-\infty}$ increases with height so that an average value of the veloity used to determine k will also increase as T is increased. Hence further results were obtained for $T = 12$ km and $k = 3.5$ and 4.5. No agreement with Farley's phenomenon was achieved, however, and these flow patterns are not presented.

One possible explanation for the lack of agreement with Farley's observations when $T = 12$ km is that, while the assumption (2.1) holds in the lower troposphere, it becomes inadequate for the upper layers.

Flow patterns for larger k were obtained but look unrealistic and are not presented.

In practically all cases examined an anticlockwise rotor is formed directly in the lee of the step. The fluid goes down the step but is soon forced upwards and backwards by the heavier fluid downstream. Blocking thus occurs downstream, the blocked region taking the form of a rotor.

Another feature which is characteristic of all

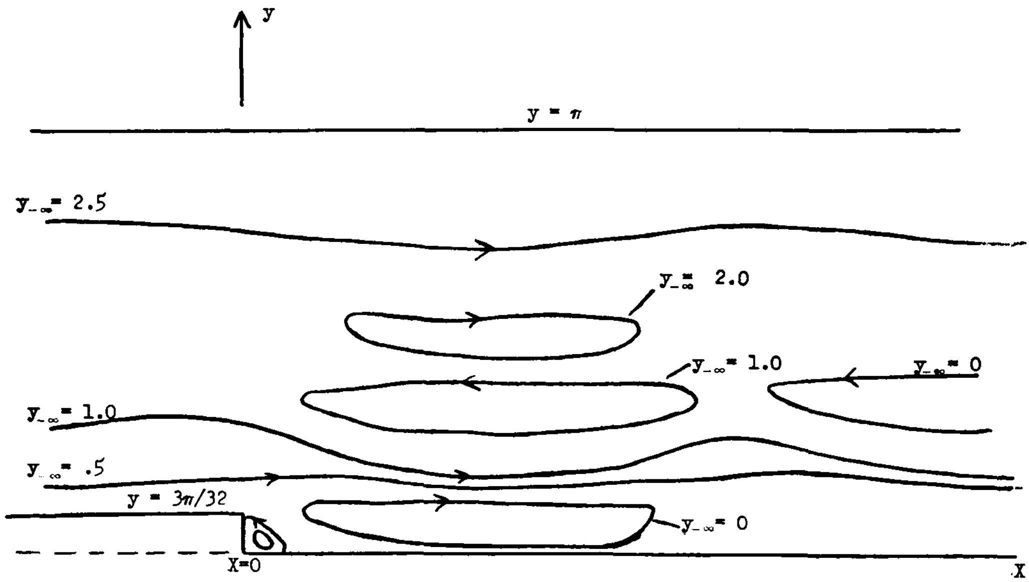

Fig. 8. Flow over a vertical step. $T = 12$ km, $d = 29\pi/32$, $k = 7.2$.

the flow patterns obtained is that a stagnation point occurs at or very near the top corner of the step. This is in agreement with Farley's observations. He says the air was going down the Jura escarpment but rising several kilometres above it.

A double vortex, as envisaged by Farley, does not seem to occur but rotors are developing and Figs. 4, 5 and 7 agree reasonably with the phenomenon Farley described.

Finally it should be pointed out that the boundary condition at $x = 0$, $0 \leqslant y \leqslant \pi - d$ has been chosen because Farley observed the thin layer of stratus moving down the Jura escarpment. This boundary condition produces Farley's phenomenon for certain upstream condi-

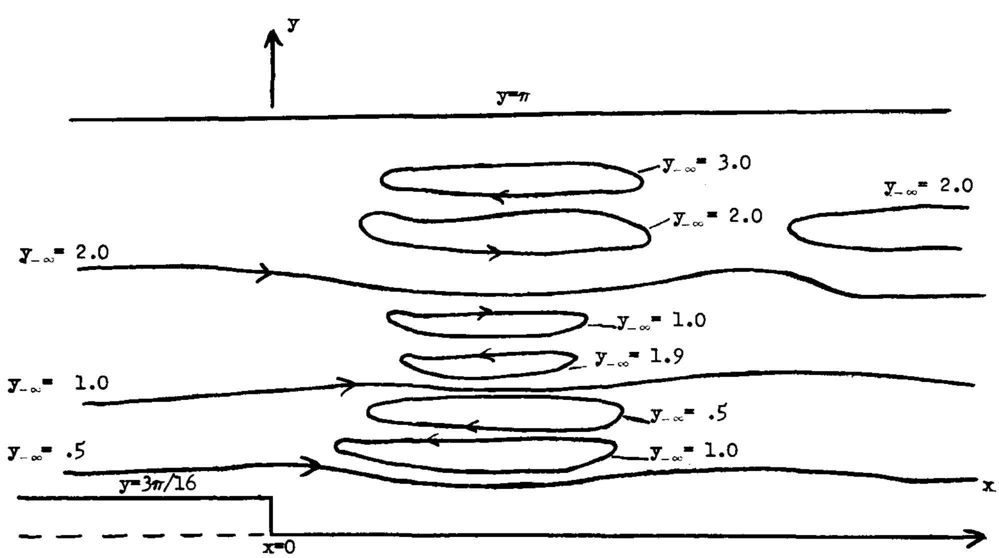

Fig. 9. Flow over a vertical step. $T = 12$ km, $d = 29\pi/32$, $k = 9.2$.

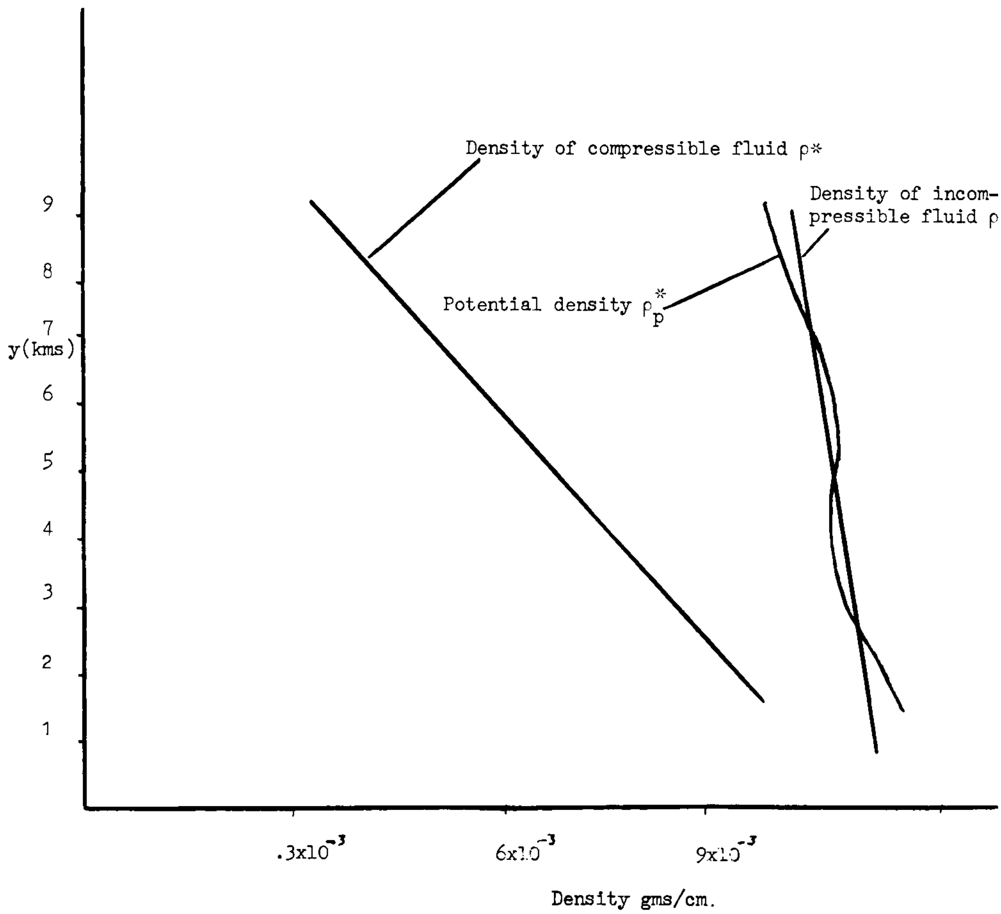

Fig. 10. ϱ^*, ϱ_p^* and ϱ plotted against y.

tions, i.e. for certain values of the Richardson number. However, it might not always be the correct boundary condition and this would explain some of the strange flow patterns which were obtained for certain values of T and k, and which are not presented.

4. Calculation of $U'_{-\infty}$

From the results of Section 3 it is seen that fairly good agreement with Farley's observations is obtained if, for $T = 3$ km, $k = 1.5 - 1.8$. If the density and velocity distributions upstream were known it would be possible, from (3.4), to determine how realistic these values of k are, and also to see if Long's upstream conditions (2.1), was satisfied on the day in concern. Unfortunately Farley's readings of wind velocity and temperature, given in Table 1, were taken 25 km in the lee of the Jura and cannot be relied upon to represent upstream conditions.

In order to have some check on the results the following procedure is adopted. The temperature distribution downstream (Table 1) is used to determine the density distribution and this is assumed to represent conditions upstream. Since a compressible fluid (the atmosphere) is being represented by an incompressible one, it is necessary to decide what density distribution is to be used in the compressible theory. Claus (1964), using the same density distribution in both cases, found one lee wave predicted by the compressible theory while the incompressible theory predicted nine. The reason for this is that gravity acts differently on a compressible fluid than on an incompressible one.

Table 1. *Meteorological data given by Farley*

Height (metres)	Wind		Temperature (degress absolute
	Knots	Direction	
500 (ground)	10–20	270	
1 500	25	280	281
3 000	30	290	273
5 000	40	310	260
9 000	60	320	233

That the potential density,

$$\varrho_p^* = \varrho^*(p/p^*)^{1/\gamma} \quad (4.1)$$

is the correct distribution to be used in the incompressible case is well known and can be shown quite simply as in Yih's book (1965, chapter 2, equations 31 and 66b). Here ϱ^* and p^* are the density and pressure of the compressible fluid, p_0 is a reference pressure and γ is the ratio of the specific heats. The non-dimensional value of k to be used is defined as

$$k^2 = \frac{T^2}{\pi^2}\left|\frac{g}{\varrho_p^*\, U'^2}\frac{d\varrho_p^*}{dy'}\right|_{-\infty} \quad (4.2)$$

Values of $U'_{-\infty}$ are then calculated from (4.2) which give $k = 1.5$ and 1.8.

The temperature T^* plotted against height from Table 1 reveals that

$$\frac{dT^*}{dy'} \approx \frac{6}{11} \times 10^{-4} \text{ degrees absolute/cm, a constant} \quad (4.3)$$

The density stratification can be determined using the relationships

$$\frac{dp^*}{dy'} = -g\varrho^* \quad (4.4)$$

and

$$p^* = \varrho^* RT^* \quad (4.5)$$

where R is the real gas constant. It follows that

$$\varrho^* = \varrho_0 (T^*/_0)^{-1-g/R\cdot(dy/dT)} \quad (4.6)$$

where $\varrho_0 = 1.23 \times 10^{-3}$ gm/c^3, and $T_0^* = 290$ degrees absolute, the density and temperature at sea-level.

ϱ^* is determined from (4.6), and (4.1) and (4.5) are then used to give ϱ_p^*. The reference pressure p_0^* is taken to be 95×10^6 dynes/cm^2, the pressure at sea-level. The density ϱ^* and the potential density ϱ_p^* are shown plotted against height in Fig. 10. $d\varrho_p^*/dy'$ is taken to be $3/20 \times 10^{9-}$ gm/cm^4.

If k is taken to be 1.5 with $T = 3$ km, $U'_{-\infty}$ is found to be 7.3 m/sec = 14 knots while if k is taken to be 1.8, $U'_{-\infty}$ is found to be 6.1 m/sec = 11.5 knots.

5. Discussion

The preceding results show that Farley's phenomenon occurs for quite realistic wind velocities. Air is rising above the edge of the step as well as flowing down it. Regions of enclosed stream lines, which represent the observed rotor clouds, are also formed. Unfortunately Long's model assumes that all stream lines originate upstream. Long (1955) asserted that a necessary condition for stability is that

$$\phi_y < 1 \quad (5.1)$$

at every point in the flow and that when this condition is violated the flow becomes statically unstable. Now $\phi_y > 1$ within finite regions of enclosed stream lines (actually ϕ has not been defined within such regions) so that the flow is, at least locally, turbulent. Farley, in his observations taken in the lee of the Jura plateau, did report turbulence within the rotor cloud, though conditions were smooth outside.

It could be that the closed stream lines are

Table 2. *Calculated density distributions*

Height (metres)	Temperature	ϱ^*(gm/cm^3)	p*(dynes/cm)	ϱ_p^*(gm/cm^3)
1 500	281°A	1.047×10^{-3}	0.788×10^6	1.198×10^{-3}
3 000	273°A	0.8696×10^{-3}	0.6339×10^6	1.1161×10^{-3}
5 500	260°A	0.6585×10^{-3}	0.4571×10^6	1.104×10^{-3}
9 000	233°A	0.3516×10^{-3}	0.2188×10^6	0.9991×10^{-3}

analytic continuations of streamlines from far upstream and that they somehow fit into a broader flow, and do not affect the qualitative character of such a flow. However, Long's experiments indicate that when condition 5.1 is violated the theoretically predicted lee-wave field is then not realized. Miles (1968) suggested that such a violation throws serious doubt on the physical significance of such a field.

Comparison of the results obtained here with Farley's observations suggest that it is plausible that the predicted rotors will not affect the external field. However, there is no obvious way of arguing that results predicted by Long's model are valid within regions of enclosed stream lines.

[1] This was pointed out to the author by J. W. Miles.

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ТЕЧЕНИЕ СТРАТИФИЦИРОВАННОЙ ЖИДКОСТИ НАД ВЕРТИКАЛЬНОЙ СТУПЕНЬКОЙ

Линеаризированная модель Лонга использована для изучения двумерного течения несжимаемой, невязкой, устойчиво стратифицированной жидкости над ступенькой конечной высоты и длинном канале, ограниченном сверху твердой горизонтальной крышкой. Высота этой крышки менялась с тем, чтобы можно было определить ее влияние на течение. Результаты, полученные для различных значений числа Ричардсона, сравниваются с наблюдениями с планера, предпринятыми в Швейцарии с подветренной стороны плато Юра.