The effect of weak shear and rotation on internal waves

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ABSTRACT

The effect of constant shear on internal waves is studied in a simplified oceanic model with constant Brunt-Väisälä frequency and high dynamical stability. As observed in the non-rotating case (Phillips, 1966), the wave number is rotated, the energy transferred to the mean flow and the frequency decreased. A lower bound at inertial frequency is found as expected, but a shear instability will probably occur before the frequency reaches this limit.

Equilibrium frequency spectra are derived and compared to relevant observations. A good agreement is observed with the horizontal kinetic energy spectrum obtained at Site D, below the thermocline, whereas no inconsistency is found in comparison with other (less relevant) oceanic data. It is concluded that the interaction between shear and internal waves could play an important role in some region of the ocean.

1. Introduction

A major problem in oceanography is the interpretation of the inertio-gravitational range of the frequency spectra recorded below the sea surface. This range is bounded by the inertial frequency at the lower limit and the Brunt-Väisälä frequency at the upper limit. Different interpretations of this data have been suggested. These are based either on the assumption of a turbulence field or a random internal wave field.

Because a tractable theory of turbulence in a stratified fluid is not yet available, results valid for isotropic homogeneous turbulence have been used for comparison with the data. Although this is based on unjustified assumptions (Webster, 1969), the agreement between theory and observations is good. However, recent observations suggest that the observed motion may be most effectively represented by a field of internal waves (Fofonoff, 1969). Studies of non-linear interactions between different components of a random internal wave field are not yet able either to provide the shape of an equilibrium spectrum or to evaluate their practical importance. However, linear models can be used to predict a frequency spectrum.

Such a model which is based on the linear interaction between internal waves and a mean shear flow, was investigated by Phillips (1966) for a non-rotating case. For a constant stratification, he found that the effect of the weak constant shear on a simplified type of wave in infinitely deep medium was to rotate the wavenumber towards the vertical, to reduce the frequency and to lead to a gradual transfer of the wave energy to the mean flow. Assuming the existence of a steady source of internal waves, Phillips derived the high-frequency part of several equilibrium spectra. There is, however, an algebraic error in his results. In the present paper, we have conserved Phillips' model, but taken the Coriolis force into account; hence the derived results are valid throughout the whole internal wave range.

In sections 2 and 3, the equations of motion are presented and simplified. An asymptotic wave solution is derived and discussed in section 4. Equilibrium spectra are then derived (§ 5). A carefull discussion is devoted to the representativity of the calculated frequency spectra and to the restrictions imposed by the simplifying hypotheses. Results are then compared to some available observations (§ 6).

2. Derivation of the equations

We assume that the mean velocity $\mathbf{U} \equiv (U(z), V(z), 0)$ is horizontal and depends only upon the depth. Furthermore, we hypothesize (Stern,

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1969) that a stress field of unknown mechanics balances the Coriolis force which acts on the mean current. With these assumptions and making the inviscid Boussinesq approximation, the first-order equations of motion for a small velocity perturbation $\mathbf{v} \equiv (u, v, w)$ are

$$Du + wU' + \Omega_2 w - \Omega_3 v + \frac{1}{\varrho_0}\frac{\partial p}{\partial x} = 0 \quad (1)$$

$$Dv + wV' + \Omega_3 u + \frac{1}{\varrho_0}\frac{\partial p}{\partial y} = 0 \quad (2)$$

$$Dw - \Omega_2 u + \frac{1}{\varrho_0}\frac{\partial p}{\partial z} - b = 0 \quad (3)$$

$$Db + N^2 w = 0 \quad (4)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (5)$$

where $\mathbf{\Omega} \equiv (0, \Omega_2, \Omega_3)$ is the rotation vector, p the difference between the actual and the hydrostatic pressure in an ocean at rest with constant density ϱ_0, $N(z)$ the Brunt-Väisälä frequency and b the fluctuation in buoyancy per unit volume. The operator D is equal to $(\partial/\partial t) + U'(\partial/\partial x) + V'(\partial/\partial y)$ and $'$ indicates the z-derivative.

Elimination of p and b between (1)–(5) leads to

$$D^2(\nabla^2 w) - D\left(U''\frac{\partial}{\partial x} + V''\frac{\partial}{\partial y}\right) w + D\left(\Omega_2\frac{\partial}{\partial y} + \Omega_3\frac{\partial}{\partial z}\right)\omega_3 + N^2\nabla_h^2 w = 0 \quad (6)$$

where ∇_h^2 denotes the horizontal Laplacian. ω_3 is the vertical component of the vorticity and satisfies the equation

$$D\,\omega_3 = (\mathbf{\Omega}\nabla)\, w \quad (7)$$

Equations (6) and (7) can be combined to give

$$D^3(\nabla^2 w) - D^2\left(U''\frac{\partial}{\partial x} + V''\frac{\partial}{\partial y}\right) w + N^2 D(\nabla_h^2 w) + D(\mathbf{\Omega}\nabla)^2 w - \Omega_3(\mathbf{U}'\nabla)(\mathbf{\Omega}\nabla) w = 0 \quad (8)$$

In many situations of interest, the steady shear $\mathbf{U}'(z)$ is nearly constant and the Brunt-Väisälä almost independent of depth. Hence for simplicity we assume that:

$$U'(z) \equiv U' \equiv \text{cons.} \quad (9)$$

$$N(z) \equiv N \equiv \text{cons.} \quad (10)$$

For these hypotheses to be valid the vertical scale of the motion must be rather small in a diffuse thermocline but could be larger in the region below the thermocline where $N(z)$ is roughly constant. If, as is also usually done, we neglect the horizontal component of the earth rotation vector and choose the x-axis in the direction of the mean flow, equation (8) reduces to

$$\frac{d^3}{dt^3}(\nabla^2 w) + \frac{d}{dt}\left(f^2\frac{\partial^2}{\partial z^2} + N^2\nabla_h^2\right) w - U' f^2\frac{\partial^2 w}{\partial x\,\partial z} = 0 \quad (11)$$

with $(d/dt) \equiv (\partial/\partial t) + U(\partial/\partial x)$ and where $f \equiv \Omega_3$ is the usual Coriolis parameter notation.

3. The wave model

In his study of the behaviour of internal gravity waves in non-rotating fluids with spatial variations in the mean flow, Jones (1969)[1] found that the general effect of a deformational mean flow is to shorten wavelengths and to transfer energy towards higher wave-numbers. For a linear mean velocity profile directed along the x-axis he found that under certain conditions, which stated in terms of the present formalism reduce to

$$U' < [N^2 - (n - \mathbf{kU})^2]^{\frac{1}{2}} \quad (12)$$

the horizontal components of the wave-number $\mathbf{k} = (k, l, m)$ are constant, whereas m satisfies

$$m = m]_{t=0} - kU't \quad (13)$$

This result can be interpreted as a direct effect of the geometry: the lines of constant phase are tilted when the oscillating particles are advected at speeds which change with depth. For positive U' the effect of the shear is to rotate the wave-number vector towards the vertical in the clockwise direction. As soon as $\mathbf{k}$ is horizontal, its magnitude increases continually.

[1] Jones' study has been recently extended to a rotating case by Mooers (1970).

In the following the time origin is chosen[1] when the wave-number is horizontal. Consequently the general solution of (11) is of the form

$$w = W \exp\{i(kx + ly - kU'zt)\} \qquad (14)$$

We furthermore assume (Phillips, 1966), as is consistent with the previous assumptions which avoid any depth dependence, that W is independent of z. Substitution of (14) into (11) leads to

$$\frac{\partial^3}{\partial t^3}[k^2 + l^2 + k^2 U'^2 t^2)\, W] + \frac{\partial}{\partial t}[\{(k^2 + l^2)N^2 + f^2 k^2 U'^2 t^2\}\, W] + f^2 k^2 U'^2 t W = 0 \qquad (15)$$

We have replaced U by $U'z$. We introduce the non-dimensional variable

$$T = U't \cos \Phi \qquad (16)$$

where $$\cos \Phi = \frac{k}{(k^2 + l^2)^{\frac{1}{2}}} \qquad (17)$$

and Φ specifies the angle between the horizontal wave-number and the mean current direction. Also we let

$$F = \frac{f}{N} \qquad (18)$$

$$K^2 = \frac{N^2}{U'^2 \cos^2 \Phi} = \frac{\mathrm{Ri}}{\cos^2 \Phi}, \qquad (19)$$

where Ri is the Richardson number corresponding to the mean flow. Equation (15) may now be written

$$K^{-2} \frac{\partial^3}{\partial T^3}[(1 + T^2)\, W] + \frac{\partial}{\partial T}[(1 + F^2 T^2)\, W] + F^2 T W = 0 \qquad (20)$$

4. Asymptotic solution for large K^2

An asymptotic solution to equation (20) can be found when K^2 is large. This case corresponds either to a very large dynamical stability ($Ri \gg 1$) or to a wave-number which is nearly perpendicular to the direction of the mean flow ($\cos \Phi \ll 1$). This was solved by Phillips (1966) in the non-rotating case. The first order oscillatory solution is

$$W \sim G(1 + T^2)^{-\frac{3}{4}} \exp\{\pm i \psi(T)\} \qquad (21)$$

with $$\psi(T) = K \int^T \left(\frac{1 + F^2 T^2}{1 + T^2}\right)^{\frac{1}{2}} dT \qquad (22)$$

where G is an arbitrary constant. Using equation (5), one finds to first order in K^{-1} when only positive frequencies are kept that

$$w \sim G(1 + T^2)^{-\frac{3}{4}} \exp\left\{i\left(kx + ly - \frac{kzT}{\cos \Phi}\right)\right\} \times \exp\{-i \psi(T)\} \qquad (23)$$

$$u \sim G \cos \Phi\, T(1 + T^2)^{-\frac{3}{4}} \exp\left\{i\left(kx + ly - \frac{kzT}{\cos \Phi}\right)\right\} \times \exp\{-i \psi(T)\} \qquad (24)$$

$$v \sim G \frac{l}{k} \cos \Phi\, T(1 + T^2)^{-\frac{3}{4}} \exp\left\{i\left(kx + ly - \frac{kzT}{\cos \Phi}\right)\right\} \times \exp\{-i \psi(T)\} \qquad (25)$$

The frequency[2] n of the wave motion changes continually

$$n = N\left(\frac{1 + F^2 T^2}{1 + T^2}\right)^{\frac{1}{2}} \qquad (26)$$

$$\frac{\partial n}{\partial t} = N U' \cos \Phi\, T(F^2 - 1)\,(1 + T^2)^{\frac{3}{2}}\,(1 + F^2 T^2)^{-\frac{1}{2}} \qquad (27)$$

or $$\frac{\partial n}{\partial t} = -kU' Cg(^3) \qquad (28)$$

where $Cg \equiv \partial n / \partial m$ is the vertical component of the group velocity.

The dispersion relation is not affected by the shear and has the form

$$\frac{N^2 - n^2}{n^2 - f^2} = \frac{m^2}{k^2 + l^2} \qquad (29)$$

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[1] We emphasize that this choice of the time origin just simplifies the writing and does not depend upon the initial conditions; negative times are allowed.

[2] When the horizontal component of the earth rotation vector is taken into account, the internal wave range is enlarged and

$$n = \left(\frac{N^2 + (\Omega_3 T - \Omega_2 \frac{l}{k} \cos \Phi)^2}{1 + T^2}\right)^{\frac{1}{2}}$$

[3] A more general expression for $\partial n / \partial t \theta$ is given by Mooers (1970).

When $\mathbf{k}$ is horizontal, $n = N$ and $\partial n/\partial t = 0$, this is as expected from the assumption of an infinite medium with constant N and from the definition of $N(z)$ as the frequency of small oscillations of a vertical column around its equilibrium position. Combined with the observed fact that the ocean is layered on a very small scale (of the order of a few meters or less), this emphasizes the inadequacy of our model to represent the ocean at very high frequencies. As soon as $\mathbf{k}$ is horizontal, the frequency diminishes constantly and tends asymptotically to the inertial frequency. The time scale of this process is discussed in § 6.

The behavior of the amplitude of the wave velocity is not affected by the rotation and Phillips' conclusions are valid in the whole internal wave range. The mean horizontal, vertical and total kinetic energy densities diminish constantly in time, as soon as $\mathbf{k}$ is horizontal, and have the respective behavior

$$E_h \sim \tfrac{1}{4} G^2 \varrho_0 T^2 (1+T^2)^{-\frac{3}{2}} \tag{30}$$

$$E_v \sim \tfrac{1}{4} G^2 \varrho_0 (1+T^2)^{-\frac{3}{2}} \tag{31}$$

$$E \sim \tfrac{1}{4} G^2 \varrho_0 (1+T^2)^{-\frac{1}{2}} \tag{32}$$

This energy loss is compensated by a small energy gain in the mean flow which is neglected here as U is supposed to be time-independent. It must be emphasized that this process occurs in the absence of dissipation and without consideration of non-linear effect.

Booker & Bretherton (1967) have shown that this gradual transfer of the wave energy to the mean flow is identical to the critical-layer absorption, though the approach is quite different. A critical level is a level at which the fluid velocity equals the phase velocity or, in the rotating case (Jones, 1967), a level at which the Doppler-shifted wave frequency is equal to plus or minus the Coriolis frequency. In their paper, Booker & Bretherton were interested in the upwards and downwards propagation of Lee waves of fixed frequency (in a non-rotating case). They found, as is also obtained here, that, for constant shear and when T is large, the amplitude of the vertical velocity was decreasing as $t^{-3/2}$ whereas the amplitude of the horizontal velocity was decreasing as $t^{-1/2}$. At each frequency corresponds a critical level where most of the momentum of the wave is transferred, but such level does not appear here as our wave components have changing frequency, which corresponds to a continuous frequency spectrum and hence to a continuous distribution of critical level.

As a critical level has no meaning in the present study, another spacial characteristic must be introduced. More adapted to an infinite medium is the vertical scale of the propagation of the wave energy. To define it, we shall use the concept of group velocity, though its signification is not clear here. Nevertheless, it must be remembered that our simple model represents a more elaborated situation, for instance a large wave train propagating in deep water. Calling d_t the vertical distance over which energy at any point runs from a time t-characterizing the frequency and the phase velocity of the wave—to an infinite time—when all energy has been transferred to the mean flow, one has

$$d_t = \int_t^\infty Cg\, dt \tag{33}$$

A typical value allowing for an estimation of the mean vertical scale of the energy transfer—and the validity of the omission of boundary conditions—is d_0 which corresponds to an initially horizontal wave number

$$|d_0| = \frac{N-f}{U'k} \tag{34}$$

A specific example will be discussed later (§ 6).

Phillips (1966) noticed that the effect of the shear was to diminish the vertical scale of the wave and to increase the vertical gradient of the velocity field. More precisely, if we define a minimum Richardson number

$$\mathrm{Ri} = \frac{N^2}{\left[U' + \left(\dfrac{\partial u}{\partial z}\right)_{\max}\right]^2 + \left[\left(\dfrac{\partial v}{\partial z}\right)_{\max}\right]^2} \tag{35}$$

it is seen from (24) and (25) that R_i decreases in time and behaves as t^{-1} for large t. According to Miles (1961), dynamical instability of a steady inviscid shear flow may occur for $R_i \leq \frac{1}{4}$. It appears that a shear instability will occur before the disappearance of the wave even if the steady flow has a large dynamical stability. This gives rise to small scale turbulence.

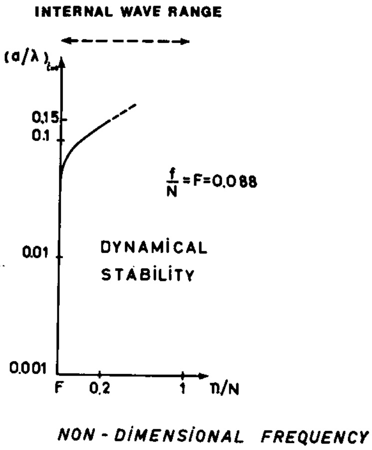

Fig. 1. Curve relating the maximum frequency for shear instability to the ratio a/λ of the wave amplitude to its wave length at $t=0$. The value for f and N are observed at Site D (39°20′ N, 70° W) at a depth of 2,000 m (for simplicity, $1=0$).

A rough estimation of the critical ratio of the wave amplitude to the wave-length can easily be made. (35) may be written, at the critical value $R_i = \frac{1}{4}$, as

$$T \simeq \frac{4N^2 \cos^2 \phi}{k^2 G^2} \tag{36}$$

if

$$T \gg 1 \tag{37}$$

The vertical displacement, ζ, of a fluid particle is, to the first-order in K^{-1}

$$\zeta \sim -\frac{iG}{N}(1+T^2)^{-\frac{1}{4}}(1+F^2T^2)^{-\frac{1}{4}}$$

$$\times \exp\left\{i\left(kx+ly-\frac{kzT}{\cos\Phi}\right)\right\}\exp\{-i\Psi(T)\} \tag{38}$$

If n_i is the frequency under which instability may occur, it is then found

$$\left(\frac{n_i}{N}\right)^2 \simeq F^2 + \frac{\pi^4 a^4}{\lambda^4 \cos^4 \phi} \tag{39}$$

where a is the wave amplitude at $t=0$ and λ its horizontal wave-length. As an example, the critical curve for stability is drawn (Fig. 1) for value corresponding to Site D (see below). Clearly the shear instability will occur very near the inertial frequency (for small values of a/λ) because the critical frequency is lower than indicated by the curve which is valid for an inviscid steady flow.

5. Frequency spectrum

The model used here and first developed by Phillips is clearly a simplified one. As there is no depth dependence and hence no boundary conditions, the waves must fill the whole space and, if this condition is understood literally and not as a mathematical simplification, this can bring in some confusion about the source of the waves. Thus it should be kept in mind that our model is only an oversimplification of some real situation. On the other hand, the use of the form (14), different from the usual normal mode decomposition, has great advantage in the sense that it allows for a variation of the frequency as an effect of the mean shear, leading to a possibility of evaluating the influence of the shear on the frequency spectra.

Phillips (1966) gave a simple way to derive equilibrium frequency spectra by assuming that there was a steady source of internal waves which has been acting long enough for a steady spectrum to develop. This hypothesis, necessary for mathematical rigor, is of course irrealistic, but provides a good way to approximate, in the mean, the real situation. As very little is actually known on the generation of internal waves in the deep ocean, far from boundaries, let us consider a large wave train propagating and entering in a region where the model applies approximately. Then, whatever the initial conditions, the shear will act on the wave and produce the investigated energy transfer, always in a similar way. Hence, except for wave-number nearly vertical (waves are then subject to shear instability), the wave will be rotated and its frequency will go from n to f, or from n to N and then to f, depending upon the orientation of $\mathbf{k}$ relative to $\mathbf{U}$. A given wave will undergo part of the process described here and cover a certain range of frequency. Supposing an isotropy (in the mean) of the directions of propagation of the waves, it is easy to see that, after a time long enough, the frequency spectra due to a large number of different waves will be similar to the frequency spectrum obtained by assuming the existence of an unique steady source. We must nevertheless consider carefully the very high frequencies where, as noticed before, the model is not very realistic. For waves much larger than the oceanic fine structure (say $\lambda > 50$ m), it seems plausible to suppose that, no matter what happens near N, the wave number will be rotated through the horizontal, without final modification. The range of representative results is hence only restricted: the very low and very high frequencies are excluded. In fact, this limitation is of little importance as anyway other motions such as turbulence, internal tides or inertial oscillations are always present, contributing to the frequency ends of the internal wave spectrum.

Assuming that a steady source of internal waves has been acting long enough for a steady spectrum to develop, the frequency spectrum $\Phi(n)$ of the mean square displacement $\overline{\zeta^2}$ can be derived simply because as each wave is rotated, its amplitude and frequency are modified. At any given time, the waves contributing to the frequency band n, $n+dn$ have been generated previously during a time interval dt

$$dn = \frac{\partial n}{\partial t} dt \tag{40}$$

The more slowly the frequency changes in the interval $[n, n+dn]$, the larger will be the frequency spectrum because the source is steady and waves tend to accumulate. The contribution to the total mean squared displacement from waves generated during dt is, at a subsequent time t

$$\overline{d\,\zeta^2} = \overline{\zeta^2}(t)\,dt \tag{41}$$

whereas the contribution from waves with frequency in the range $[n, n+dn]$ is, by the definition of $\Phi(n)$,

$$\overline{d\,\zeta^2} = \Phi(n)\,dn \tag{42}$$

Combining these relations and using

$$\zeta^2 \sim \frac{G^2}{2N^2}(1+T^2)^{-\frac{1}{2}}(1+F^2T^2)^{-1} \tag{43}$$

one gets

$$\Phi(n) \propto (N^2 - n^2)^{-\frac{1}{2}} n^{-1} \tag{44}$$

This result does not depend on the rotation and was found previously by Phillips (1966), except for an algebraic error. The potential energy density spectrum $P_{\zeta\zeta}$ is proportional to $\Phi(n)$ and hence is also of the form (44).

Similarly, the shape of vertical and horizontal kinetic energy density spectra are of the form

$$E_v \propto n(N^2 - n^2)^{-\frac{1}{2}} \tag{45}$$

$$E_h \propto n(N^2 - n^2)^{\frac{1}{2}}(n^2 - f^2)^{-1}. \tag{46}$$

The temperature fluctuation $\Delta\theta$ at a fixed point is approximately given by

$$\Delta\theta = -\frac{\partial T}{\partial z}\zeta \tag{47}$$

where $\partial T/\partial z$ is the local and instantaneous value of the vertical component of temperature gradient. Hence the spectral function of temperature variation is (Fofonoff, 1969).

$$P_{\theta\theta} = \left(\frac{\partial T}{\partial z}\right)^2 P_{\zeta\zeta} \tag{48}$$

which again is of the form (44) since $\partial T/\partial z$ is constant in our model provided that the salinity variation is constant or negligible.

6. Discussion

The results of this simplified model can be expected to be valid where the Brunt-Väisälä frequency and the mean shear are approximately constant, and topography does not play an important role. Consequently the results of theories in which such assumptions are made (Phillips, 1966) cannot be compared to the observations of Haurwitz, Stommel & Munk (1959). These measurements were obtained by recording temperature fluctuations on the bottom at depths of 50 and 500 m within 2–4 km of Bermuda. These depths correspond respectively to the level of the summer and the main thermocline as measured further off-shore and are characterized by large variation of N.

Some measurements which are more suitable for a comparison were obtained below the main thermocline at Site D (39°20′ N, 70° W). This is located in 2,640 m of water, 50 km south of the continental shelf and approximately 175 km north of the mean axis of the Gulf Stream. Consequently the assumption of horizontal homogeneity is valid for scales up to 10 km or so.

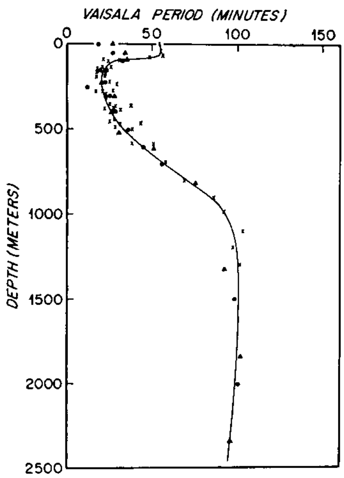

Fig. 2. Vertical profile of the Väisälä period near Site D, after Voorhis (1968).

The smoothed vertical profile of the Brunt-Väisälä period shown in Fig. 2 indicates that N is roughly constant below the main thermocline. A few simultaneous measurements of the mean velocity at different depths are available. The progressive vector diagram (Fig. 3) allows some estimation of the daily mean shear between 1000 and 2000 m to be made. In particular a long period of shear, which was roughly constant and unidirectional, was observed from 67-III-1 to 67-III-12. The shear is $\sim 0(7 \cdot 10^{-5}\ \mathrm{sec}^{-1})$. In order to establish whether this shear can be considered as steady for the time scales we are interested in, we have evaluated the time needed for a wave to be modified, at Site D (2,000 m), for shear of this order. Fig. 4 shows that the energy transfer process occurs in a few days. The vertical scale of the process can be evaluated by formula (34). Using values corresponding to Site D (2000 m) and the estimated shear, one has $|d_0| \simeq 2.16\,\lambda$. For not too large wave lengths, the vertical displacement of the energy is rather small; if $\lambda \sim 0$ (500 m), its maximum value is ~ 0 (2 km). This could partly justify the omission of the boundary conditions (valid for not too large scales) and shows that the transfer process can be completed in a real situation. As, according to a rough calculation,

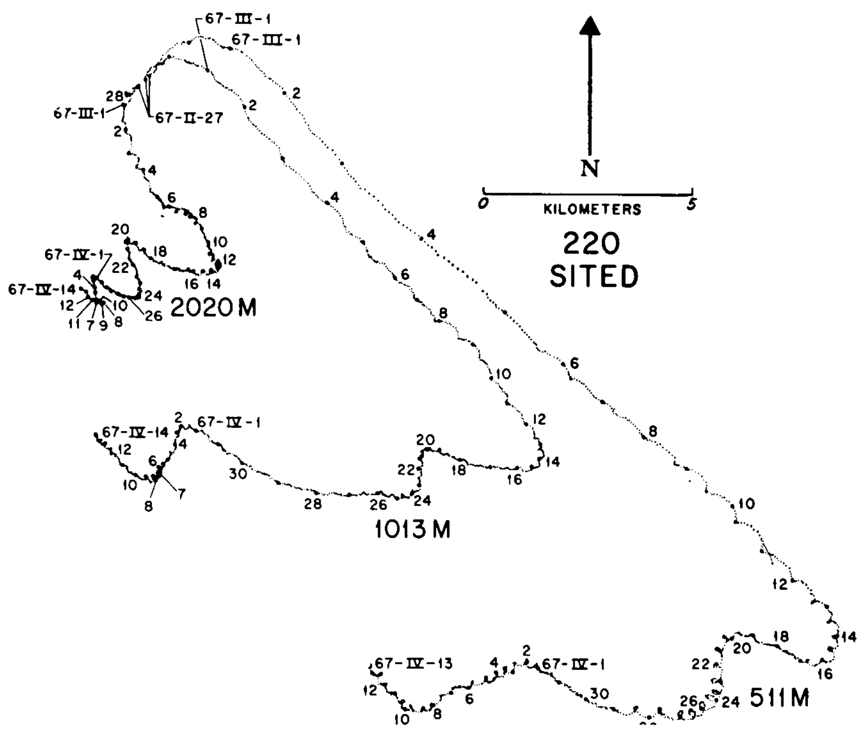

Fig. 3. Progressive vector diagram computed at Site D at three different depths (511, 1013 and 2002 m; 1967) (Unpublished data; courtesy of Dr. F. Webster).

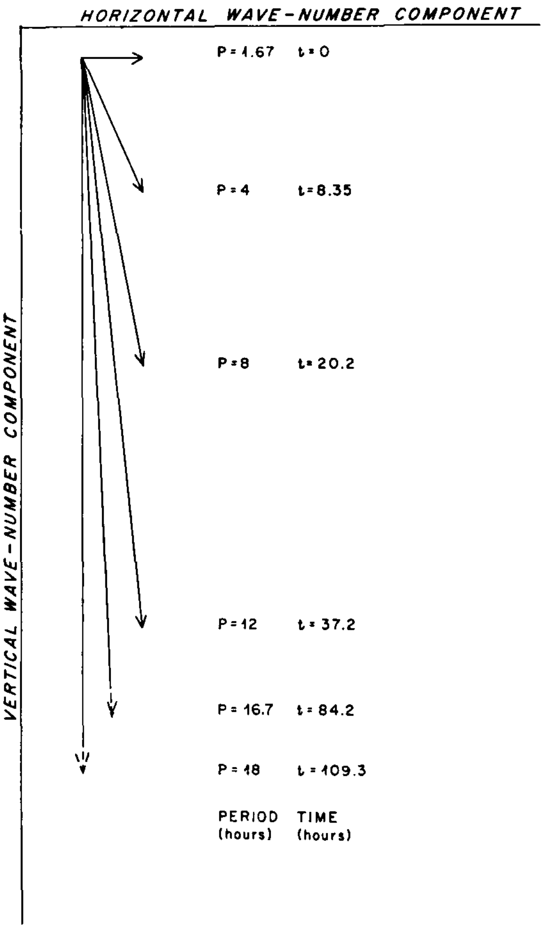

Fig. 4. Evolution of the wave vector $\mathbf{k} = (k, 0, m)$ and of the period P of the waves. Time t indicates the time interval elapsed since $\mathbf{k}$ was horizontal. N and f correspond to Site D (2000 m), $U' \simeq 7.10^{-5}$ sec^{-1}.

the overall dynamical stability is large (this hypothesis is necessary for obtaining analytically the spectra but it is easy to show that the transfer process also occurs for small value of R_i by looking for an asymptotic solution of equation (20), in the case of large T), the conditions at Site D are such that a reasonable comparison can be made between the results of this theory and the observations.

Before comparing the computed spectrum of horizontal kinetic energy with oceanic observations, it should be noticed that the frequency n given by (26) is the intrinsic frequency, or frequency relative to a frame of reference moving with the mean flow, whereas observations of horizontal velocity are computed from data recorded on instrumented moored buoys. The mean drift causes non negligible Doppler shifts in the observed frequencies (given by $\mathbf{kU}$). Nevertheless, if one still hypothesizes the statistical isotropy of the wave propagation directions, all Doppler effects cancel in the mean. This is reasonable so that the computed curve (46) seems representative of observations made during a long enough time. Fig. 5 shows a comparison between the calculated spectrum with observations made at Site D (2000 m). The agreement is very good. The peaks near the

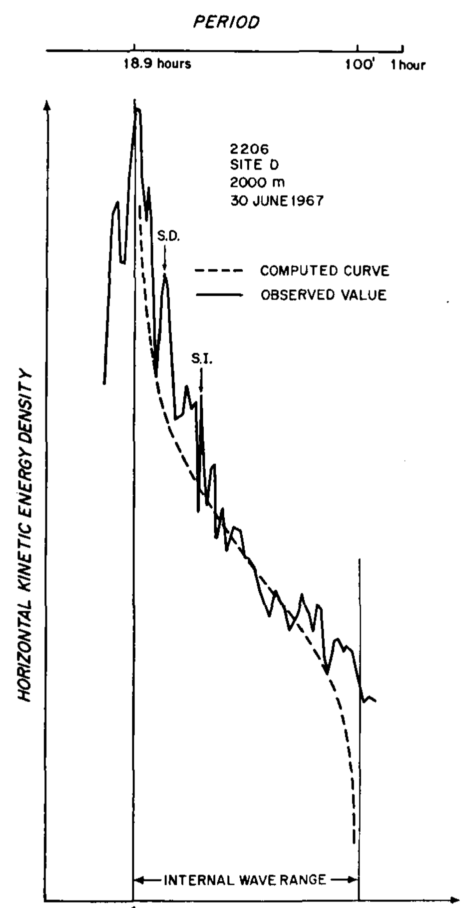

Fig. 5. A comparison of the horizontal kinetic energy density spectrum observed at Site D (*2206*; 2000 m, 30 June 1967) with the calculated spectrum whose arbitrary magnitude has been chosen to fit the observations.

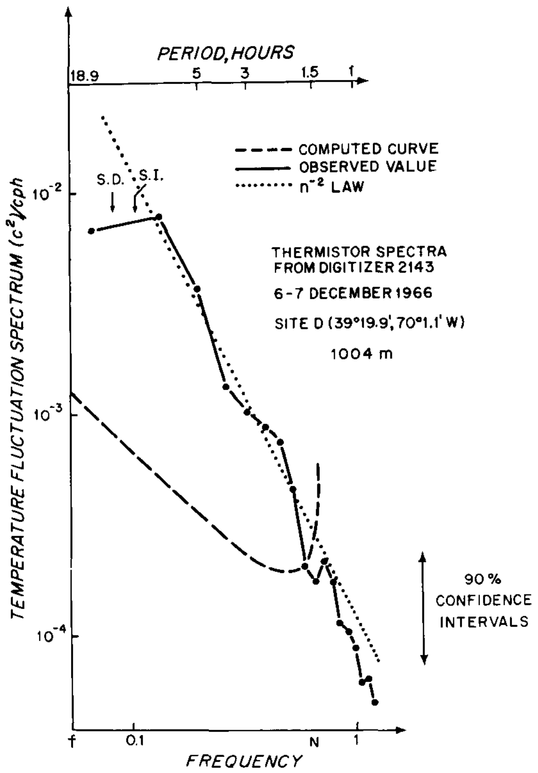

Fig. 6. A comparison of temperature fluctuations recorded at Site D (*2143;* 1004 m, Dec. 1966) with the calculated spectrum whose amplitude has been obtained by fitting corresponding measurements of horizontal kinetic energy (Site D, *2205;* 1013 m, Feb. 1967).

inertial frequency (diurnal, semi-diurnal, inertial, semi-inertial, ...) are associated with different processes. The very high frequency part of the observed curve does not tend to zero, since other motions are present (turbulence, internal waves trapped in thin layers of higher Brunt-Väisälä frequency, ...).

Though they are not really relevant, other observed spectra can be used to check the validity of our model. Measurements of vertical kinetic energy have been made by Voorhis (1968) at Site D. Using neutrally-buoyant rotating floats, Voorhis obtained direct measurements of vertical motions. Unfortunately his deepest measurement is at 960 m which is at the lower edge of the thermocline, in a region in which N is not constant. Also the accuracy is weak so that a comparison is not very representative. Let us only notice that Voorhis' observed peak is at frequency a little lower than the mean N, and that the slope of the low frequency part is rather in agreement with (45).

As discussed in detail by Cox (1966), the determination of the spectrum of water motion from the spectrum of temperature fluctuation is difficult, especially at high frequency. This occurs because only an average temperature gradient can be used in (48), which disregards the layered fine structure which is present in the oceans. This consists of a series of irregular jumps, a few meters or centimeters apart, in the temperature structure. This effect, whose importance cannot be estimated, is neglected in our model. Consequently, good agreement in the spectra of temperature fluctuations cannot be expected. Nevertheless, a comparison has been attempted in Fig. 6 with the deeper temperature observations (1004 m) recorded at Site D. This, again, is in a region where N is not constant. The amplitude of the computed curve was estimated from a comparison between the recorded and calculated horizontal kinetic energy spectra and by using an averaged ratio for $E_h/P_{\theta\theta}$ (Fofonoff, 1969). It should be noted that the agreement between the horizontal kinetic energy spectra at 1013 m was not as good

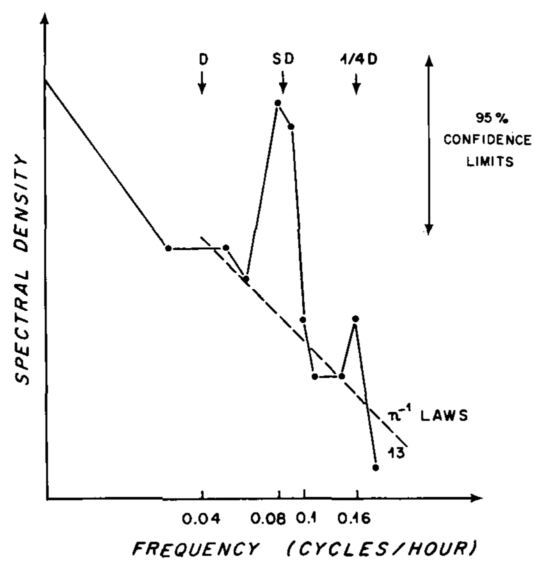

Fig. 7. Comparison of temperature fluctuations recorded by Cox *et al.* (1965) (30°44′ N, 120°40′ W, 1990 m) with the slope of the computed curve. Arrows indicate tidal constituents. The total depth is 3836 m.

as it was at 2000 m. The temperature observations are better fitted by a n^{-2} curve. However, we emphasize that the low frequency part of this spectra is poorly known as a consequence of the very short observational period ($\simeq 40$ hours), and that the experimental error is too large for drawing any conclusion on the presence of a peak near N. Fofonoff (1969) also found that the observed slope was larger than expected from a simple but different internal wave model.

More relevant to our model are the low frequency spectra of temperature fluctuation obtained by Cox *et al.* (1965) in the northeastern Pacific. There is no information about the shear. However, the temperature measurements obtained below the thermocline indicate a better agreement with our n^{-1} law. The spectrum obtained at 1990 m and recorded during 307 hours is outlined[1] in Fig. 7.

7. Conclusions

The simplified model we have examined allows us to make an estimation of the effect of the mean shear on internal waves for special oceanic conditions. Under such conditions, comparison of observed and computed frequency spectra of horizontal kinetic energy indicates a good agreement, except at very high frequencies where the assumption of an infinitely deep medium with uniform stratification is most open to criticism. Hence, the interaction between shear and internal waves could play a predominant role in determining these deep sea spectral shapes. Studies of more general situations are required to evaluate the overall importance of this mechanism. However, this study supports the idea that internal waves are the most important feature of the inertio-gravitational frequency range.

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[1] Detailed data was not available.

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ВЛИЯНИЕ СЛАБОГО ПОПЕРЕЧНОГО ГРАДИЕНТА СКОРОСТИ И ВРАЩЕНИЯ НА ВНУТРЕННИЕ ВОЛНЫ

Изучается влияние постоянного поперечного градиента скорости на внутренние волны в упрощенной океанической модели с постоянной частотой Брэнта–Вяисяля и большой динамической устойчивостью. Как было замечено в случае отсутствия вращения (Филлипс, 1966). Энергия передается среднему течению и частота убывает. Нижняя граница при инерциальной частоте находится на прежнем месте, но при наличии поперечного градиента скорости неустойчивость возможно имеет место до того, как частота достигает этого предела. Выводится равновесный частотный спектр и сравнивается с соответствующими наблюдениями. Наблюдается хорошее согласие со спектром горизонтальной кинетической энергии, полученными ниже термоклина, в то же время при сравнении с другими океаническими данными не наблюдается никакой несогласованности. Делается вывод, что влияние поперечного градиента скорости на внутренние волны может играть важную роль в некоторых областях океана.