Optical properties of thick fog layers

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ABSTRACT

The reflection and transmission properties of thick plane layers of fog, which is a highly anisotropically scattering medium, are investigated. First, the function $R(\mu, \varphi; \mu_0, \varphi_0)$ describing the law of reflection for a semi-infinite plane layer is determined. A table based upon the scattering function of Spencer is presented. With the help of this function, the angular distribution of the emerging light in the case of the Milne problem, and the extrapolation length, are computed. Finally, the angular distributions of the light reflected and transmitted by a thick plane layer of fog are expressed in terms of the function R and of the angular distribution for the Milne problem.

1. Introduction

Monochromatic radiative transfer and one-speed neutron transport in plane geometry have been treated by numerous authors, and several monographs in this field are available (Busbridge, 1960; Chandrasekhar, 1950; Davison, 1957; Hopf, 1934; Kourganoff, 1952; Sobolev, 1963). Exact analytical methods for solving problems in a semi-infinite medium have been developed: the Wiener-Hopf technique, the method of invariance principles (Chandrasekhar, 1950) and singular eigenfunction expansion (Case, 1960). In several ways, these methods have been generalised to anisotropic scattering (Busbridge, 1960; Chandrasekhar, 1950; Davison, 1957; Kuščer, 1956, 1958; McCormick & Kuščer, 1966; Mika, 1961; Mullikin, 1964; Romanova, 1960; Sobolev, 1963).

Problems in slab geometry are more difficult and cannot be solved exactly in closed forms. However, simple analytical approximations for a thick slab can be obtained in terms of the semi-infinite medium solutions. In the present paper, these approximations will be used to determine the reflection and transmission properties of a thick layer of fog.

The basic equations of the general theory will be briefly reviewed, with the use of the standard astrophysical notation (e.g. Mullikin, 1964). Radiative transfer in plane geometry is described by the equation

$$\left(\mu \frac{\partial}{\partial x} + 1\right) I(x, \mu, \varphi) = \frac{1}{4\pi} \int_{-1}^{1} \int_{0}^{2\pi} p(\mu, \varphi; \mu', \varphi') \times I(x, \mu', \varphi')\, d\mu'\, d\varphi', \tag{1.1}$$

where $I(x, \mu, \varphi)$ is the specific intensity, x is the distance measured in units of mean free path, and μ is the cosine of the inward polar angle.

It will be assumed that the scattering function $p(\mu, \varphi; \mu', \varphi')$ can be approximated by a finite series of Legendre polynomials,

$$p(\mu, \varphi; \mu', \varphi') = p(\cos\theta) = \sum_{l=0}^{N} \tilde{\omega}_l P_l(\cos\theta), \tag{1,2}$$

with

$$\cos\theta = \mu\mu' + (1-\mu^2)^{\frac{1}{2}} (1-\mu'^2)^{\frac{1}{2}} \cos(\varphi - \varphi'). \tag{1.3}$$

Also we shall limit ourselves to the nonabsorbing medium ($\tilde{\omega}_0 = 1$), which is justified in the case of natural fog.

The Fourier expansion of $p(\cos\theta)$ can be written as

$$p(\mu, \varphi; \mu', \varphi') = p^0(\mu, \mu') + 2 \sum_{m=1}^{N} p^m(\mu, \mu') \times (1-\mu^2)^{m/2} (1-\mu^{1,2})^{m/2} \cdot \cos m(\varphi - \varphi') \tag{1.4}$$

with $$p^m(\mu, \mu') = \sum_{l=m}^{N} c_l^m p_l^m(\mu) p_l^m(\mu'), \tag{1.5}$$

$$p_l^m(\mu) = \frac{d^m}{d\mu^m} P_l(\mu), \tag{1.6}$$

$$c_l^m = \tilde{\omega}_l \frac{(l-m)!}{(l+m)!}. \tag{1.7}$$

For each m we define a function $\lambda^m(z)$ (Mullikin, 1964*a*, Eq. 3.26), which is analytic in the whole complex plane cut form -1 to 1,

$$\lambda^m(z) = 1 - \frac{z}{2} \int_{-1}^{+1} \frac{g^m(\mu, \mu)}{z - \mu} dm(\mu), \quad z \notin (-1, 1) \tag{1.8}$$

where $dm(\mu)$ stands for $(1-\mu^2)^m d\mu$. Herein $g^m(\mu, \mu)$ is a polynomial given by

$$g^m(\mu, \mu) = \sum_{l=m}^{N} c_l^m g_l^m(\mu) p_l^m(\mu), \tag{1.9}$$

$$\mu(2l + 1 - \tilde{\omega}_l) g_l^m(\mu) = (l + 1 - m) g_{l+1}^m(\mu)$$

$$+ (l+m) g_{l-1}^m(\mu), \tag{1.10}$$

$$g_m^m(\mu) = p_m^m(\mu). \tag{1.11}$$

The roots of the characteristic equations,

$$\lambda^m(z) = 0, \tag{1.12}$$

play an important part in the theory of radiative transfer (Mullikin, Pahor & Kuščer, 1966).

Let us first consider the albedo problem. In this case, a layer or a semi-infinite medium is illuminated from outside (at $x = 0$) by a parallel beam of light of direction (μ_0, φ_0) and of unit net flux per unit area normal to the beam. The specific intensity will be denoted as $I(x, \mu, \varphi; \mu_0, \varphi_0; x_1)$, with x_1 expressing the thickness of the slab. Hence, I is the solution of Eq. (1.1) for the boundary conditions

$$I(0, \mu, \varphi; \mu_0, \varphi_0; x_1) = \delta(\mu - \mu_0)\delta(\varphi - \varphi_0), \ \mu > 0, \tag{1.14}$$

$$I(x_1, -\mu, \varphi; \mu_0, \varphi_0; x_1) = 0, \quad \mu > 0, \tag{1.15}$$

or, in case of a semi-infinite medium ($x_1 = \infty$),

$$\lim_{x \to \infty} I(x, \mu, \varphi; \mu_0, \varphi_0; \infty) < \infty.$$

We will also need the solution of the Milne problem. In this case we have a plane source at infinity, and no light is falling on the surface of the semi-infinite medium. The corresponding specific intensity will be denoted as $I(x, \mu)$. This is the solution of Eq. (1.1), satisfying the boundary condition

$$I(0, \mu) = 0, \quad \mu > 0, \tag{1.17}$$

and normalized to unit net flux. The asymptotic behaviour of $I(x, \mu)$ for large x is given by (Chandrasekhar, p. 92)

$$I(x, \mu) \to \frac{3 - \tilde{\omega}_1}{2}(x + q) - \frac{3}{2}\mu, \quad x \to \infty, \tag{1.18}$$

where q is the extrapolation length (see also Section II).

2. Solutions for a semi-infinite medium

In the albedo problem we shall be interested only in the distribution of the reflected intensity

$$I(0, \ -\mu, \ \varphi; \ \mu_0, \ \varphi_0; \ \infty).$$

It is convenient to introduce Chandrasekhar's R-function

$$\frac{\mu\mu_0}{\mu + \eta_0} R(\mu, \varphi; \mu_0, \varphi_0) = 4\pi\mu I(0, -\mu, \varphi; \mu_0, \varphi_0; \infty), \quad \mu > 0 \tag{2.1}$$

and the expansion

$$R(\mu, \varphi; \mu_0, \varphi_0; \infty) = R^{\circ}(\mu, \mu_0) + 2 \sum_{m=1}^{N} R^m(\mu, \mu_0)$$

$$\times (1-\mu^2)^{m/2} (1-\mu_0^2)^{m/2} \cos m(\varphi - \varphi_0). \tag{2.2}$$

The fact that the reflected net flux equals the unit incident one imposes on the R-function the normalization condition:

$$\frac{1}{2} \int_0^1 \frac{\mu R_0(\mu, \mu_0)}{\mu + \mu_0} d\mu = 1 \tag{2.3}$$

The coefficients $R^m(\mu, \mu_0)$ satisfy non-linear integral equations (Pahor & Kuščer, 1966, Eq. 9)

$$R^m(\mu, \mu_0) = \sum_{l=m}^{N} (-1)^{l+m} c_l^m \psi_l^m(\mu) \psi_l^m(\mu_0), \tag{2.4}$$

with

$$\psi_l^m(\mu) = p_l^m(\mu) + (-1)^{l+m} \frac{\mu}{2} \int_0^1 R^m(\mu, \mu') p_l^m(\mu')$$

$$\times \frac{dm(\mu')}{\mu' + \mu} \tag{2.5}$$

The uniqueness of solutions of these equations depends on the number of roots of the related characteristic equations (1.12). If there are no roots, we have a unique solution, while for each pair of roots there exists a linear constraint (Pahor, 1966, Eq. 20), which determines the solution obeying Eqs. (1.1), (1.14) and (1.16).

The angular distribution function $I(0, -\mu)$ for the emerging light in the case of the Milne problem can be expressed in terms of $R^0(\mu, \mu_0)$ according to the equation (Chandrasekhar, 1950, p. 92)

$$I(0, -\mu) = \frac{3}{2}\left(\mu + \frac{1}{2}\int_0^1 \frac{R^0(\mu, \mu')}{\mu' + \mu}\mu'^2 d\mu'\right). \qquad (2.6)$$

Another quantity of interest is the extrapolation length q. At this distance from the medium surface the asymptotic density of radiation ($\int I \, d\mu \, d\varphi$) is equal zero. We quote its ratio to the transport mean free path $(1 - (\varpi_1/3))^{-1}$

$$\left(1 - \frac{\tilde{\omega}_1}{3}\right) q = \frac{3}{4}\int_0^1 \int_0^1 \mu\mu'^2 R^0(\mu, \mu')\, d\mu\, d\mu'. \qquad (2.7)$$

3. Approximate solutions for a thick layer

The angular distribution of the reflected and transmitted light in the case of a thick layer can be formaly written in terms of Chandrasekhar's R and T functions. These two satisfy a system of two coupled nonlinear integral equations (Chandrasekhar, p. 170).

However, it is more convenient to solve a pair of linear integral equations for R and T which are derived from the solution for the half space albedo problem by using principles of invariance (Pahor, 1967). These equations may be solved by iteration. In the lowest order of approximation the reflected and transmitted angular distributions are given by[1]

$$I(0, -\mu; \mu_0; x_1) = \frac{1}{4\pi\mu}\frac{\mu\mu_0}{\mu + \mu_0} R^0(\mu, \mu_0) - I(0, -\mu)\, j(\mu_0; x_1), \qquad (3.1)$$

$$I(x_1, \mu; \mu_0; x_1) = I(0, -\mu)\, j(\mu_0; x_1), \qquad (3.2)$$

where $j(\mu_0; x_1)$ is the net flux transmitted by the atmosphere,

$$j(\mu_0; x_1) = \frac{2\mu_0}{3} \frac{I(0, -\mu_0)}{\left(1 - \frac{\tilde{\omega}_1}{3}\right)(x_1 + 2q)}. \qquad (3.3)$$

If we are interested only in the transmission problem the above result can be derived in a still shorter fashion. Owing to the great thickness of the atmosphere, the angular distribution of the transmitted light is almost independent of μ_0 and is nearly the same as the distribution of the emergent light in Milne's problem. That is, $I(x_1, \mu; \mu_0; x_1)$ is approximately equal to $I(0, \mu)$, multiplied by a function of μ_0 and x_1. However, since $I(x, \mu; \mu_0; x_1)$ is a symmetric function of μ and μ_0 (Chandrasekhar, 1950, p. 21) we obtain the expression (3.2) with (3.3).

4. The scattering function for fog

The scattering function for fog could be calculated (Chu & Churchill, 1955; Clarc, Chu & Churchill, 1957; van de Hulst, 1957), if the size distribution of water droplets were given. In practice, however, this distribution is not well known, so it is better to rely upon measured scattering characteristics. A series of such data, summarized by Spencer (1960), has shown that the scattering function of fog does not vary much from case to case, so that a certain average scattering function may be used for calculations. We shall follow this recommendation, supposing moreover that within the visible region the scattering function does not depend upon the wavelength.

Since the data of Spencer cover only scattering angles from 20° to 160°, an extrapolation had to be made. Consistent values (within 7% at 0°) were obtained in various ways graphically and numerically. A check of normalization, instead of giving $2\int_0^\pi F(\theta)\cdot \sin\theta \cdot d\theta = 1$ (according to the definition of Spencer), led to the value 0.9386 ± 0.0070 for this integral, where the estimated error is due to uncertainty in extrapolation. The difference is too great to be reasonably explained otherwise than by assuming an error in the original normalization. Consequently we divide the values of Spencer by 0.9386, in order to obtain a normalized scattering function. Taking into account that fog is practically free of absorption in the visible region, we have

$$p(\cos\theta) = 4F(\theta)/0.9386. \qquad (4.1)$$

[1] For different derivation, see Kuščer, 1958.

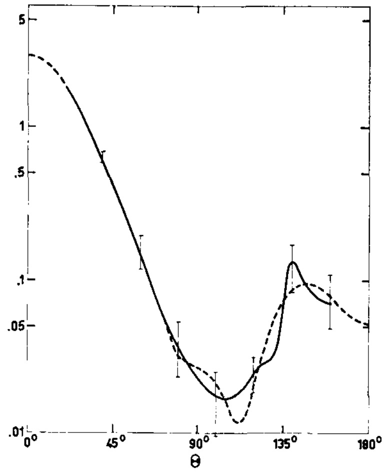

Fig. 1. The average scattering function of fog. The full curve represents the renormalized data of Spencer, with the maximal deviations of the individual scattering functions indicated by bars. The dashed curve refers to the approximation (4.2).

The coefficients $\tilde{\omega}_l$, occurring in the Legendre development of $p(\cos\theta)$, have been evaluated as follows:

$\tilde{\omega}_0 = 1$	$\tilde{\omega}_5 = 1.0716$
$\tilde{\omega}_1 = 2.1053$	$\tilde{\omega}_6 = 0.4803$
$\tilde{\omega}_2 = 2.7424$	$\tilde{\omega}_7 = 0.3615$
$\tilde{\omega}_3 = 2.1929$	$\tilde{\omega}_8 = 0.1587$
$\tilde{\omega}_4 = 1.5578$	$\tilde{\omega}_9 = 0.0075$

The lowest partial sum, which reproduces the scattering function to within the deviations given by Spencer (see also Fig. 1), is

$$p(\cos\theta) = \sum_{l=0}^{N=8} \varpi_l P_l(\cos\theta). \qquad (4.2)$$

This expression, with the above values of $\tilde{\omega}_l$, will be used as the basis for the computation of R^m and related functions.

Since the scattering function for natural fog is strongly peaked in the forward direction one could argue that our extrapolated values for $\Theta \in (0°, 20°)$ are not reliable. The same could be said also for the approximation (4.2). However, it will be shown that the transmission properties of thick layers, which are most interesting in meteorology, depend only slightly on the scattering function, if there is no true absorption. Therefore, we claim that at least the transmission properties of thick layers, derived by using approximation (4.2) are valid also for a natural fog.

As a specific property of our approximation for the scattering function the roots of the corresponding characteristic equation (1.12) should be quoted. For $m = 0$ we have the double root at infinity, which is characteristic of any nonabsorbing medium. However, it has been found that another pair of roots exists. It is located at $z = \pm 1.2148$, as has been determined by numerical evaluation. The appearance of this second pair of roots is quite unusual property of the scattering functions derived from measurements. An analysis for $m = 1, 2, \ldots, 8$ has shown that in these cases the characteristic equations have no roots.

5. Computations

Numerical values of $R^m(\mu, \mu_0)$ with $m = 1, 2, \ldots, 8$ have been calculated by iterating Eqs. (2.4) and (2.5).

Suitable initial approximations can be obtained, if we remember that the fraction of light which has been scattered more than once does not strongly depend on the scattering function (Chandrasekhar, p. 146). Since the part of $R^m(\mu, \mu_0)$, belonging to the first-order scat-

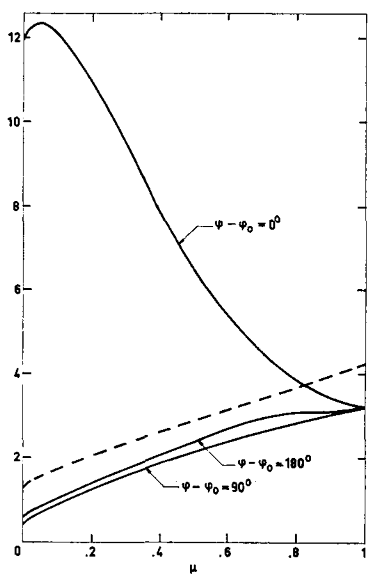

Fig. 2. The function $R(\mu, \varphi; \mu_0, \varphi_0)$ for a semi-infinite layer of fog for $\mu = 0.2$ and $\varphi - \varphi_0 = 0$, 90°, 180° (full curves) and the same function for isotropic scattering (dashed curve).

tering, is equal to $p^m(-\mu, \mu_0)$, we say that in a first approximation, the difference $R^m(\mu, \mu_0) - p^m(-\mu, \mu_0)$ is independent of the scattering function.

Practical use of this statement is made by observing that this difference vanishes for isotropic scattering, so that the solution of general problem is approximated by

$$R^m(\mu, \mu_0) \cong p^m(-\mu, \mu_0), \quad m = 1, 2, \ldots, 8 \quad (5.1)$$

If scattering is far from being isotropic, the above approximations cannot be expected to be very accurate. Some improvement could be achieved by choosing a more suitable reference case (Sobolev, Ch. X. 1). Unfortunately few cases with well tabulated solutions are known and none for strongly peaked scattering functions. For this reason, (5.1) have been used as the initial approximations for iterating Eqs. (2.4) and (2.5). The convergence has been satisfactory.

The values of $R^0(\mu, \mu_0)$ have been calculated by iterating a modified system of Eqs. (2.4) and (2.5),

$$\psi_l^0(\mu) = \frac{p_l^0(\mu) + \dfrac{\mu}{2} \sum_{k=0,\, k \neq l}^{8} (-1)^{l+k} \tilde{\omega}_k \displaystyle\int_0^1 \frac{\psi_k^0(\mu)\, \psi_k^0(\mu')}{\mu + \mu'} \times p_l^0(\mu')\, d\mu'}{1 - \dfrac{\mu}{2} \tilde{\omega}_l \displaystyle\int_0^1 \frac{\psi_l^0(\mu')}{\mu + \mu'} p_l^0(\mu')\, d\mu'} \quad (5.2)$$

The following initial approximations, based upon expansions around $\mu = 0$, have been used

$$\psi_0^0(\mu) = \left(1 + \frac{1}{2} p^0(0, 0)\, \mu \ln \frac{1 + \mu}{\mu}\right)(1 + a\mu), \quad (5.3)$$

$$\psi_2^0(\mu) = P_2(\mu) + \frac{1}{4} p^0(0, 0)\, \mu \left(1 - \ln \frac{1 + \mu}{\mu}\right), \quad (5.4)$$

$$\psi_3^0(\mu) = P_3(\mu) + \tfrac{1}{4}\mu, \quad (5.5)$$

$$\psi_l^0(\mu) = P_l(\mu), \quad l = 4, \ldots, 8. \quad (5.6)$$

In Eq. (5.3) the parameter a is adjusted by condition (2.3). In addition to these approximations we have the exact relation following from (2.3)

$$\psi_1^0(\mu) = 0. \quad (5.7)$$

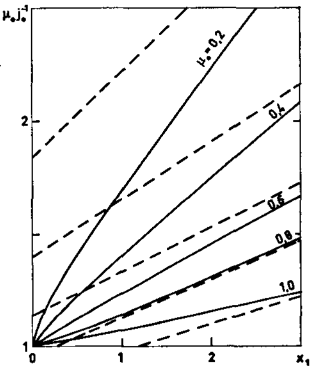

Fig. 3. The reciprocal values of the transmitted net flux $j_0(\mu_0, x_1)$ multiplied by μ_0 versus thickness of the layer (measured in mean free paths) for various incident angles. Dashed lines represent thick layer approximations. Derivatives of $j_0(\mu_0, x_1)$ with respect to x_1 were calculated at $x_1 = 0$ in order to get graphical estimations for $\mu_0 j_0^{-1}$ at intermediate thicknesses (full curves).

Regarding the uniqueness of obtained solutions following investigation has been done. For $m \geq 1$, no further checks were needed because uniqueness of the solutions is guaranteed in view of what has been said before. For $m = 0$, where the solution is not unique, the iterated values of $R^0(\mu, \mu_0)$ have been tested by the corresponding constraint.

In order to have a check, $R^0(\mu, \mu_0)$ has also been calculated by factorizing it as

$$R^0(\mu, \mu_0) = H(\mu) H(\mu_0) \sum_{k=0}^{8} (-1)^k \tilde{\omega}_k q_k(\mu) q_k(\mu_0) \quad (5.8)$$

where $H(\mu)$ is Chandrasekhar's H-function, and $q_k(\mu)$ are the polynomials introduced by Busbridge (1960). The integral equation and linear constraint, which have to be obeyed by the H-function, can be reduced to a homogeneous Hilbert problem. So the function $H(\mu)$ has been calculated from a closed-form expression by numerical integration. Then the polynomials $q_k(\mu)$ have been determined by a method proposed by Pahor (1966, Eq. 2.24).

The values of $R^0(\mu, \mu_0)$ obtained by these two methods agreed to five significant places. Comparing the usefulness of the two methods, the following conclusions can be made. If the solution $R^m(\mu, \mu_0)$ is unique, the straightforward iteration of Eqs. (2.4) and (2.5) is simpler, especially for large N. On the other hand, if

$N = 3$ or 4 or if the solution is not unique, the factorization method seems advisable.

The values of $R^0(\mu, \mu_0)$, tabulated in Appendix II (Table 1), are expected to be correct to within one unit of the last decimal place. Tables for the less important $R^m(\mu, \mu_0)$, $m = 1, ..., 8$ are omitted (available in preprint form). For illustration the azimuth-dependent R-function is represented for some special values of $\varphi - \varphi_0$ (Fig. 2).

The angular distribution function $I(0, -\mu)$ for the emerging light in the case of Milne's problem is given in Appendix II (Table 2) together with the values of $I(0, -\mu)$ for isotropic scattering.

For the extrapolation length the following value is obtained

$$(1 - \tfrac{1}{3}\tilde{\omega}_1)\, q = 0.7146.$$

A comparison of these results with those for isotropic scattering ($\tilde{\omega}_0 = 1$, $q = 0.7104$) again shows (cf. Mark, 1943) that neither $(1 - \frac{1}{3}\tilde{\omega}_1)q$ nor $I(0, -\mu)$ are strongly dependent upon the scattering function in cases with no absorption.

The dependence of the transmitted flux upon the incident angle and upon the thickness of the layer was calculated according to Eq. (3.3) and is shown by Fig. 3.

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ОПТИЧЕСКИЕ СВОЙСТВА ТОЛСТЫХ СЛОЕВ ТУМАНА

Исследованы отражательные и поглощательные свойства толстых слоев тумана, который является анизотропно рассеивающей средой. Определена функция $R(\mu, \varphi; \mu_0, \varphi_0)$, описывающая закон отражения для полубесконечного плоского слоя. Представлена таблица, основанная на функции рассеяния Спенсера. С помощью этой функции вычислено угловое распределение выходящего света в случае проблемы Милна и длина экстраполяции. Наконец, угловое распределение света, отраженного и пропущенного толстым плоским слоем тумана выражено через функцию R и угловое распределение для проблемы Милна.

APPENDIX

Table 1. *Chandrasekhar's* $R^0(\mu, \mu_0)$*-function*

μ ... μ_0	0	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	1.963	2.240	2.374	2.456	2.381	2.219	2.028	1.850	1.706	1.605	1.550	1.583
0.05	2.240	2.550	2.699	2.797	2.732	2.584	2.411	2.254	2.128	2.034	1.982	2.031
0.1	2.374	2.699	2.857	2.973	2.933	2.818	2.686	2.567	2.472	2.399	2.358	2.419
0.2	2.456	2.797	2.973	3.144	3.185	3.170	3.141	3.118	3.102	3.092	3.097	3.165
0.3	2.381	2.732	2.933	3.185	3.336	3.441	3.528	3.608	3.684	3.759	3.834	3.903
0.4	2.219	2.584	2.818	3.170	3.441	3.669	3.869	4.053	4.230	4.404	4.564	4.655
0.5	2.028	2.411	2.686	3.141	3.528	3.869	4.178	4.467	4.746	5.022	5.277	5.446
0.6	1.850	2.254	2.567	3.118	3.608	4.053	4.467	4.860	5.241	5.612	5.965	6.280
0.7	1.706	2.128	2.472	3.102	3.684	4.230	4.746	5.241	5.717	6.177	6.632	7.136
0.8	1.605	2.034	2.399	3.092	3.759	4.404	5.022	5.612	6.177	6.727	7.296	7.954
0.9	1.550	1.982	2.358	3.097	3.834	4.564	5.277	5.965	6.632	7.296	7.974	8.658
1	1.583	2.031	2.419	3.165	3.903	4.655	5.446	6.280	7.136	7.954	8.658	9.210

Table 2. *Solutions for Milne's problem for fog and for an isotropically scattering medium*

μ	$I(0, -\mu)$ Fog	$I(0, -\mu)$ Isotropic scattering
0	0.1088	0.1379
0.1	0.1521	0.1693
0.2	0.1861	0.1975
0.3	0.2173	0.2243
0.4	0.2469	0.2502
0.5	0.2751	0.2757
0.6	0.3025	0.3008
0.7	0.3291	0.3257
0.8	0.3552	0.3504
0.9	0.3810	0.3749
1	0.4065	0.3994