Short term variations of a glacier-fed river

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ABSTRACT

The main factors determining the short term variations of the river Thjorsa in Iceland are rainfall and the melting of snow and glaciers. Time dependent spectral analysis was applied to investigate seasonal variations in the second order properties of the variations of Thjorsa and their relationships with temperature and precipitation.

In late summer a linear model of Thjorsa and the temperature and precipitation at the meteorological station at Haell provides a good fit of the observations. Because of non-linear effects connected with snow melting linear models are less appropriate during the winter although they can still explain a significant proportion of the variations of Thjorsa.

The most important information for predictions a few days ahead are past values of the river itself. Some improvement can be achieved by also taking meteorological observations into account.

Introduction

The objective of this study is to demonstrate the use of time dependent spectral analysis for investigating seasonal changes in the variations of a river and their relationships with the weather. The river is Thjorsa in Southern Iceland (Fig. 1). Its drainage area is about 7500 km^2, 50 % of it above 630 m. The average flow at our point of observation, Urridafoss, is about 32 Gl/day. About 20 % of the flow is drainage from the two glaciers, Vatnajökull and Hofsjökull. Continuous records are available from 1.9.1953. Before that date readings on a mounted scale were carried out about twice per week.

The western part of the drainage area is covered by grey basalts of early Quaternary age. The eastern half is in the Neovolcanic zone. The total area of glaciers on the drainage area is about 1100 km^2. The effect of geographical factors on the characteristics of Icelandic rivers is discussed by Kjartansson (1965) and Rist (1956).

The meteorological data consists of daily averages of temperature and precipitation at Haell (Fig. 1). The average temperature is 4.1°C and the precipitation 2.8 mm/day. The altitude is only 130 m which implies that most of the drainage area has a lower temperature. The distance from the sea is probably of considerable importance to the temperature, especially during the winter when the temperature of the sea is much higher than the air temperature. A, large proportion of the precipitation, therefore, falls as snow. Because of the low temperature evaporation is not a major factor in determining the relationship between Thjorsa and the weather. Vegetation is sparce and there are no forests which further reduces the evaporation.

Theoretical considerations

A linear model of the relationship between river, R, temperature, T, and precipitation, P, at the time t can be written as follows:

$$R(t) = \int_{-\infty}^{\infty} w_T(t,\tau)\,T(t-\tau)\,d\tau + \int_{-\infty}^{\infty} w_P(t,\tau)\,P(t-\tau)\,d\tau + Z(t) \qquad (1)$$

$w_T(t, \tau)$ and $w_P(t, \tau)$ are weight functions which describe how $R(t)$ depends upon values of $T(t)$ and $P(t)$, τ days apart from t. $Z(t)$ is an error term to account for such variations in $R(t)$ as our model fails to explain.

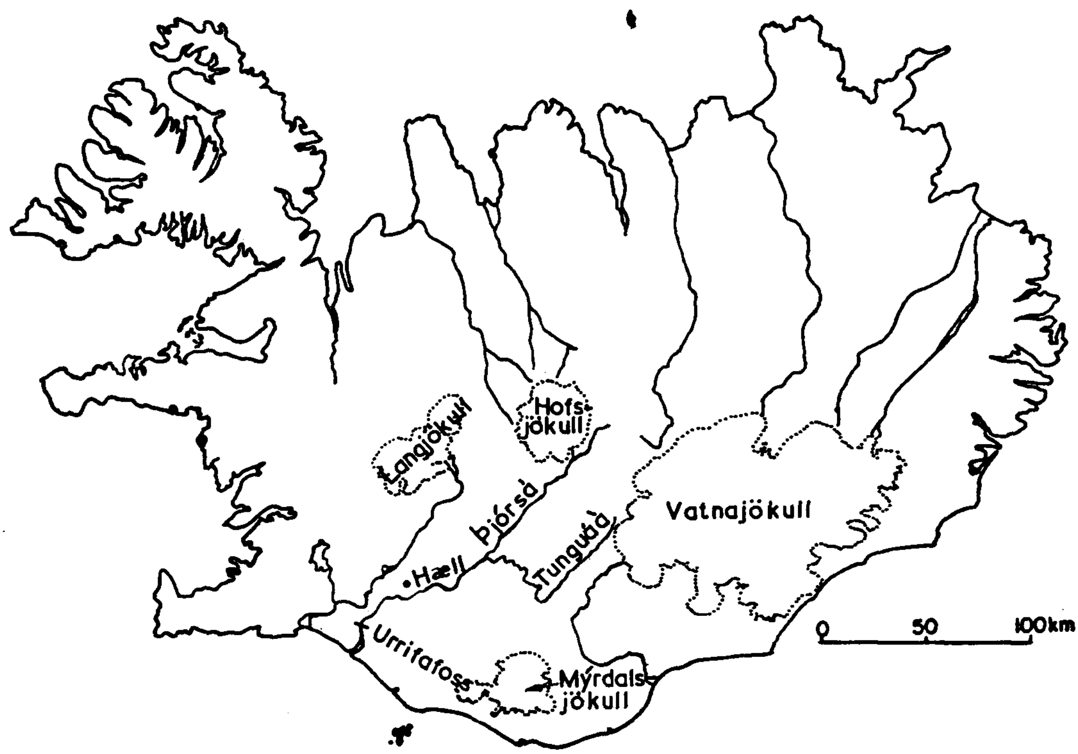

Fig. 1. A map of Iceland, showing locations referred to in this paper.

The river does not depend on future values of precipitation and temperature, nor is there any significant feedback from the river to the meteorological variables. It would, therefore, be natural to use zero rather than $-\infty$ as the lower limit in the integration. The integration from $-\infty$ is mainly a mathematical convenience, but as we shall see later it has certain other advantages as well.

If R, T and P are stationary series and w_T and w_P are independent of t we have a system which has been studied extensively in the literature (Akaike, 1965). We use the notations

$$W_T(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} w_T(\tau) \exp(-iu\tau)\, d\tau \qquad (2)$$

for the Fourier transform of $w_T(\tau)$,

$S_{RR}(u)$ = the power spectrum of R,

$R_{RP}(u)$ = the cross spectrum of R and P

and an analogous notation for other spectra and weight functions. If Z is independent of T and P, $W_T(u)$ and $W_P(u)$ can be obtained from the equations

$$S_{RT}(u = 2\pi W_T(u) S_{TT}(u) + 2\pi W_P(u) S_{PT}(u)$$

$$S_{RP}(u) = 2\pi W_T(u) S_{TP}(u) + 2\pi W_P(u) S_{PP}(u). \qquad (3)$$

Unless other references are provided all aspects of the theory and estimation of spectra relevant to this paper are described in detail in the book by Jenkins & Watts (1968). The review paper by Kisiel (1969) deals with the application of spectral analysis to hydrology. It also contains an introduction to the estimation of spectra.

The partial power spectrum of R with T eliminated is defined as

$$S_{RR\cdot T}(u) = S_{RR}(u) - S_{RT}(u) S_{TR}(u) / S_{TT}(u). \qquad (4)$$

It describes the distribution of that part of the power of series R which cannot be accounted for by a linear relationship with T. In the same way the partial cross spectrum of R and P with T eliminated is defined as

$$S_{RP\cdot T}(u) = S_{RP}(u) - S_{RT}(u)S_{TP}(u)/S_{TT}(u). \tag{5}$$

The solutions to equations (3) can then be written:

$$2\pi W_T(u) = S_{RT\cdot P}(u)/S_{TT\cdot P}(u),$$

$$2\pi W_P(u) = S_{RP\cdot T}(u)/S_{PP\cdot T}(u). \tag{6}$$

The weight functions $w_P(\tau)$ and $w_T(\tau)$ can be recovered from equation (6) by the inverse Fourier transform. Much of the following discussion will be concerned with the estimation and interpretation of w_P and w_T. The second order properties of the series themselves, expressed by the power spectra, are also of interest. Certain aspects of the relationships between the series are better expressed by coherences and phases. The coherence of R and T is given by

$$C_{RT}(u) = |S_{RT}(u)|^2/S_{RR}(u)S_{TT}(u). \tag{7}$$

It takes values between zero and one and is one in the case of a perfect linear relationship. The phase is defined as

$$\phi_{RT}(u) = \arg S_{RT}(u). \tag{8}$$

It can be converted into a time lag Δt at each frequency by

$$\Delta t = \phi/u + n\lambda, \quad n = 0, \pm 1, \pm 2, \ldots, \tag{9}$$

where λ is the wavelength, given by

$$\lambda = 2\pi/u. \tag{10}$$

In the lowest frequencies the ambiguity involved in selecting n in equation (9) is usually easily resolved by physical considerations. Partial phases and coherences are obtained by replacing the spectra in equations (7) and (8) by the partial spectra.

The simplifying assumption that w_P and w_T are independent of t is obviously rather unrealistic. It takes the snow which falls in January longer on average to reach the river than the rain in May. A change in temperature by one degree has a different effect in February than August.

Both the river and the meteorological variables have seasonal variations in the mean These will be regarded as deterministic components here and are supposed to have been subtracted from the variables in our model. The weight functions, w_P and w_T also vary seasonally. But, unlike the seasonal components in the means of R, P and T, this variation cannot, with impunity, be assumed away in a linear model.

With observations extending over several years at our disposal we could estimate $w_P(t, \tau)$ and $w_T(t, \tau)$ for each day of the year by methods described by Stefansson & Gudmundsson (1969). The number of observations behind each estimate would be small. It is obviously essential to impose further restrictions on w_T and w_T, enabling us to carry out a smoothing over adjacent t's as well as averaging over the years.

If our model is to provide a useful approximation to the actual situation, $w(t_1, \tau)$ and $w(t_2, \tau)$ should not differ much unless the seasonal change in the meteorological factors between t_1 and t_2 is comparable with the standard deviation of P or T. The temperature is the main factor in determining the seasonal pattern of the weight functions. Its standard deviation is of the order of 3°C and the amplitude of the annual variation is 6.4°. This suggests that averaging over two months would hardly introduce a serious bias in estimates of w_P and w_T.

The time it takes the precipitation to reach the river differs greatly depending upon where and when it falls. Some of the snow that falls on the glaciers remains there for months or years. The contribution from the January precipitation to the flow of Thjorsa in August will be much the same whether it is all accumulated during a heavy snowstorm on the 13th or evenly spread over the whole month. Post glacial lava beds are porous and cause long delays between precipitation and runoff whereas rain falling in the more watertight grey basalt areas quickly reaches the river. The phase lag between the diurnal variations of temperature and glacial runoff is a matter of hours.

These considerations suggest that the same factors which cause long delays between meteorological variations and the corresponding response in the rivers also act as high frequency filters. In so far as rapid variations in Thjorsa can be attributed to meteorological variations of the same frequency, i.e. follow a linear model, their phase difference will not correspond to a long time lag. For low values of τ, estimates of $w_P(t, \tau)$ and $w_T(t, \tau)$ will, thereore, not be much affected if we only use values of $P(t)$ and $T(t)$ from the same interval as included in the smoothing of the weight functions.

The second order properties of our series, especially R and T, vary seasonally as well as in the mean. These variations are of interest themselves in addition to their influence upon the relationship between river flow and the weather.

Priestley (1965) shows that provided the statistical properties of the series change slowly with time an evolutionary spectrum can be defined which expresses the same main characteristics of the series as the ordinary power spectrum, but varying with time. Evolutionary spectra are based on a generalisation of the spectral representation of a stationary process, which is given by

$$X(t) = \int_{-\infty}^{\infty} \exp[iut]\, dZ(u), \qquad (11)$$

where $Z(u)$ is an orthoganal process, i.e.

$$E|dZ(u)|^2 = dF(u) \qquad (12)$$

$$E dZ(u) d\tilde{Z}(v) = 0 \text{ uness } u = v$$

If the derivative of $F(u)$ exists we have

$$S_{XX}(u) = F'(u) \qquad (13)$$

The evolutionary spectrum is based on a spectral representation of the form

$$X(t) = \int_{-\infty}^{\infty} A(t,u) \exp[iut]\, dZ(u), \qquad (14)$$

where $Z(u)$ is still an orthogonal process. The evolutionary spectrum at time t is then given by

$$S_{XX}(t,u) = |A(t,u)|^2 S_{XX}(u), \qquad (15$$

provided the functions $A(t, u)$ satisfy certain conditions—see Priestley (1965, 1966).

A study of the seasonal changes of the second order properties, therefore, implies an investigation of $|A(t)|^2$ which we have stipulated to be periodic with the period one year.

The mathematical theory of evolutionary spectra has not yet been extended to cross spectra. Brillinger & Hatanaka (1969) investigated systems of equations similar to our equation (1) and conclude that the estimation of the impulse response can be carried out in a similar way as in the stationary case, using estimates of time dependent spectra and cross spectra.

Estimation

Power spectra of series from the natural sciences, including our series, usually decrease with increasing frequency. In such cases reduction of power in the lowest frequencies may be necessary in order to prevent bias in estimates at higher frequencies. Here the filter

$$Y(k) = X(k) - \frac{1}{225}\sum_{j=-14}^{14} (15-|j|)\, X(k+j) \qquad (16)$$

was applied to all series. $X(k)$ denotes the original daily observation and $Y(k)$ the filtered value. This is a powerful low frequency filter. At 1 c/48 days it eliminates about 50% of the power and beyond 1 c/30 days its effect is small.

Our observations are from the 12 years from 1954–1965. Each year is divided into six over lapping intervals of 96 days. The observations in each interval are weighted by multiplication by the function

$$C(m) = 0.5 - 0.5 \cos(2\pi m/97), \qquad (17)$$

where m takes the values from 1 to 96. The intervals start at 1/1, 3/3, 3/5, 3/7, 2/9 and 2/11 which implies that our estimates of spectra and impulse responses will be centered around February, April, June, August, October and December.

We call the weighted and filtered daily value of the river on the day m in the interval j of the year k $R_{jk}(m)$ and calculate the complex Fourier series

$$A_{Rjk}(n) = \sum_{k=1}^{96} R_{jk}(m) \exp[2\pi i mn/96] \qquad (18)$$

for n from 0 to 48 with a similar notation for the other series. From the Fourier series we obtain the periodograms and cross periodograms:

$$I_{RRjk}(m) = A_{Rjk}(m)\, A_{\tilde{Rjk}}(m)$$

$$I_{RPjk}(m) = A_{Rjk}(m)\, A_{\tilde{pjk}}(m) \qquad (19)$$

The spectra and cross spectra were estimated by averaging over the years and two frequencies, e.g.

$$S_{RPj}(u) = \frac{1}{K}\sum_{k=1}^{12}\sum_{m=m(u)}^{m(u+1)} I_{RPjk}(m) \qquad (20)$$

where

$$K = 48\pi \sum_{1}^{96} C(m) \qquad (21)$$

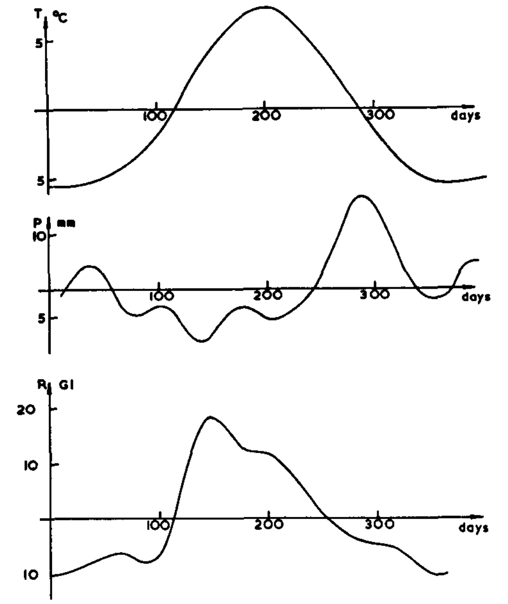

Fig. 2. Seasonal variations of Thjorsa and the temperature and precipitation at Haell. Divide by 10 into the values on R-axis.

$$u = (m(u) + 0.5)\ \text{c/day}. \tag{22}$$

The estimates were carried out for

$$m(u) = 0, 2, 4, \dots \tag{23}$$

All the power spectra and cross spectra of our series were estimated in this way for each of the 6 overlapping intervals indicated by j.

These estimates differ from those described in most of the literature in that the averaging is mainly over time and not frequencies. A notable exception is Bartlett's paper (1948). With 12 years data at our disposal we could in fact have avoided altogether averaging over frequency. The reduction in the output from the computations were at least as important in the decision to include two periodogram ordinates from each year in each estimate of the spectrum, as the reduction of the variability.

The distribution of our variables is very skew; observations more than six standard deviations above the average occur every now and then for both R and P. The confidence intervals that have been worked out for spectra and phases and coherences of Gaussian series are, therefore, of doubtful validity here. This is, however, not a matter of great concern. With 24 periodogram ordinates behind each estimate of the power spectra these become smooth and as we have over 20 estimates with little overlapping of the frequency bands of adjacent estimates, a peak in the spectrum needs not be large to be detected with confidence. It is usually advisable to consider the phases and coherences together. Estimates of the phase, accompanied by coherences of 0.2 or even smaller, appear to be fairly accurate.

The effect of the weighting function of equation (17) is to increase somewhat the variability of the estimates, sharpen the definition of their position in the year, produce a correlation of about 0.25 between a periodogram ordinate and each of its neighbours on either side on the frequency axis and reduce the interdependence of estimates farther apart in the frequency do-

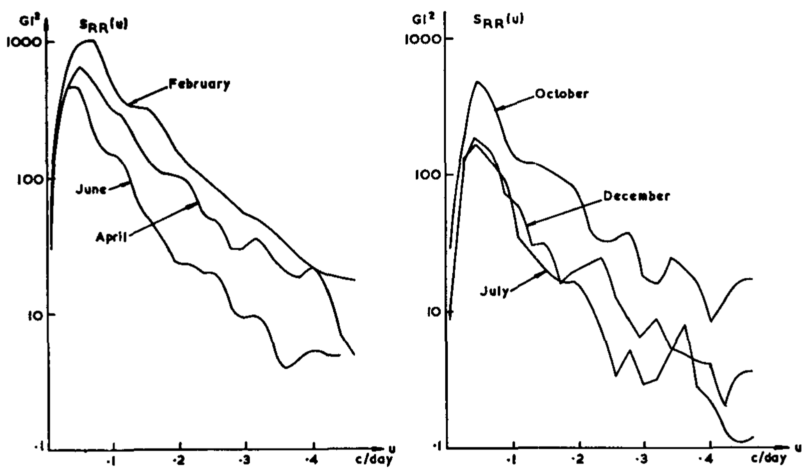

Fig. 3. Power spectra of the flow of Thjorsa.

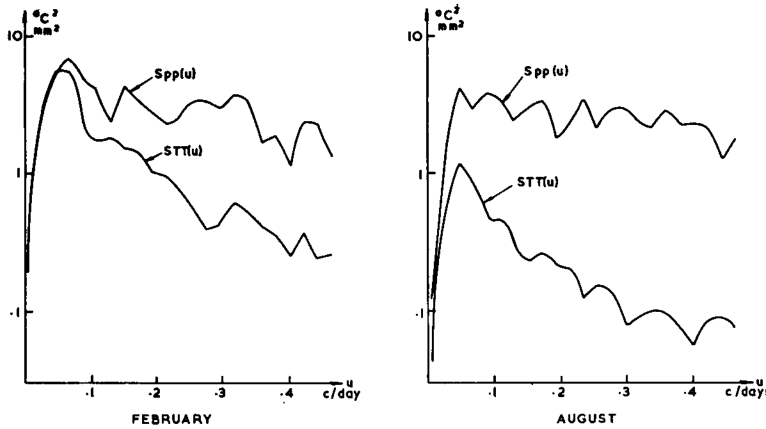

Fig. 4. Power spectra of temperature and precipitation in February and August.

main. A detailed discussion of its effect on spectral estimates is given by Cooley, Lewis & Welch (1969).

Results

The filter in equation (16) eliminates the seasonal variations in the mean. These variations are of course closely related to the seasonal pattern of second order properties. The results of an estimation of the seasonals by means of the Fourier series at 1, 2, 3, 4, 5 and 6 c/year, using data from 1949–1965, are given in Fig. 2. The largest contribution to the annual variation of Thjorsa comes from the melting of the glaciers. It follows the annual variation of the temperature during the summer, but is fairly constant during several months in the winter when surface melting is negligible. The effect of the glaciers is modified by snow melting which produces a maximum at the end of May and, in a smaller degree, by the autumn peak in the precipitation.

The estimates of the power spectra of Thjorsa

Table 1.

	σ_R (Gl)	σ_T (°C)	σ_P (mm)
February	12.4	3.4	5.0
April	11.2	2.8	4.4
June	9.1	1.6	4.4
August	5.1	1.5	5.7
October	8.4	3.0	6.8
December	6.1	3.8	5.1

are given in Fig. 3. Figure 4 shows the power spectra of the meteorological variables in the intervals centered around February and Augst. Apart from the effect of the filter in the lowest frequencies, the main characteristic of the power spectra of Thjorsa is a gradual decrease towards the higher frequencies of the order of a factor of 50 from 1 c/21 days to 1 c/2.4 days. The power spectra of the temperature behave in a similar manner, but the abatement is somewhat smaller. The power spectra of the precipitation are almost constant except for stochastic variations, but a slight decrease takes place towards the higher frequencies. This indicates that much of the seasonal variation of the power spectra can be described by a multiplicative factor which is proportional to the variance. Table 1 gives the standard deviation of the filtered values in each interval.

The standard deviation of the river was estimated by

$$\sigma_R = \frac{\sum_{k=1}^{12} \sum_{m=1}^{96} R_{jk}(m)^2}{12 \sum_{m=1}^{96} C(m)^2} \tag{24}$$

and the standard deviation of T and P was estimated in the same way.

The large variations of Thjorsa in Feburary are due to snow melting. The average temperature in most of the drainage area is considerably below the freezing point. Precipitation in the form of snow can, therefore, be accumulated for long periods of time. But the standard deviation of T is large enough to enable the temperature occasionally to rise adequately to melt a large proportion of it in a short time. The conditions are fairly similar in the April interval.

Snow melting is still taking place at the higher altitudes in June and the glaciers have now entered the picture. The sharp reduction in the standard deviation of T from the mid-winter values is a major factor in limiting variations of Thjorsa during this period.

In August practically all snow has vanished except from the glaciers which dominate the flow. As in the June interval the small standard deviation of T keeps down the variations. The main factor in determining the effect of precipitation of Thjorsa is whether it is rain or snow, but the speed with which it reaches the river also depends upon the state of the ground. When it is frozen the water flows rapidly along the surface. In late summer when there is no frost in the ground more rain is captured as ground water. This helps reducing the fluctuations of Icelandic rivers in August.

In the interval centered around October a great reduction has taken place in the contribution from the glaciers. But due to the increased variability of T, considerable melting can still take place in exceptionally hot days. The large values of the mean and standard deviation of the precipitation further augment the variations of the river.

By December melting from the surface of glaciers is negligible and the precipitation is back to normal. The great increase in the spectrum of R from December to February underlines the effect of few large floods on the second order properties of Thjorsa during the winter. In December the higher average temperature of the preceeding weeks, together with the large standard deviation of T, prevents the necessary accumulation of snow for floods comparable to those taking place later in the winter.

The seasonal pattern of the power spectra are fairly similar at all frequencies, i.e. $A(t, u)$ does not greatly depend upon u. A few nuances are, however, worth mentioning.

A narrow peak occurs in the August spectrum at 1 c/2.8 days. Its height is only about 0.4 on the $\log_{10}$ scale, but it is clearly significant. There is no sign of this peak in any of the other spectra of R nor in the August spectra of T or P. The cause of a narrow distinct peak at this high frequency must be something which first effects an increase of the river for about 24 hours and then turns into the opposite effect or vice versa, beginning with the decrease. I have no doubt that the peak is caused by the combined effect of melting from the glaciers and evaporation on other parts of the drainage area. Urridafoss is about 150 km from the glaciers. This entails a delay between the impact of the weather on the glaciers and our records of the river. The evaporation affects the whole drainage area and is, therefore, observed at once at Urridafoss. The peak is reduced, but by no means eliminated, in the partial spectrum $S_{RR \cdot T}(u)$. The main reasons for this are probably the effect of the wind which affects both melting and evaporation and the fact that the temperature has little effect on evaporation in wet conditions. In all the other time intervals considered here, melting of snow closer to Urridafoss than the glaciers masks the effect of the evaporation. The presence and small size of this peak is a good demonstration of the relative unimportance of evaporation in explaining the characteristics of Icelandic rivers and the power of spectral analysis for revealing them.

The largest values of the power spectrum are in February. This spectrum is also the smoothest, especially in the higher frequencies. This may be a coincidence, but I think it can all be traced to the dominating influence of a few large floods. In the floods $R(t)$ can change by several standard deviations in a single day. These big jumps exert a dominating influence on the spectral estimates at the higher frequencies. In spite of the steep increase of Thjorsa in the floods the largest values are not reached in a single day. Nor does the return to normal take place in one day. This accounts for the decline of $S_{RR}(u)$ with increasing frequency.

Consider now two series of 96 values, both with the same variance and zero average. In the first all except one value are euqal, the second consists of independent normally distributed random variables. In the first case all the periodogram ordinates have the same value; in the second they are approximately exponentially distributed. The properties of the daily values of Thjorsa in mid-winter lie between these extremes. Their affinity with the first provides an explanation of the smoothness of the February spectrum.

The preceeding argument is corroborated by the shape of the October spectrum which pre-

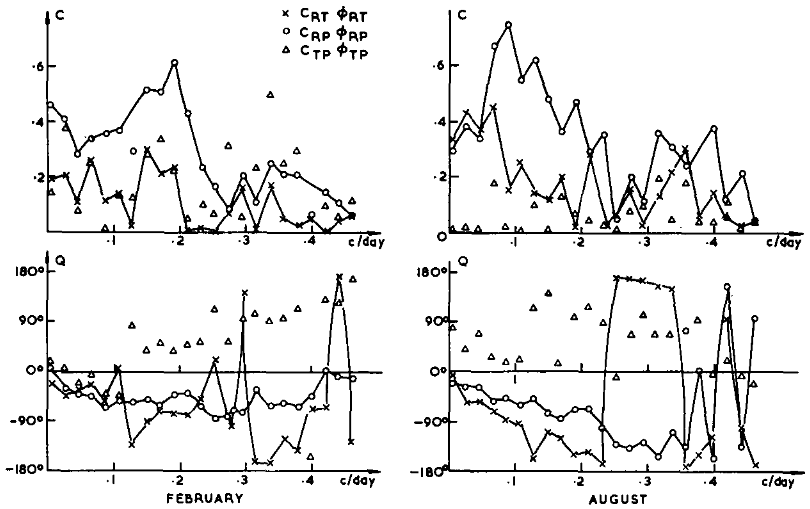

Fig. 5. Phases and coherences in February and August.

sents fairly large variations, although smaller than the spectra from February to June. But in October no huge snow melting floods take place. The shape of the largest snow melting floods is affected by the exhaustion of the snow supply, and the occurrence of one flood reduces the probability of having another in the near future. This applies neither to melting from glaciers nor precipitation which dominate the October spectrum which is closer to the hypothetical case of normal variates. The spectrum is fairly irregular in shape and abates little beyond 1 c/3.4 days.

We proceed now to more direct means of investigating the relationship between variations in the river and the meteorological variables. Fig. 5 shows the estimates of the phases and coherences of our series in the February and August intervals. Considering the dominating effect of the floods on the second order properties of Thjorsa it may seem surprising that its coherences with precipitation are larger than with the temperature. At this time of the year the average temperature is more than one standard deviation below the freezing point on most of the drainage area. The indifference of show melting to what happens in T below the freezing point reduces the coherences of R and T. Precipitation is accompanied by southerly winds and a high temperature which is reflected in significant coherences between T and P. The coherences of R with T indicate that for predicting snow-melting temperatures the inaccuracy of the rain gauge at the upper end of the temperature scale, as compared with the thermometer, is more than compensated for by its more sensible attitude during cool periods. Contribution of the raw material for floods enhances the coherence of Thjorsa with the precipitation in the lowest frequencies.

In Fig. 5 the sign of the phases is such that a negative value implies that the meteorological series are leading the river when $n = 0$ in equation (9). In the lowest frequency band the estimate shows the river leading the precipitation by 4.1°. This anomaly is not too large to be blamed on the stochastic element and as the wavelength is 192 days it is perhaps not surprising that estimation from 96 days intervals produces peculiar results. However, keeping in mind the connection between positive temperatures and P, a physical explanation is also at hand. In response to a temperature variation of the shape of a sine-wave the snow supply is reduced during the rising part of the positive half which leads to a diminished flow of meltwater during the decline of T. The response of the river would then not be in the form of a sine-wave and its peak would appear before the peak of the temperature. The response of the river can be represented by a Fourier series. The component at the same frequency as the temperature would be such that the river was leading the temperature.

After the lowest frequency band the phase difference between R and P increases slowly towards the higher frequencies. At 1 c/3.4 days it corresponds to a time lag of about 0.7 days. Beyond that the lag shortens and becomes less regular. Thjorsa has similar phase differences with both temperature and precipitation at frequencies where the coherence is large enough to produce meaningful estimates of the phase.

Both temperature and precipitation exhibit large coherences with Thjorsa in August. The temperature has a slightly greater effect in the lowest frequencies, but in frequencies over 1 c/20 days precipitation is more important in shaping the fluctuations of Thjorsa. One reason for the decline of the coherence of R and T at the higher frequencies is the contribution of the wind to the melting. The power spectrum of the strength of the wind declines more slowly towards higher frequencies than $S_{TT}(u)$.

The phase difference between Thjorsa and both temperature and precipitation increases gradually up to about 1 c/2.8 days where the peak in the power spectrum of R occurs. The time lag between R and T decreases slightly with the frequency and is about 1.7 day at 1 c/3 days. The phase lag between R and P is smaller.

The coherences do not signify any relationship between temperature and precipitation in August, but more than half of the phase estimates fall into the first quadrant.

We now turn to the investigation of the estimates of the weight functions in the linear model of equation (1). Together with the power spectra, either the phases and coherences or the weight function contain all information about linear relationships between Thjorsa and the temperature and precipitation. Fig. 6 shows the estimates of the weight functions.

When the integrals in equation (1) are replaced by sums over daily values, w_T and w_P have the dimensions Gl/°C and Gl/mm respectively. The weight functions can be normalised through multiplication by the ratio of

Table 2.

	w_T	w_P
February	0.06	0.25
April	0.15	0.13
June	0.15	0.22
August	0.14	0.30
October	0.14	0.24
December	0.12	0.17

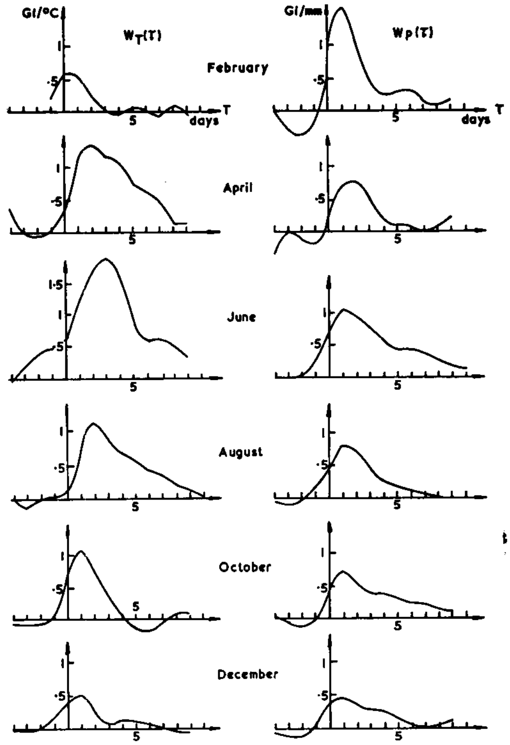

Fig. 6. Weight functions in a linear model of Thjorsa, temperature and precipitation.

the standard deviations σ_T/σ_R and σ_P/σ_R respectively. Table 2 shows the average value of the three largest values of the normalised weight functions for each of our intervals. The normalised values are a measure of how far the variations of R can be accounted for by a linear relationship with T and P.

It has already been pointed out that both T and P serve as indicators of snow melting in February. The maxima of $w_T(\tau)$ and $w_T(\tau)$ are both in the interval $0 \leqslant \tau \leqslant 1$ day which shows the relative importance of melting in the lower parts of the drainage area, closest to Urridafoss, at this time of the year.

In April the rain gauge is of less importance as a thaw-meter. The coherences between T and P are lower than in February which impairs the reliability of the rain-gauge for the purpose. At the same time the thermometer has become more useful because of the increase in the mean temperature. The shape of $w_T(t, \tau)$ (Fig. 6) is fairly similar in the three intervals of April, June and August and so are the values in Table

2. The sharp rise of the seasonal average of Thjorsa during the latter half of the April interval marks the beginning of the melting of the snow on glaciers and high ground. This continues in the June interval. Apart from the delay caused by the distance to Uridafoss the August value of the weight function can be regarded as an estimate of the impulse response of the glaciers to changes in temperature.

Table 2 shows that in these intervals the largest values of $w_T(\tau)$ explain about the same proportion of the variations of R as the largest values in the October and December intervals. However, in April–August $w_T(t, \tau)$ declines more slowly as τ increases beyond the maximum of the weight function and thus accounts for more of the variation of Thjorsa than in autumn and early winter.

The April and October intervals are rather similarly placed with regard to the mean value of T (Fig. 2), but in April the effect of the temperature takes longer to reach its maximum and extends over a longer interval of time. I shall endeavour to mention some factors which might explain the great difference in the shape of the two weight functions, but feel rather uncertain as to their relative quantitative importance.

The average value of Thjorsa rises sharply during the latter half of the April interval. Snow on the lowest part of the drainage area is liable to melt away at any time and thus does not greatly affect the seasonal averge. The increase of R shows that a significant inroad is being made on the snow at the higher altitudes which takes longer to reach our point of observation. Contrariwise, snow is accumulating at the higher altitudes in the October interval.

There is more frost in the ground in April than in October, but when there is no snow cover to insulate the ground, the few uppermost centimeters freeze quickly which is all that is needed to ensure a quick delivery of water to the river.

As explained in connection with the power spectra the variations of R in October are more Gaussian in character than those in April. This should lead to more reliable estimates of the cross spectra. The weight functions are calculated from the partial spectra. Equation (5) shows that if there is no linear relationship between T and P the partial spectra are identical with the ordinary spectra. The coherences of T and P in October are slightly, but not much, smaller than in January. The phase differences of R and T obtained from $S_{RT}(u)$ are larger than those of $S_{RT\cdot P}(u)$. In April the coherences of R and T are low and there is accordingly little difference between $S_{RT}(u)$ and $S_{RT\cdot P}(u)$. Estimates of partial spectra are less accurate than the ordinary spectra, I do not know any investigation of sample properties of partial spectra.

The model of equation (1) is a linear approximation to a highly non-linear relationship. The main non-linear factors in the relationship between Thjorsa and the temperature are the different effects of variations in temperature above and below the freezing point and the amount of snow on the ground. The singularity in the behaviour of water at the freezing point is also the main cause of non-linearity in the relationship between the river and precipitation. As the temperature in the April interval is about 1°C higher and snow more abundant than in October, non-linear effects are probably smaller.

This comparison of spring and autumn has touched on many important factors for the interpretation of the results of this study. I only add one more: even at the lowest altitude of the drainage area there is a great variation in the time that elapses until precipitation which falls as snow melts and reaches Thjorsa. This random element annulls the coherency between snowfall and R in frequencies where the standard deviation of the time lag is of the same order of magnitude as the wavelength. As most of our wavelengths are < 10 days the estimates of $w_P(\tau)$ should largely be regarded as representing the relationship between Thjorsa and rainfall.

Prediction

A comprehensive search for optimal methods of prediction for Thjorsa lies outside the scope of this paper. But because of the practial importance of river prediction I shall present a short digression on the relevance of the results from the spectral analysis for this purpose.

The strongest linear relationship of R with T and P is in the August interval. As a numerical example the observations of R from 7/8–2/9 in the last three years of observations included here, 1963–1965, were predicted by $\widehat{R}(m)$ defined by the equation

$$\hat{R}(m) = \sum_{k=1}^{6} w_T(k)\, T(m-k) + \sum_{k=1}^{6} w_P(k)\, P(m-k) \tag{25}$$

where w_T and w_P are the estimates of these functions presented by Figure 6. The seasonal averages were subtracted from the observations. The standard deviation of $R(m)$ was 7.3 Gl and the standard deviation of the error,

$$Z_1(m) = \hat{R}(m) - R(m), \tag{26}$$

was 2.5 Gl.

In order to assess the value of the relationship between the river and the meteorological variables in predicting the river we need a predictor based only on past values of the river itself. Suitable methods for constructing such predictors are obtained by regarding the river as a mixed autoregressive and moving average process, i.e.

$$R(m) - \sum_{k=1}^{M} c_k R(m-k) = Z_2(m) + \sum_{k+1}^{N} d_k Z_2(m-k), \tag{27}$$

where $Z_2(m)$ is supposed to be a process of uncorrelated residuals, c_k and d_k are parameters and M and N are integers defining the size of the model. A detailed discussion of the use of this type of process and the estimation of the parameters is provided by Box & Jenkins (1966). Given the values of $R(m-k)$ and $Z_2(m-k)$ for $k \geqslant 1$ the predictor of $R(m)$ is

$$\hat{R}(m) = -\sum_{k=1}^{M} c_k R(m-k) + \sum_{k=1}^{N} d_j Z_2(m-j). \tag{28}$$

Comparison with equation (27) shows that $Z_2(m)$ represents the error of this predictor.

The serial correlation between $R(m)$ and $R(m-1)$ in the data used to test the predictor of equation (25) was 0.955. If we use this as $-c_1$ in equation (28), assuming that the other parameters are zero, the standard deviation of $Z_2(m)$ is 2.2 Gl. By adding one more term to the moving average part by putting $d_1 = 0.4$ the standard deviation is reduced to 2.0 Gl.

The error of this relatively simple predictor derived from past values of the river is thus smaller than the standard deviation of $Z_1(m)$. There is, however, a serial correlation of 0.676 between $Z_1(m)$ and $Z_1(m-1)$ and by introducing the model

$$Z_1(m) = 0.676\, Z_1(m-1) + \varepsilon(m), \tag{29}$$

where $\varepsilon(m)$ represents the residuals, the standard deviation is reduced to 1.8 Gl. It seems likely that both this predictor and the one using only past values of the river can be improved by a few per cent by introducing more terms into the mixed autoregressive and moving average model or some other contrivance. But I think they provide a fair assessment of the order of magnitude of the advantages of including meteorological observations in the predictions. This amounts to a reduction of the order of 10 per cent in the standard deviation compared with the predictor based on the river alone.

The models of equations (25) and (27) can also be used for prediction more than one day ahead.

The performance of the predictors is in accordance with the results from the spectral analysis. The reason why the meteorological observations only provide a modest improvement on the results derived from the river alone lies in the statistical properties of the series. The effect on Thjorsa of changes in the meteorological conditions is felt immediately, but its impact also extends over a couple of days. This introduces relationships between the values of R at different days which the model of equation (27) takes into account. These relationships are also reflected in the sharp decline of the spectrum with increasing frequency.

During the winter the linear model of the relationship between the river and the meteorological variables is less appropriate and it was argued in the previous section that a relatively small number of large floods dominate the results of the spectral analysis. Experiments with models similar to that of equation (25) indicate that in winter time the linear relationships with T and P are of little value in predicting Thjorsa at levels below the seasonal average plus one or two standard deviations. The winter spectra of R still show that the values are far from independent and this can be used for predicting R. But, as a result of the skewness of the distribution and non-linear effects, a single predictor derived from equation (27) is likely to fall considerably short of the optimum predictor obtainable from past values of R. Possible means of improvement could be either to use more than one linear predictor to be selected according to the value of R on the day of prediction, or to introduce non-linear terms.

Discussion

Time dependent spectral analysis seems to provide a worthwhile contribution towards a quantitative description of Thjorsa and its relationship with meteorological variables. But in spite of significant coherences at all seasons, the linear model leaves much of the variations of Thjorsa unaccounted for, especially during the winter. I shall here discuss briefly some of the shortcomings of the model and possibilities for improvement.

Our series are subject to perturbations, stochastic and systematic. The rain gauges collect less in stormy weather (Rodda, 1968); both Thjorsa itself and measurements of it can be disturbed by slush and ice. These errors are all prevalent in the months of low coherences. Observations from a single meteorological station do not contain all the relevant information about temperature and precipitation from a drainage area of 7500 km^2. This reduces the coherences, especially at higher frequencies, but I see no obvious reason why it should have more effect in one season than another. In August the linear model explains about 70% of the variance of the filtered values of R and a part of the remaining variation could be accounted for by including the wind in the equation. This success of the linear model is a vindication of the value of the observations at Haell for hydrological research in this area.

The linear model still explains about 50% of the variations of Thjorsa during the winter. Nonlinear effects and the distribution of R make linear models and spectra less appropriate to describe the actual relationships than in August. A possible way of extending the model of equation (1) is to introduce terms like

$$\iint_{-\infty}^{\infty} w_{TP}(\tau_1, \tau_2)\, T(t-\tau_1)\, P(t-\tau_2)\, d\tau_1\, d\tau_2.$$

The weight function $w_{TP}(\tau_1, \tau_2)$ would contain information about the influence of the temperature at $t-\tau_1$ in deciding the effect of the precipitation at $t-\tau_2$ on $R(t)$. Higher order spectra provide a mathematically attractive generalisation of power spectra and cross spectra. Brillinger (1969) shows how weight functions like the w_{TP} above may be estimated from the higher order spectra. Unfortunately for our problem this procedure relies heavily upon the assumption of normality of the independent variables, an assumption which cannot even be contemplated as a first approximation in the case of precipitation.

The weight functions could, of course, be estimated without using the higher order spectra The sample properties of estimates of ne dimensional weight functions are rather awkward and it is a great advantage to have recourse to the phases and coherences. The estimates at $\tau < 0$ also provide a valuable indication of the reliability of our estimates of $w_T(\tau)$ and $w_P(\tau)$. In the case of two dimensional weight functions our a priori knowledge of what a reasonable result should look like is more limited and, because of the non-Gaussian character of the precipitation, there are no spectral estimates from which to judge the significance of the weight functions.

Another way of dealing with the non-linear effects is to obtain new series by non-linear transformations of P and T and then use the second order properties of these series in the same way as we have investigated the original observations. Examples of such transformations could be:

$$T_1 = \begin{matrix} T(t) \text{ if } T(t) > T_0 \\ 0 \text{ otherwise} \end{matrix}$$

and

$$\begin{matrix} P_1(t) = P(t)\; F_1(T(t)) \\ P_2(t) = P(t)\; F_2(T(t)) \end{matrix} \quad \text{where } F_1(t) + F_2(t) = 1.$$

The functions F_1 and F_2 could indicate the proportion of the drainage area where the temperature is above or below 0°C respectively.

Because the transformations are not linear we can produce a linear model analogus to that of equation (1) describing R as a function of 3 or more series without other observations than those of T and P. The series would, however, tend to be highly correlated and I think comparison of different models, each with only two series on the right-hand side, would provide a better insight into the hydrological situation than estimation of a single big model. Phases and coherences would be more suitable quantities to use in the comparison than weight functions.

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КРАТКОЕ СОДЕРЖАНИЕ

Главными факторами, определяющими кратковременные изменения реки Тиорзы в Исландии являются осадки и таяние снега и ледников. Чтобы исследовать сезонные изменения во второстепенных свойствах изменений Тиорзы и их зависимости от температуры и осадков, был применен спектральный анализ, зависящий от времени.

В конце лета линейная модель Тиорзы, температуры и осадков на метеорологической станции в Хелле дает хорошее соответствие наблюдений. Из-за нелинейных эффектов, связанных с таянием снега, линейные модели менее подходящи во время зимы, хотя они все же могут объяснять значительную часть изменений Тиорзы.

Самой главной информацией для предсказаний на несколько суток вперед являются прошлые данные о самой реке. Некоторое улучшение может быть достигнуто, если принимаются во внимание также и метеорологические наблюдения.