Transport of mean zonal kinetic energy in the atmosphere

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ABSTRACT

The transport integrals across the free boundaries of a polar cap region for mean zonal kinetic energy are formulated from physical considerations. These are then incorporated as boundary terms in a zonal kinetic energy balance equation recently presented by V. P. Starr and N. E. Gaut. Various properties of the final equation are then discussed.

Introduction

The material in this paper is in essence a continuation of the discussion of the zonal kinetic energy equation presented recently in this journal by Starr & Gaut (1969). The system treated, the notation and the general methods remain the same. To avoid confusion the numbering of equations and of integrals (the latter enumerated within curly brackets) also remains the same and forms a consecutive extension from the prior paper indicated henceforth as S & G. In the interests of saving space, since the basic assumptions are the same as in the previous work, they are not repeated here. It may be remarked, nevertheless, that the assumption that $\varrho = \varrho(R)$ is unnecessarily restrictive for both the prior and present discussions, the horizontal uniformity of ϱ being required by the treatments only in the zonal direction. This is of some account in thinking of actual atmospheric conditions.

Interest in S & G centers around (17), a form of the mean zonal kinetic energy equation. This equation as a whole expresses the fact that if on the average there exists a dissipation of the kinetic energy of the mean differential rotation of the atmosphere in its entirety, then its thermally induced macroscopic motions must act in the fashion of a predominantly negative eddy viscous action, transporting angular momentum from less rapidly to more rapidly rotating zonal regions. The observational question that arises is which particular terms are of most importance. This we still do not propose to treat presently, reserving that aspect of our efforts for a forthcoming publication. Rather, for the moment, we address our efforts to the problem of the physical interpretation of the boundary integrals that arise and related questions.

In S & G there was presented a version of the more customary form of the zonal kinetic energy equation in (20), for comparison with (17). In making such a comparison it is to be observed that the boundary integrals of the more traditional formulation differ from those appearing in (17). As was stated in S & G it is also possible to write the equation (still for an open region) without any boundary integrals. Furthermore, as will be apparent from what follows, the equation may be written in innumerable forms differing in regard to the boundary terms. It therefore follows that the boundary terms in either (17) or (20) cannot, without further study, be assumed to represent the physical transport of mean zonal kinetic energy across the bounding surfaces of an open region.

The question of specifying a set of boundary integrals which would represent the physical transport of mean zonal kinetic energy across the free boundaries is not capable of unique resolution simply through the mere mathematical manipulation of the energy equation. It requires the consideration of added, purely physical, information. This is what we attempt to do below.

Formulation of transport terms

As is recognized in hydrodynamic theory (see for example, Reynolds 1895) the transport of kinetic energy of a specific kind is accomplished on the one hand through work done by the stress components acting across the boundary surface, and on the other through the advection of the specific form of kinetic energy under consideration by the normal component of velocity across the bounding surface. As defined in S & G the boundary surfaces are stationary and consist of a conical sheet of constant latitude $\phi = \phi_1$, and of a spherical sheet at $R = R_1$, the two defining a symmetrical polar cap region bounded also by a solid surface at the bottom.

We restrict the treatment to the stress caused by macroscopic motions. Across the boundary at ϕ_1, this stress may be written in terms of the flux of absolute zonal momentum as $\varrho vM/R \cos\phi_1$, and analogously as $\varrho wM/R_1 \cos\phi$ for the boundary at R_1. Since such fluxes are physical forces, their magnitudes remain the same in the relative coordinates. We may therefore simply multiply each of these stresses by $[u]$ to compute the work done by them in transferring $[u]^2/2$ kinetic energy in the relative system, and integrate over the areas involved. The advection of this kinetic energy is $\varrho v[u]^2/2$ and $\varrho w[u]^2/2$, in order for the two respective bounding surfaces. We may now write directly that

$$T(\phi_1) = \iint \frac{\varrho vM}{R\cos\phi_1}[u]\, R\cos\phi_1\, d\lambda\, dR + \iint \frac{\varrho v[u]^2}{2} R\cos\phi_1\, d\lambda\, dR \quad (22)$$

$$T(R_1) = \iint \frac{\varrho wM}{R_1\cos\phi}[u]\, R_1^2\cos\phi\, d\lambda\, d\phi + \iint \frac{\varrho w[u]^2}{2} R_1^2\cos\phi\, d\lambda\, d\phi \quad (23)$$

where $T(\phi_1)$ and $T(R_1)$ are the transports of mean zonal kinetic energy across the two bounding sheets in the previously mentioned order.

Incorporation of transports in energy equation

Our next object is to seek a form of the mean zonal kinetic energy equation which would contain the transports $T(\phi_1)$ and $T(R_1)$ as boundary integrals without any others, thus fixing a unique form of this equation. Seemingly the direct method for doing this is to start with the equation of motion (2) of S & G, and to proceed with the manipulations until the desired result is obtained. The way to do just this will be outlined later. However, since it is hard to see in advance just what should be done in pursuing this method successfully, and since other facts not revealed by this course of action should be made apparent, an alternative approach is first followed immediately below.

The terms expressing the work done by the hydrodynamic stresses in $T(\phi_1)$ and $T(R_1)$ may be transformed in various ways. By use of the several definitions and conventions stated in S & G, it is possible to recast them in the symbolism of equation (17) of that paper, or of equation (26) of the present discussion. When this is done, it is immediately clear that the boundary terms of (17) are exactly these stress work terms, but that the advective parts of $T(\phi_1)$ and $T(R_1)$ are missing. These advective parts are added as boundary integrals in (26), which equation therefore embodies boundary effects which represent physically the desired transports of mean zonal kinetic energy.

It is of course necessary to make certain changes in the internal integrals of (17), if the advective parts of $T(\phi_1)$ and $T(R_1)$ are added to the equation as is done in the form $\{A\}$ and $\{B\}$ in (26). Thus we may subtract the same quantities in these places, noting that

$$\int 2\pi\varrho R\cos\phi_1[v]\frac{[u]^2}{2}\, dR \equiv -\iint \frac{\partial}{\partial\phi} 2\pi\varrho R\cos\phi[v]\frac{[u]^2}{2}\, d\phi\, dR \quad (24)$$

$$\int 2\pi\varrho R_1\cos\phi[w]\frac{[u]^2}{2} R_1\, d\phi \equiv \iint \frac{\partial}{\partial R}\left(2\pi\varrho R\cos\phi[w]\frac{[u]^2}{2} R\right) dR\, d\phi \quad (25)$$

Through the utilization of previous methods and of equation (21) of S & G, the double integrals of the identities (24), (25) may be put in the forms $\{a\}$ and $\{b\}$ appearing in (26), after subtraction. Other forms are of course also possible.

Equation (26) is our desired result, containing not only a set of boundary terms which together represent the total flux of mean zonal kinetic energy across these free boundaries, but also the proper corresponding internal integrals.

$$\frac{d}{dt}\int_{\tau}\varrho\frac{[u]^2}{2}\,d\tau = +\iint 2\pi[u][F]R^2\cos\phi\,d\phi\,dR$$

Internal horizontal integrals

$$+\iint 2\pi\varrho R^2\cos^2\phi(\Omega R\cos\phi)[v]\frac{\partial}{R\partial\phi}\times\left\{\frac{[u]}{R\cos\phi}\right\}R\,d\phi\,dR \qquad \{1\}$$

$$+\iint 2\pi\varrho R^2\cos^2\phi[u][v]\frac{\partial}{R\partial\phi}\times\left\{\frac{[u]}{R\cos\phi}\right\}R\,d\phi\,dR \qquad \{2\}$$

$$+\iint 2\pi\varrho R^2\cos^2\phi[u'v']\frac{\partial}{R\partial\phi}\times\left\{\frac{[u]}{R\cos\phi}\right\}R\,d\phi\,dR \qquad \{3\}$$

$$+\iint 2\pi\varrho R\cos\phi[v]\frac{\partial}{R\partial\phi}\left\{\frac{[u]^2}{2}\right\}R\,d\phi\,dR \qquad \{a\}$$

Internal vertical integrals

$$+\iint 2\pi\varrho R^2\cos^2\phi(\Omega R\cos\phi)[w]\frac{\partial}{\partial R}\times\left\{\frac{[u]}{R\cos\phi}\right\}R\,d\phi\,dR \qquad \{4\}$$

$$+\iint 2\pi\varrho R^2\cos^2\phi[u][w]\frac{\partial}{\partial R}\times\left\{\frac{[u]}{R\cos\phi}\right\}R\,d\phi\,dR \qquad \{5\}$$

$$+\iint 2\pi\varrho R^2\cos^2\phi[u'w']\frac{\partial}{\partial R}\times\left\{\frac{[u]}{R\cos\phi}\right\}R\,d\phi\,dR \qquad \{6\}$$

$$+\iint 2\pi\varrho R\cos\phi[w]\frac{\partial}{\partial R}\left\{\frac{[u]^2}{2}\right\}R\,d\phi\,dR \qquad \{b\}$$

Vertical boundary integrals at $\phi = \phi_1$

$$+\int 2\pi\varrho R^2\cos^2\phi_1(\Omega R\cos\phi_1)[v]\frac{[u]}{R\cos\phi_1}\,dR \qquad \{7\}$$

$$+\int 2\pi\varrho R^2\cos^2\phi_1[u][v]\frac{[u]}{R\cos\phi_1}\,dR \qquad \{8\}$$

$$+\int 2\pi\varrho R^2\cos^2\phi_1[u'v']\frac{[u]}{R\cos\phi_1}\,dR \qquad \{9\}$$

$$+\int 2\pi\varrho R\cos\phi_1[v]\frac{[u]^2}{2}\,dR \qquad \{A\}$$

Horizontal boundary integrals at $R = R_1$

$$-\int 2\pi\varrho R_1^2\cos^2\phi(\Omega R_1\cos\phi)[w]\frac{[u]}{R_1\cos\phi}R_1\,d\phi \qquad \{10\}$$

$$-\int 2\pi\varrho R_1^2\cos^2\phi[u][w]\frac{[u]}{R_1\cos\phi}R_1\,d\phi \qquad \{11\}$$

$$-\int 2\pi\varrho R_1^2\cos^2\phi[u'w']\frac{[u]}{R_1\cos\phi}R_1\,d\phi \qquad \{12\}$$

$$-\int 2\pi\varrho R_1\cos\phi[w]\frac{[u]^2}{2}R_1\,d\phi \qquad \{B\}$$

$$\tag{26}$$

Discussion

The following points may be noted in regard to the material presented in the foregoing sections.

(1) From the manner in which the energy equation (26) is here derived from (17) of S & G, it is clear that the added internal integrals $\{a\}$, $\{b\}$ must (in combination) vanish for the entire atmosphere. The conclusions reached that zonal kinetic energy in the atmosphere cannot be generated without countergradient transports of angular momentum is apparent from (26) as well as from (17); however this energy may be transported from place to place internally.

(2) From the manner in which (26) was derived from (17) via the identities (24) and (25), it follows that innumerable other forms of the equation may be formulated by changing the integrands in (22) and (23). Although through this purely mathematical process results without physical interest may also be gotten, the possibility of obtaining new, as yet unknown, forms of the equation showing new important physical concepts also exists. These remarks apply equally to equation (20) of S & G, if it is desired to use it as a starting point.

(3) The method used for obtaining (26) from (17) is illuminating in certain respects. It illustrates how boundary integrals may be canceled by suitable portions of the internal integrals. Nevertheless the procedure, although rigorous, may seem somewhat artificial to certain readers who might prefer a derivation of (26) directly from the initial basic equations of S & G. This may indeed be done, although the entire process will not be carried through here. The key to the manipulations is to obtain an expression for $\varrho d/dt$ $([u]^2/2)$ from the equation

of motion (2). The expansion of this derivative with the aid of the continuity equation (5) will ultimately yield the advective boundary terms lacking in (17).

(4) It is instructive, and something of a check of the work, to consider the special case of an atmosphere in absolute rest, thus giving relative easterlies everywhere increasing with R. Since the relative angular velocity, $[u]/R \cos \phi$, is then uniform, all the terms in (26) containing derivatives of this quantity will vanish. Since the absolute angular momentum M per unit mass is identically zero, all the work stress terms at the boundaries will also vanish. Since $[v]$ and $[w]$ are zero, the integrals $\{a\}$, $\{b\}$, $\{A\}$ and $\{B\}$ will likewise vanish. The only possible way to change the mean zonal kinetic energy would then be a spin-up effect due to surface friction, because $[u]$ is everywhere negative.

(5) If all the special conditions stipulated in discussion (4) are retained, except that $[v]$ and $[w]$ are not supposed to be zero, then each of the transports $\{A\}$ and $\{B\}$ may not be zero. In that case their contributions, however real they may be, are exactly canceled by the internal integrals $\{a\}$ and $\{b\}$, as is clear from the mode of derivation given for the latter. (In this special case $\{7\}$ cancels $\{8\}$, and $\{10\}$ cancels $\{11\}$).

(6) In the discussion (5) the advective transports were considered non-zero, while the stress work transports were zero. The opposite case may also be conceived. If it is postulated that $[u]$ and $[w]$ are everywhere zero, but that motions are not restricted otherwise, then the stress-work transports $\{9\}$ and $\{12\}$ need not vanish while the advective transports will be zero. In such a case the internal integrals $\{3\}$ and $\{6\}$ may not vanish, but there is now no implied necessity for them to cancel the remaining boundary work terms.

(7) It is of interest to note that, save for a factor of one half, the advective transport $\{A\}$ is of the same mathematical form as the stress-work transport $\{8\}$, and could therefore be combined with it. The same applies to $\{B\}$ and $\{11\}$. The implication is thus given that if one effect is present, so must also be the other. However, from a physical point of view the two actions are distinct, as is evident from the method which was used to set up the transport integrals.

(8) Formally at least, equation (26) may be considered to include all macroscopic motions. Molecular motion effects are symbolically included in the term containing F. This term could be expanded in Navier–Stokes form to give added molecular stress-work terms at the boundaries and added dissipative internal integrals.

(9) The utilization of physically formulated specific boundary integrals in (26) lends some weight to the designation of the internal integrals as generation terms, not only for the whole atmosphere but also for a polar cap having free boundaries. However, these terms may be combined and arranged in many different ways.

(10) We may note that a term of the type $\{A\}$ is included in the energy equation by Obasi (1965). His corresponding internal integral is written in an alternative possible form as compared with $\{a\}$.

(11) Equation (26) may be generalized for other fluid systems in addition to the earth's atmosphere. Since no assumption is made as to the radial thickness of the fluid stratum considered, the differential rotation of deep shells, let us say of stellar interiors, could therefore be treated. However in the case of celestial bodies such as the sun, magnetic and other effects might be included giving, among added actions, such things as Maxwell stress-work transports across boundaries (see Starr 1968 and references there cited)

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ПЕРЕНОС СРЕДНЕЙ ЗОНАЛЬНОЙ КИНЕТИЧЕСКОЙ ЭНЕРГИИ В АТМОСФЕРЕ

Интегралы переноса средней зональной кинетической энергии через свободные поверхности в области полярной шапки формулируются из физических соображений. Затем они вставляются в качестве граничных членов в уравнение баланса зональной кинетической энергии, недавно полученное Старром и Готом. Далее обсуждаются различные свойства получившегося уравнения.