Cnoidal waves in a compressible fluid over a toroidal channel

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ABSTRACT

In this paper we have discussed the problem of cnoidal waves at the free surface of a compressible fluid in a toroidal channel. We have utilised the technique of expanding the dependent variables in terms of a parameter, to analyse the problem. The expression for the cnoidal wave profile has been obtained, and also some particular cases have been discussed at the end. The accuracy of the results presented here has been established by correlating them with those of Shen (1967).

1. Introduction

Cnoidal waves are non-linear periodic waves of permanent type which were first observed by Korteweg & De Vries (1895). Its profile can be represented by the square of Cn-Jacobian elliptic function. It is known that Cnoidal waves, like solitary waves, occur in flows with speeds in the neighbourhood of the critical speed (i.e. Froude number near unity), Stoker (1957). In fact, Cnoidal waves for small departures of the flow speed from the critical speed have large wave lengths, and in the limit as the wave length tends to infinity, they tend to well known solitary waves. It is known that for cnoidal waves the depth of the trough below the mean height is less than the height of the crest above the mean height, and the crest is narrower than the trough. Thus cnoidal waves are unsymmetric about the mean height; but they are symmetric about a vertical line due to the permanency of the form of the wave.

Modifying the method of Rayleigh (1876), Korteweg & De Vries (1895) succeeded in establishing the existence of cnoidal waves. Levi-Civita (1925) presented a formal mathematical proof of the existence of cnoidal waves when the depth of the fluid is infinite. Later a proof of the existence of these waves, based on the variational principle was given by LaVrientiev (1947). Keller (1948), applying the technique of Friedrichs (1948) to the non-linear shallow-water theory, gave an approximate description of the cnoidal waves. A rigorous proof of the two-dimensional cnoidal waves was presented by Littman (1957) by adopting a method similar to that of Friedrichs & Hyers (1954). Hunt (1953) studied two-dimensional rotational cnoidal waves. Benjamin & Lighthill (1954) discussed cnoidal waves in connection with the formation of Hydraulic jumps and bores. A study of cnoidal waves was also made by Iwasa (1955), and Keulegan & Patterson (1940). Laiton (1960) while studying higher order approximations of solitary and cnoidal waves, pointed out that the pressure is no longer hydrostatic and the vertical motions are no longer negligible in the higher order approximations. A presentation of cnoidal wave theory for the ready use of the engineers was made by Wiegel (1960). Benny (1966) indicated the possibility of the existence of cnoidal and solitary waves in a thin viscous film flowing down an inclined plane.

There has been growing interest in the waves of permanent type in stratified media, with the belief that they may play an important role in certain meteorological and other phenomena where density changes are important. It has been pointed out by Abdullah (1955) that the so-called bubbles observed by Fawbush & Miller (1954) in connection with the formation of tornados, have many similarities with the atmospheric solitary waves (see also Tepper (1954). Peters & Stoker (1960) have pointed out the existence of cnoidal waves in media where density variations are present, while Ter-Krikorov (1963) has studied propagation of permanent waves in heterogenous fluids.

To gain an insight into the non-linear waves in a gaseous or liquid ocean confined to a channel parallel to the latitude of a sphere, Peters (1966) has studied cnoidal waves in a toroidal channel. He has discussed three-dimensional aspects of these waves and has arrived at similar conclusions to those for a three dimensional solitary wave in a straight channel (Peters, 1966). Shen (1967) has discussed cnoidal waves while treating waves of permanent type in a rotating fluid with a free surface and supported by a cylinder. He has treated two cases of geophysical importance, in the first the fluid is outside the cylinder and in the other the fluid is inside the cylinder. Recently Shen (1968) has presented a new method of analysis for the study of the steady long waves in a stratified medium in a straight channel of arbitrary cross section. It has been observed, that in addition to density stratification there are other factors which affect the formation of internal waves. Here Shen (1967) has taken into consideration the current speed and bottom topography of the channel which seem to be more important than others. He has derived the speed of the cnoidal waves for a given distribution of density, shear and also the cross section of the channel.

We have studied here cnoidal wave propagation at the free surface of a compressible fluid in a toroidal channel of arbitrary cross section. This is to bring out the effect of density variation and bottom topography of the channel on the propagation of cnoidal waves. For the sake of simplicity we have assumed a specific relation between the density and pressure of the fluid. To analyse the problem, we have developed the dependent variables in the power series of a small parameter. The expressions for the critical speed and the cnoidal wave profile have been obtained for general case (sect. 3). Section 4 is devoted to the discussion of the propagation of cnoidal waves in a straight channel of arbitrary cross section, as a limiting case of the results obtained in section 3. The particular case of a toroidal channel with a rectangular cross section has also been discussed in section 5. The main results of this paper are precisely stated in section 6.

2. Formulation

Consider an inviscid compressible fluid confined over a toroidal channel by a gravitational force g acting perpendicular to the axis of the channel. We follow the notation of Peters (1960), for convenience, and introduce cylindrical coordinates (r, θ, x_3) where θ is measured in the anticlockwise direction. The inner wall of the toroidal channel considered, can be generated by rotating a piecewise smooth curve $r = a + y$ about the x_3-axis, and r is in the radial direction. Here a is the distance of the equilibrium free surface from the x_3-axis. We assume that the pressure on the free surface of the compressible fluid is constant, say G, and there are no geometric constraints on this surface. Introduce a stretching parameter ε, and a typical length h, to nondimentionalise the variables, see reference Raman (1960). The cylindrical coordinates in the dimensionless form are ϱ, σ and ζ. Now the free surface in the perturbed state is represented by $\varrho = \alpha + \eta$ and in the unperturbed state by $\varrho = \alpha/\lambda$. In terms of the dimensionless quantities, the equations governing the flow are:

$$\frac{\varepsilon}{\varrho^2}\frac{\partial}{\partial\sigma}(\phi_\sigma) + \frac{\partial}{\partial\varrho}(\delta\varphi_\varrho) + \frac{\partial}{\partial\zeta}(\delta\varphi_\zeta) + \frac{\delta\varphi_\varrho}{\varrho} = 0, \quad (2.1)$$

$$\varepsilon\left\{\frac{\varphi_\sigma^2}{2\varrho^2} + \varrho + \int\frac{d\pi}{\delta}\right\} + \frac{1}{2}(\varphi_\varrho^2 + \varphi_\zeta^2) = C, \quad (2.2)$$

where C is a constant.

The necessary boundary conditions are: the Kinematic condition at the free surface $\varrho = \alpha + \eta(\sigma, \zeta)$

$$\varepsilon\frac{\varphi_\sigma\eta_\sigma}{\varrho^2} = \varphi_\varrho - \varphi_\zeta\eta_\zeta, \quad (2.3)$$

and at the channel wall $\varrho = \alpha + y(\zeta)$

$$\phi_\varrho = \phi_\zeta y_\zeta. \quad (2.4)$$

In the above equations subscripts denote partial differentiation, π and δ are the pressure and density of the fluid respectively. Here equation (2.1) is the equation of continuity, (2.2) is the energy equation, and (2.4) ensures that the velocity normal to the channel wall, $\varrho = \alpha + y(\zeta)$, is zero. It is assumed in the derivation of the above equations that the initial flow is irrotational and also the flow is homoentropic. We note that in the absence of dissipative mechanism the flow continues to be irrotational.

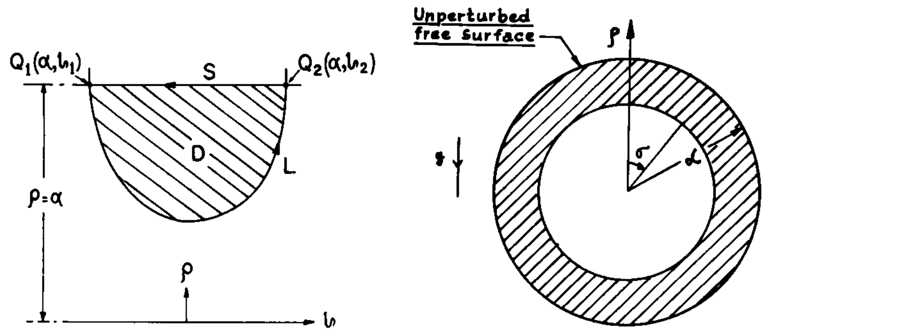

Fig. 1. (*a*) Upper part of the cross section of the channel in the plane of ϱ and ζ in the unperturbed state. (*b*) Cross section of the channel in the unperturbed state in a plane perpendicular to the axis (ζ) of the channel.

For simplicity we consider a barotropic fluid, for which the pressure is a function of density alone. Let the pressure density relation of the compressible fluid be

$$\pi = A\delta^{\gamma}, \tag{2.5}$$

where A is an absolute constant and γ is adiabatic exponent. ($\gamma = cp/cv$, cp and cv being the specific heats at constant pressure and volume respectively.) Now considering the equation (2.2) at the free surface, where the pressure is G, we have

$$\varepsilon\left\{\frac{\varphi_\sigma^2}{2(\alpha+\eta)^2} + \alpha + \eta + \frac{G\gamma}{\gamma-1}\frac{1}{\delta}\right\} + \frac{1}{2}(\varphi_\varrho^2 + \varphi_\zeta^2) = C \quad \text{at } \varrho = \alpha + \eta \tag{2.6}$$

3. Analysis

To analyse the problem formulated in the earlier section, we expand the dependent variables in the form

$$\left.\begin{aligned} &\varphi = \varphi_0 + \varepsilon\varphi_1 + \varepsilon^2\varphi_2 + \cdots, \\ &\pi = \pi_0 + \varepsilon\pi_1 + \pi_2\varepsilon^2 + \cdots, \\ &\delta = \delta_0 + \varepsilon\delta_1 + \varepsilon^2\delta_2 \cdots, \\ &\eta = \varepsilon\eta_1 + \varepsilon^2\eta_2 + \cdots, \\ &\text{and} \\ &C = C_0 + \varepsilon C_1 + \varepsilon^2 C_2 + \cdots, \end{aligned}\right\} \tag{3.1}$$

and substitute (3.1) in the equations of the earlier section, and equate the coefficients of like powers of $\varepsilon\lambda$ to get sets of equations which we would solve.

After substituting (3.1) in the equations of section 2, we get to the zeroth order of approximation

$$\frac{\partial}{\partial\varrho}(\delta_0\varphi_{0\varrho}) + \frac{\partial}{\partial\zeta}(\delta_0\varphi_{0\zeta}) + \frac{\delta_0\varphi_{0\varrho}}{\varrho} = 0, \quad \text{in } D, \tag{3.2}$$

$$\left.\begin{aligned} &\tfrac{1}{2}(\varphi_{0\varrho}^2 + \varphi_{0\zeta}^2) = C_0, \\ &\varphi_{0\varrho}(\alpha, \sigma, \zeta = 0 \end{aligned}\right\} \quad \text{on } S, \tag{3.3}$$

$$\phi_{0n} = 0 \quad \text{on } L, \tag{3.4}$$

with

$$\pi_0 = A\delta_0^{\gamma}, \tag{3.5}$$

where subscript n denotes the normal derivative. Here D is the region bounded by the curve L ($\varrho = \alpha + y$), and S, the part of the line $\varrho = \alpha$ included between the points Q_1 and Q_2 (Fig. 1*a* and *b*). It is assumed that the curve L is a piecewise smooth curve and meets the line $\varrho = \alpha$ in precisely two points $Q_1(\alpha, \zeta_1)$ and $Q_2(\alpha, \zeta_2)$ such that $\zeta_2 - \zeta_1 = b > 0$, b being the breadth of the equilibrium free surface. The last equation of (3.3) and (3.4) show that the normal derivative of ϕ_0 is zero throughout on the boundary of D. This with equation (3.2) shows thet ϕ_0 is a function of σ alone, say $\phi_0 = F_0(\sigma)$, where F_0 is an arbitrary function of σ. Now the first equation of (3.3) gives that $C_0 = 0$

Considering the equations of the first order of approximation, we have

$$\frac{1}{\varrho^2}\frac{\partial}{\partial\sigma}(\varphi_{0\sigma}) + \frac{1}{\varrho}\frac{\partial}{\partial\varrho}(\varrho^2\delta_0\varphi_{1\varrho}) + \frac{\partial}{\partial\zeta}(\delta_0\varphi_{1\zeta}) = 0 \quad \text{in } D, \tag{3.6}$$

$$\frac{\varphi_0^2\sigma}{2\alpha^2} + \alpha + \frac{G\gamma}{\gamma-1}\frac{1}{\delta_0(\alpha)} = C_1 \quad \text{on } S, \tag{3.7}$$

$$\left.\begin{array}{ll}\varphi_{1\varrho}(\alpha,\sigma,\zeta)=0 & \text{on } S,\\ \phi_{1n}=0 & \text{on } L,\end{array}\right\} \tag{3.8}$$

and

$$\pi_1=\delta_1\frac{d\pi_0}{d\delta_0} \tag{3.9}$$

Also equation (2.2) gives to the first order of approximation.

$$\frac{\varphi_{0\sigma}^2}{2\varrho^2}+\varrho+\frac{1}{\gamma-1}\frac{d\pi_0}{d\delta_0}=C_1 \tag{3.10}$$

Substituting for C_1 from (3.7) in (3.10) we have

$$\delta_0^{\gamma-1}(\varrho)=\frac{\gamma-1}{A\gamma}\left\{t+\alpha-\varrho+\frac{\varphi_{0\sigma}^2}{2\alpha^2}\left(1-\frac{\alpha^2}{\varrho^2}\right)\right\}, \tag{3.11}$$

Where $t=G\gamma/(\gamma-1)\,\delta_0(\alpha)$.

We see from (3.7) that F_0' is a constant and for φ_1 we have the set of equations,

$$\left.\begin{array}{ll}\dfrac{\partial}{\partial\varrho}(\varrho\delta_0\phi_{1\varrho})+\dfrac{\partial}{\partial\zeta}(\varrho\delta_0\varphi_{1\zeta})=0 & \text{in } D,\\ \varphi_{1\varrho}(\alpha,\sigma,\zeta)=0 & \text{on } S,\\ \varphi_{1n}=0 & \text{on } L.\end{array}\right\} \tag{3.12}$$

The solution of the set (3.12) can easily be written as

$$\phi_1(\sigma)=F_1(\sigma) \tag{3.13}$$

where F_1 is an arbitrary function of σ to be determined.

The equations to the second order of approximation are:

$$\frac{1}{\varrho}(\delta_{1\sigma}\varphi_{0\sigma}+\delta_0\varphi_{1\sigma\sigma})+\frac{\partial}{\partial\varrho}(\varrho\delta_0\varphi_{2\varrho})+\frac{\partial}{\partial\zeta}(\delta_0\varrho\varphi_{2\zeta})=0 \quad \text{in } D, \tag{3.14}$$

$$\delta_0(\alpha)\left\{\eta_1+\frac{\varphi_{0\sigma}\phi_{1\sigma}}{\alpha^2}-\frac{\varphi_{0\sigma}^2}{\alpha^3}\eta_1\right\}=C_2\delta_0(\alpha)+t\left\{\eta_1\frac{d\delta_0}{d\varrho}(\alpha)+\delta_1(\alpha,\sigma,\zeta)\right\} \quad \text{on } S, \tag{3.15}$$

$$\left.\begin{array}{ll}\varphi_{2\varrho}(\alpha,\sigma,\zeta)=\dfrac{\varphi_{0\sigma}}{\alpha^2}\eta_{1\sigma} & \text{on } S,\\ \varphi_{2n}=0 & \text{on } L.\end{array}\right\} \tag{3.16}$$

And equation (2.2) gives to the second order of approximation.

$$\delta_1\delta_0^{\gamma-2}A\gamma+\frac{\varphi_{0\sigma}\varphi_{1\sigma}}{\varrho^2}=C_2 \tag{3.17}$$

Substituting δ_1 from (3.17) in (3.15) we get

$$C_2=\frac{\varphi_{1\sigma}\varphi_{0\sigma}}{\alpha^2}+\eta_1\frac{t_3\alpha^3-F_0'^2}{\alpha^3(1+t_1)}. \tag{3.18}$$

where $t_1=t\dfrac{d\delta_0}{d\pi_0}(\alpha)$ and $t_3=1-t\dfrac{d\delta_0}{d\varrho}(\alpha)\dfrac{1}{\delta_0(\alpha)}$.

Now first equation (3.16) with (3.18) gives

$$\varphi_{2\varrho}(\alpha,\sigma,\zeta)=-\frac{R_0^2F_1''(\sigma)\,t_2}{\alpha(\alpha^3gh\,t_3-R_0^2)}, \tag{3.19}$$

where

$$t_2=1+t\frac{d\delta_0}{d\pi_0}(\alpha),\ R_0^2=F_0'^2gh.$$

Here dashes denote differentiation with respect to σ.

From (3.17), (3.19), and (3.14), (3.15) we have for φ_2, the set of equations

$$\begin{aligned}&\frac{\partial}{\partial\varrho}(\varrho\delta_0\varphi_{2\varrho})+\frac{\partial}{\partial\zeta}(\delta_0\varrho\varphi_{2\zeta})=-\frac{F_1''(\sigma)\delta_0}{\varrho}\times\left(1-\frac{F_0'^2}{\varrho^2}\frac{d\delta_0}{d\pi_0}\right) \quad \text{in } D,\\ &\varphi_{2\varrho}(\alpha,\sigma,\zeta)=-\frac{R_0^2F_1''t_2}{\alpha(\alpha^3ght_3-R_0^2)} \quad \text{on } S,\\ &\varphi_{2n}=0 \quad \text{on } L\end{aligned} \tag{3.20}$$

Applying divergence theorem

$$\iint_D\left\{\frac{\partial}{\partial\zeta}(\varrho\chi_\varrho)+\frac{\partial}{\partial\varrho}(\varrho\zeta)\right\}d\zeta d\varrho=\int_{L+S}\varrho\chi_n\,dD, \tag{3.21}$$

where x_n is the outward normal derivative of χ at the boundary of D, and ds is the line element, to (3.20) we get

$$F_1''(\sigma)\left[\iint_D\frac{\delta_0}{\varrho}\left(\frac{d\delta_0}{d\pi_0}\frac{F_0'^2}{\varrho^2}-1\right)d\zeta d\varrho+\frac{t_2\delta_0(\alpha)R_0^2b}{\alpha^3ght_3-R_0^2}\right]=0 \tag{3.22}$$

Equation (3.22) shows a bifurcation phenomenon. Since the choice $F_1''(\sigma)=0$ leads to a uniform flow in which we are not interested, we have the other alternative

$$\iint_D \frac{\delta_0}{\varrho}\left(\frac{d\delta_0}{d\pi_0}\frac{F_0'^2}{\varrho^2}-1\right)d\zeta d\varrho + \frac{t_2\,\delta_0(\alpha)\,F_0'^2\,b}{\alpha^3\,t_3-F_0'^2}=0, \tag{3.23}$$

which is an implicit equation for the determination of F_0'. Writing

$$\varphi_2(\varrho,\sigma,\zeta) = -\frac{F_1''(\sigma)\,\Psi(\varrho,\zeta)}{\alpha^2}+F_2(\sigma), \tag{3.24}$$

where F_2 is an arbitrary function of σ, we have Ψ as the solution of the system:

$$\begin{aligned}(\varrho\,\delta_0\,\Psi_\varrho)_\varrho + (\varrho\,\delta_0\,\Psi_\zeta)_\zeta &= \frac{\delta_0}{\varrho}\alpha^2\left(1-\frac{F_0'^2}{\varrho^2}\frac{d\delta_0}{d\pi_0}\right) && \text{in } D,\\ \Psi_\varrho(\alpha,\zeta) &= \alpha\,\mu\,t_2 && \text{on } S,\\ \Psi_n &= 0 && \text{on } L,\end{aligned} \tag{3.25}$$

also we assume that $\Psi(\alpha, 0) = 0$ to make Ψ unique, and here $\mu = R_0^2/(\alpha^3 ght_3 - R_0^2)$.

Now from (3.17) and (3.18)

$$\left.\begin{aligned}F_1'(\sigma) &= -\frac{F_0'\,\eta_1}{\alpha\mu t_2}+\frac{C_2\,\alpha^2}{F_0'},\\ \delta_1(\sigma,\varrho) &= t_4(\varrho)\left\{\frac{F_0'^2}{\varrho^2}\frac{\eta_1}{\alpha\mu t_2}+C_2\left(1-\frac{\alpha^2}{\varrho^2}\right)\right\},\end{aligned}\right\} \tag{3.26}$$

where $t_4(\varrho) = \delta_0(d\delta_0/d\pi_0)$.

This equation shows that η_1 is a function of σ alone. Determination of $F_1(\sigma)$, which determines all the flow variables to the first order of approximation, requires the consideration of the equations of the next higher order of approximation. The equations to the third order of approximation are:

$$\frac{1}{\varrho^2}\frac{\partial}{\partial\sigma}\{\delta_0\,\varphi_{2\sigma}+\delta_2\,\varphi_{0\sigma}+\delta_1\,\varphi_{1\sigma}\}+\frac{\partial}{\partial\varrho}(\delta_0\,\varphi_{3\varrho}+\delta_1\,\varphi_{2\varrho}) + \frac{\partial}{\partial\zeta}(\delta_0\,\varphi_{3\zeta}+\delta_1\,\varphi_{2\zeta})+\frac{\delta_0\,\varphi_{3\varrho}+\delta_1\,\varphi_{2\varrho}}{\varrho}=0 \quad \text{in } D, \tag{3.27}$$

$$\begin{aligned}&\delta_0(\alpha)\left[\eta_2+\frac{\varphi_{0\sigma}\,\varphi_{2\sigma}}{\alpha^2}-\frac{2\varphi_{0\sigma}\,\varphi_{1\sigma}}{\alpha^3}\eta_1+\frac{\varphi_{1\sigma}^2}{2\alpha^2}-\frac{\varphi_{0\sigma}^2}{\alpha^3}\eta_2+\frac{\eta_1^2}{2}\frac{3\,\varphi_{0\sigma}^2}{\alpha^4}\right]+\frac{t}{\delta_0(\alpha)}\left\{\eta_1\frac{d\delta_0}{d\varrho}(\alpha)+\delta_1(\alpha,\sigma)\right\}^2\\ &= t\left[\delta_2(\alpha,\sigma,\zeta)+\eta_1\frac{\partial\delta_1}{\partial\varrho}(\alpha,\sigma)+\eta_2\frac{d\delta_0}{d\varrho}(\alpha)+\frac{\eta_1^2}{2}\frac{d^2\delta_0}{d\varrho^2}\right]+C_2\,\delta_0(\alpha) \quad \text{on } S,\end{aligned} \tag{3.28}$$

$$\left.\begin{aligned}&\varphi_{3\varrho}(\alpha,\sigma,\zeta)+\eta_1\,\varphi_{2\varrho\varrho}(\alpha,\sigma,\zeta)\\ &\quad=\frac{\varphi_{0\sigma}}{\alpha^2}\eta_{2\sigma}+\frac{\eta_1\,\varphi_{1\sigma}}{\alpha^2}-\frac{2\,\eta_1\,\eta_{1\sigma}\,\varphi_{0\sigma}}{\alpha^3} \quad \text{on } S,\\ &\varphi_{3n}=0 \quad \text{on } L\end{aligned}\right\} \tag{3.29}$$

The third approximation of (2.2) gives

$$\delta_2=\left\{C_3-\frac{\varphi_{1\sigma}^2}{2\varrho^2}-\frac{\varphi_{0\sigma}\,\varphi_{2\sigma}}{\varrho^2}-\frac{1}{2}\delta_1^2\,\delta_0^{\gamma-3}\,A\gamma(\gamma-2)\right\}\quad t_4(\varrho) \tag{3.30}$$

Differentiating (3.28) with respect to σ and substituting for δ_2, δ_1, and F_1' from (3.30) and (3.26) we get

$$\eta_{2\sigma}+\frac{\eta_1'''\,\Psi(\alpha,\zeta)}{\alpha^2}+\eta_1\,\eta_1'\,t_5+\frac{F_2''\,\alpha\mu t_2}{F_0'} - \frac{\eta_1'\,C_2\,\alpha^2}{F_0'^2}(1+2\,\mu t_2)=0 \tag{3.31}$$

where

$$\begin{aligned}t_5=\frac{1}{\alpha\mu t_2^2}\Bigg[&(\gamma-2)\frac{F_0'^2}{\alpha^2}\frac{\delta_{0(\alpha)}^{-2\gamma+3}}{A^2\,\gamma^2}-\frac{d^2\,\delta_0(\alpha)}{d\varrho^2}\frac{\alpha^4\,\mu^2\,t_2^2}{F_0'^2}\frac{t}{\delta_0(\alpha)}\\ &+4\mu\,t_2+t_2+3\mu^2\,t_2^2+\frac{\alpha^4\,\mu^2\,t_2^2}{F_0'^2}\frac{2t}{\delta_0^2(\alpha)}\\ &\times\left(\frac{d\delta_0}{d\varrho}(\alpha)+\frac{t_4(\alpha)\,F_0'^2}{\alpha^3\,\mu t_2}\right)^2-\frac{2\mu\alpha t_2 t}{\delta_0(\alpha)}\\ &\times\left(\frac{dt_4(\alpha)}{d\varrho}-\frac{2t_4(\alpha)}{\alpha}\right)\Bigg]\end{aligned}$$

Substituting for $\eta_{2\sigma}$ from (3.31) in the first equation of (3.29) we have

$$\begin{aligned}\varphi_{3\varrho}(\alpha,\sigma,\zeta)&=-\frac{\Psi(\alpha,\zeta)\,F_0'}{\alpha^4}\eta_1'''-\frac{F_2''\,\mu t_2}{\alpha}+\frac{2\eta_1'\,C_2}{F_0'}(1+\mu t_2)\\ &\quad-\eta_1\,\eta_1'\left\{\frac{t_5\,F_0'}{\alpha^2}+\Psi_{\varrho\varrho}(\alpha,\zeta)\frac{F_0'}{\alpha^3\,\mu t_2}+\frac{F_0'}{\alpha^3\,\mu t_2}+\frac{2F_0'}{\alpha^3}\right\} \quad \text{on } S.\end{aligned} \tag{3.32}$$

Also we have now

$$\left.\begin{aligned}&\frac{\partial}{\partial\varrho}(\delta_0\varrho\varphi_{3\varrho})+\frac{\partial}{\partial\zeta}(\delta_0\varrho\varphi_{3\zeta}) = -\frac{\Psi(\varrho,\zeta)\,\eta_1'''\,F_0'}{\alpha^3\mu t_2\varrho}\\ &\qquad\times\left(\delta_0-\frac{F_0'^2 t_4}{\varrho^2}\right)-\frac{1}{\varrho}\left(\delta_0-\frac{F_0'^2}{\varrho^2}t_4\right)F_2''\\ &\qquad+t_6\eta_1\eta_1'-C_2\eta_1't_7 \quad \text{in } D,\\ &\varphi_{3n}=0 \qquad\qquad \text{on } L\end{aligned}\right\}\quad(3.33)$$

where

$$t_6=-\frac{F_0'^3}{\alpha^4\mu^2t_2^2}\left\{\left(\frac{t_4\Psi_\varrho}{\varrho}\right)_\varrho+\left(\frac{t_4\Psi_\zeta}{\varrho}\right)_\zeta\right\}+\frac{3F_0'^3t_4(\varrho)}{\varrho^3\mu^2t_2^2\alpha^2}+\frac{t_4^3(\varrho)\,\delta_0^{\gamma-3}(\varrho)\,A\gamma(\gamma-2)F_0'^5}{\varrho^5\alpha^2\mu^2t_2^2},$$

$$t_7=\frac{F_0'}{\alpha^3\mu t_2}\left[\left\{t_4\Psi_\varrho\,\varrho\left(1-\frac{\alpha^2}{\varrho^2}\right)\right\}_\varrho+\left\{t_4\Psi_\zeta\,\varrho\left(1-\frac{\alpha^2}{\varrho_2}\right)\right\}_\zeta\right]+\frac{2F_0'\alpha t_4(\varrho)}{\varrho^3\mu t_2}-\left(1-\frac{\alpha^2}{\varrho^2}\right)\frac{F_0't_4(\varrho)}{\alpha\mu t_2\varrho}\times\left\{1+\frac{t_4^2(\varrho)\,A\gamma(\gamma-2)\,\delta_0^{\gamma-3}F_0'^2}{\varrho^2}\right\}.$$

Applying the divergence theorem (3.21), to (3.32) and (3.33) we have

$$\begin{aligned}\eta_1'''(\sigma)&\left[\frac{F_0'\delta_0(\alpha)}{\alpha^3}\int_{Q_1}^{Q_2}\Psi d\zeta-\frac{F_0'}{\alpha^3\mu t_2}\iint_D\left(\delta_0-\frac{t_4F_0'^2}{\varrho^2}\right)\times\frac{\Psi}{\varrho}d\zeta d\varrho\right]=\eta_1\eta_1'\left[-\frac{\delta_0(\alpha)}{\alpha}\int_{Q1}^{Q2}\left\{\frac{\Psi_{\varrho\varrho}F_0'}{\alpha\mu t_2}+\frac{F_0'}{\alpha\mu t_2}+\frac{2F_0'}{\alpha}+t_5F_0'\right\}d\zeta\right.\\&\left.-\iint_D t_6(\varrho,\zeta)\,d\zeta d\varrho\right]+\eta_1'C_2\left[\frac{\alpha}{F_0'}\delta_0(\alpha)\times 2b(1+\mu t_2)+\iint_D t_7(\varrho,\zeta)\,d\zeta d\varrho\right]\\&+F_2''(\sigma)\left[\iint_D\frac{\delta_0}{\varrho}\left(1-\frac{d\delta_0}{d\pi_0}\frac{F_0'^2}{\varrho^2}\right)d\zeta d\varrho-\delta_0(\alpha)\,bt_2\mu\right].\end{aligned}\qquad(3.34)$$

In the above equation (3.34), the coefficient of $F_2''(\sigma)$ vanishes due to (3.23), and (3.34) can now be written in the form

$$m_0\eta_1'''(\sigma)=m_1\eta_1\eta_1'+m_2\eta_1',\qquad(3.35)$$

where

$$m_0=\frac{t_2\alpha^2\mu}{F_0'}\left[\frac{F_0'\delta_0(\alpha)}{\alpha^3}\int_{Q1}^{Q2}\Psi d\zeta-\frac{F_0'}{\alpha^3\mu t_2}\times\iint_D\left(\delta_0-\frac{t_4F_0'^2}{\varrho^2}\right)\frac{\Psi}{\varrho}d\zeta d\varrho\right],$$

$$m_1=-\frac{\alpha^2\mu t_2}{F_0'}\left[\iint_D t_6(\varrho,\zeta)\,d\zeta d\varrho+\frac{\delta_0(\alpha)}{\alpha}\int_{Q1}^{Q2}\times\left\{\Psi_{\varrho\varrho}\frac{F_0'}{\alpha\mu t_2}+\frac{F_0'}{\alpha\mu t_2}+F_0'^2+t_5F_0'\right\}d\zeta\right]$$

$$m_2=\frac{\alpha^2\mu t_2}{F_0'}\left[\frac{\delta_0(\alpha)}{F_0'}b\alpha(1+\mu t_2)2+\iint_D t_7(\varrho,\zeta)d\zeta d\varrho\right].$$

We can simplify the coefficients in (3.35) to

$$m_0=\frac{1}{\alpha^3}\iint_D\varrho\delta_0(\Psi_\varrho^2+\Psi_\zeta^2)d\zeta d\varrho,$$

$$\begin{aligned}m_1=\delta_0(\alpha)&\int_{Q1}^{Q2}\Psi_{\zeta\zeta}d\zeta-\frac{F_0'^4}{\mu t_2}(\gamma-2)\iint_D\frac{t_4^2d\zeta d\varrho}{\varrho^5\delta_0}-\frac{3}{\mu t_2}\iint_D\frac{\delta_0}{\varrho}d\zeta d\varrho-b\left[\delta_0(\alpha)\left(5\mu t_2+3\mu^2t_2\right.\right.\\&\left.-\frac{\mu t_2\alpha}{\delta_0(\alpha)}\frac{d\delta_0(\alpha)}{d\varrho}\right)-\frac{2F_0'^2}{\alpha^2}t_4(\alpha)+\frac{\alpha^4\mu^2t_2t}{F_0'^2}\times\left\{\frac{2}{\delta_0(\alpha)}\left(\frac{d\delta_0(\alpha)}{d\varrho}+\frac{t_4(\alpha)F_0'^2}{\alpha^3\mu t_2}\right)^2-\frac{d^2\delta_0(\alpha)}{d\varrho^2}\right\}\\&\left.-2\mu\frac{dt_4(\alpha)}{d\varrho}+\frac{F_0'^2t_4^2(\alpha)}{\alpha^2t_2}(\gamma-2)\right].\end{aligned}$$

$$m_2=C_2R,$$

where

$$R=\frac{2\alpha^3}{F_0'^2}\iint_D\frac{\delta_0}{\varrho}d\zeta d\varrho+\frac{\alpha^3\mu^2t_2^2}{F_0'^2}\delta_0(\alpha)b-\alpha\iint_D\left(1-\frac{\alpha^2}{\varrho^2}\right)\frac{t_4}{\varrho}\left\{1+\frac{F_0'^2}{\varrho^2}\frac{d\varrho_0}{d\pi_0}(\gamma-2)\right\}d\zeta d\varrho.$$

We see here that m_0 is always positive. Integrating (3.35) with respect to σ twice, we get

$$\eta_1'^2(\sigma)=\lambda\eta_1^3+\lambda_1\eta_1^2+\lambda_2\eta_1+\lambda_3,\qquad(3.36)$$

where $\lambda_1=m_2|m_0$, $\lambda=m_1|3m_0$, and, λ_2 and λ_3 are arbitrary constants. We observe that λ_3 should be positive in order that $\eta_1'(\sigma)$ be real.

For a periodic and bounded solution (η_1) of the cubic in (3.36), in which we are interested, the three roots should be real and distinct. This is the case when the descriminant of the cubic in (3.36) is negative. Let the three real and distinct roots be ξ_1, ξ_2 and ξ_3 with magnitudes q_1, q_2, and q_3 respectively ($\xi_1 > \xi_2 > \xi_3$). As we can see from (3.36) that there are two cases to be discussed, one when $\lambda < 0$ (i.e. $m_1 < 0$) and the other when $\lambda > 0$ (i.e. $m_1 > 0$). Here we discuss only the case when $\lambda < 0$, and the other case when $\lambda > 0$ can be discussed on similar lines. Considering the case of $\lambda < 0$, we find that the cubic in (3.36) will have two negative roots (ξ_2, ξ_3) and one positive root (ξ_1), both when $\lambda_1 < 0$, and $\lambda_1 > 0$ with $\lambda_2 > 0$. In this case ($\lambda < 0$) the solution of (3.36), see Raman (1967), can be written as

$$\eta_1(\sigma) = -q_2 + (q_1+q_2)\, C_n^2\left(\sigma\sqrt{\frac{|\lambda|(q_1+q_2)}{4}},\ \sqrt{\frac{q_1+q_2}{q_1+q_3}}\right), \tag{3.37}$$

whose period is

$$T = \frac{4}{\sqrt{\varepsilon|\lambda|(q_1+q_3)}}\int_0^{\pi/2}\frac{dt}{\sqrt{1-k^2\sin^2 t}}, \tag{3.38}$$

where Cn is a Jacobian elliptic function of modulus $k = [(q_1+q_2)^{\frac{1}{2}}/(q_1+q_3)^{\frac{1}{2}}]$ writing $q = \varepsilon h q_1$, $l = 1 + (q_3|q_1)$, we have for η_1 the equation

$$\eta_1(\sqrt{\varepsilon}\,\theta) = \frac{q}{\varepsilon h}\left[1 - lk^2 + lk^2 C_n^2\left(\frac{\theta}{2}\sqrt{\frac{ql|\lambda|}{n}},\ k\right)\right] \tag{3.39}$$

and its period is

$$T = 4\sqrt{\frac{h}{ql|\lambda|}}\int_0^{\pi/2}\frac{dt}{\sqrt{1-k^2\sin^2 t}} \tag{3.40}$$

The condition that the mean of the elevation of the free surface over the mean level (η), over the period T should vanish, gives

$$(1-l)\int_0^{\pi/2}\frac{dt}{\sqrt{1-k^2\sin^2 t}} + l\int_0^{\pi/2}\sqrt{1-k^2\sin^2 t}\,dt = 0, \tag{3.41}$$

Now the equations (3.40) and (3.41) determine k and l uniquely, provided q, the amplitude of the first order cnoidal wave and T, the period of the cnoidal wave are known beforehand. This makes η_1 completely known. Knowing η_1, the first order cnoidal wave, we can calculate all the flow variables to the first order of approximation.

In the dimensional form, the cnoidal wave at the free surface of a compressible fluid over a toroidal channel, is

$$r = a + q\left[1 - lk^2 + lk^2 C_n^2\left(\frac{\theta}{2}\sqrt{\frac{ql|\lambda|}{n}},\ k\right)\right] + 0(q), \tag{3.42}$$

whose period is

$$T = 4\sqrt{\frac{h}{ql|\lambda|}}\int_0^{\pi/2}\frac{dt}{\sqrt{1-k^2\sin^2 t}}, \tag{3.43}$$

where $o(q)$ means limit of $[o(q)|q]$ as $q \to 0$, is zero. The following expressions can easily be verified. The tangential speed of the fluid is

$$u_1 = -\frac{hF_0'\sqrt{gh}}{r}\left[1 + \frac{q\alpha^2|m_1|}{3RF_0'^2h}\{3 - l(k^2+1)\} - \frac{q}{\alpha\mu t_2 h}\left\{1 - lk^2 + lk^2 C_n^2\left(\frac{\theta}{2}\sqrt{\frac{ql|\lambda|}{h}},\ k\right)\right\}\right] + 0(q),$$

the radial speed

$$v_1 = -\frac{F_0'\sqrt{gh}}{\alpha^3\mu t_2}\frac{qlk^2}{h}\Psi_\varrho(\varrho,\zeta)\sqrt{\frac{ql|\lambda|}{h}}\left[C_n\left(\frac{\theta}{2}\sqrt{\frac{ql|\lambda|}{h}},\ k\right) S_n\left(\frac{\theta}{2}\sqrt{\frac{ql|\lambda|}{h}},\ k\right) \times d_n\left(\frac{\theta}{2}\sqrt{\frac{ql|\lambda|}{h}},\ k\right)\right] + 0(q^{3/2}),$$

and the axial speed

$$w_1 = -\frac{F_0'\sqrt{gh}}{\alpha^3\mu t_2}\frac{qlk^2}{h}\Psi_\zeta(\varrho,h)\sqrt{\frac{ql|\lambda|}{h}}\left[C_n\left(\frac{\theta}{2}\sqrt{\frac{ql|\lambda|}{h}},\ k\right) S_n\left(\frac{\theta}{2}\sqrt{\frac{ql|\lambda|}{h}},\ k\right) \times d_n\left(\frac{\theta}{2}\sqrt{\frac{ql|\lambda|}{h}},\ k\right)\right] + 0(q^{3/2}),$$

also the pressure

$$p = \Delta^* gh\left[\pi_0 + \delta_0\left(1 - \frac{\alpha^2}{\varrho^2}\right)\frac{q}{h}\frac{|m_1|}{3R}\{3 - l(k^2+1)\} + \frac{q\delta_0F_0'^2}{\varrho^2\mu t_2\alpha h}\left\{1 - lk^2 + lk^2 C_n^2\left(\frac{\theta}{2}\sqrt{\frac{ql|\lambda|}{h}},\ k\right)\right\}\right] + 0(q),$$

and the density

$$\Delta = \Delta^* \left[\delta_0 + \delta_0 \frac{d\delta_0}{d\pi_0} \frac{q}{h} \frac{|m_1|}{3R} \{3 - l(k^2+1)\} + \frac{q\delta_0 F_0'^2}{\varrho^2 \alpha\mu t_2 h} \frac{d\delta_0}{d\pi_0} \left\{ 1 - lk^2 + lk^2 C_n^2 \left(\frac{\theta}{2} \times \sqrt{\frac{ql|\lambda|}{h}}, k \right) \right\} \right] + 0(q),$$

where Δ^* is the reference density, Δ the density in the dimensional form, and

$$R = \frac{2\alpha^3}{F_0'^2} \iint_D \frac{\delta_0}{\varrho} d\zeta d\varrho + \frac{\alpha^3 \mu^2 t_2^2}{F_0'^2} \delta_0(\alpha) b - \alpha \iint_D \left(1 - \frac{\alpha^2}{\varrho^2}\right) \frac{d\delta_0}{d\pi_0} \frac{\delta_0}{\varrho} \left\{ 1 + \frac{F_0'^2}{\varrho^2} \frac{d\delta_0}{d\pi_0} (\gamma - 2) \right\} d\zeta d\varrho.$$

In the above equations F_0' is defined by (3.23). In the above expressions,

$$t_4 = \delta_0 \frac{d\delta_0}{d\pi_0}, \quad t = \frac{G\gamma}{\gamma - 1} \frac{1}{\delta_0(\alpha)}, \quad t_1 = t \frac{d\delta_0(\alpha)}{d\pi_0},$$

$$t_2 = 1 + t \frac{d\delta_0}{d\pi_0}(\alpha) \text{ and } t_3 = 1 - t \frac{d\delta_0}{d\varrho}(\alpha) \frac{1}{\delta_0(\alpha)}.$$

4. Straight channel

In the case of a straight channel ($\alpha h \to \infty$), the cnoidal wave profile to the first order of approximation is

$$f = q \left\{ 1 - lk^2 + lk^2 C_n^2 \left(\frac{x_1}{2h} \sqrt{\frac{|\Lambda| ql}{h}}, k \right) \right\} + 0(q), \tag{4.1}$$

and its period is

$$T = 4 \sqrt{\frac{h}{ql|\Lambda|}} \int_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}, \tag{4.2}$$

where

$$\Lambda = \frac{3u_0^{-2} \iint_D \delta_0 \, d\zeta d\varrho + bu_0^2 \frac{d\delta_0}{d\varrho}(\alpha) - \iint_D u_0^2 \frac{d^2\delta_0}{d\varrho^2} d\zeta d\varrho}{3 \iint_D \delta_0 (\Psi_\varrho^2 + \Psi_\zeta^2) \, d\zeta d\varrho}$$

and x_1-axis (in rectangular coordinate system) is taken along the channel.

The speed of the cnoidal wave in a compressible fluid in a straight channel of arbitrary cross section is

$$\bar{u}_1 = -\frac{\sqrt{gh}}{u_0} \left[u_0^2 + \frac{q}{h} \{3 - l(k^2+1)\} K \right] + 0(q), \tag{4.3}$$

where

$$K = \frac{3\bar{u}_0^2 \iint_D \delta_0 \, d\zeta d\varrho + u_0^2 b \cdot \frac{d\delta_0}{d\varrho}(\alpha) - \iint_D u_0^2 \frac{d^2\delta_0}{d\varrho^2} d\zeta d\varrho}{6 \left[u_0^{-2} \iint_D \delta_0 \, d\zeta d\varrho \right]}$$

Here u_0, the critical speed of the cnoidal wave is defined by

$$u_0^2 \delta_0(\alpha) = \frac{1}{b} \iint_D (\delta_0 - t_4(\varrho) u_0^2) \, d\zeta d\varrho, \tag{4.4}$$

and Ψ is the solution of the problem,

$$(\delta_0 \Psi_\varrho)_\varrho + (\delta_0 \Psi_\zeta)_\zeta = \delta_0 \left(1 - u_0^2 \frac{d\delta_0}{d\pi_0}\right) \quad \text{in } D,$$
$$\Psi_\varrho = u_0^2 \quad \text{on } S,$$
$$\Psi_n = 0 \quad \text{on } L, \tag{4.5}$$

with $\Psi(\alpha, 0) = 0$.

In the case of a straight channel, when we make the wavelength (αhT) of the wave infinity, we see that the cnoidal wave develops into a solitary wave, which is represented by

$$f = q \, Sech^2 \frac{x_1}{2h} \sqrt{\frac{|\Lambda| q}{h}}, + 0(q), \tag{4.6}$$

whose speed is given by

$$\bar{u}_1 = -\frac{\sqrt{gh}}{u_0} \left[u_0^2 + \frac{q}{h} K \right] + 0(q). \tag{4.7}$$

and the critial speed u_0 is same as that defined by (4.4).

5. Rectangular channel

Let us consider a rectangular toroidal channel of breadth b and depth h. In this case the solution Ψ of the system:

$$(\Psi_\varrho \delta_0 \cdot \varrho)_\varrho + (\Psi_\zeta \delta_0 \cdot \varrho)_\zeta = \frac{\delta_0}{\varrho}\alpha^2\left(1 - \frac{F_0'^2}{\varrho^2}\frac{d\delta_0}{d\pi_0}\right)$$

$$\text{in } \begin{cases} \alpha - 1 < \varrho < \alpha, \\ -b/2 < \zeta < b/2 \end{cases}$$

$$\Psi_\varrho(\alpha - 1, \zeta) = 0 \quad \text{for} \quad -b/2 \leqslant \zeta \leqslant b/2,$$

$$\Psi_\zeta(\varrho_1 \pm b/2) = 0 \quad \text{for} \quad \alpha - 1 \leqslant \varrho \leqslant \alpha,$$

$$\Psi_\varrho(\alpha, \zeta) = \alpha\mu t_2,$$

and $\Psi(\alpha, 0) = 0$,

can be written as

$$\Psi = \alpha^2 \int_\varrho^\alpha \frac{1}{\delta_0 \beta}\left\{\int_{\alpha-1}^{\beta} \frac{\delta_0}{\xi}\left(1 - \frac{d\delta_0}{d\pi_0}\frac{F_0'^2}{\xi^2}\right)d\xi\right\}^2 d\beta \quad (5.1)$$

Now m_0, m_1, and m_2 reduce to

$$m_0 = \alpha\beta \int_{\alpha-1}^{\alpha} \frac{1}{\delta_0 \varrho}\left\{\int_{\alpha-1}^{\varrho} \frac{\delta_0}{\xi}\left(1 - \frac{d\delta_0}{d\pi_0}\frac{F_0'^2}{\xi^2}\right)d\xi\right\}^2 d\varrho,$$

$$\begin{aligned} m_1 = &-\frac{F_0'^4}{\mu t^2}(\gamma - 2)\iint_D \frac{\delta_0}{\varrho^5}\left(\frac{d\delta_0}{d\pi_0}\right)^2 d\zeta d\varrho - \frac{3}{\mu t_2} \\ &\times \iint_D \frac{\delta_0}{\varrho} d\zeta d\varrho - b\left[\delta_0(\alpha)\left(5\mu t_2 + 3\mu^2 t_2\right.\right. \\ &\left.- \frac{\mu t_2 \alpha}{\delta_0(\alpha)}\frac{d\delta_0}{d\varrho}(\alpha)\right) + \frac{\alpha^4 \mu^2 t_2 t}{F_0'^2}\left\{\frac{2}{\delta_0(\alpha)}\left(\frac{d\delta_0}{d\varrho}(\alpha)\right.\right. \\ &\left.\left.+ \frac{t_4(\alpha) F_0'^2}{\alpha^3 \mu t_2}\right)^2 - \frac{d^2\delta_0}{d\varrho^2}(\alpha)\right\} - \frac{2F_0'^2}{\alpha^2}\frac{d\delta_0}{d\pi_0}(\alpha) \\ &\left.\times \delta_0(\alpha) + \frac{F_0'^2 t_4^2(\alpha)}{\alpha^2 t_1}(\gamma - 2) - 2\mu\frac{dt_4}{d\varrho}(\alpha)\right], \end{aligned}$$

$$\begin{aligned} m_2 = \alpha b c_2 &\left[\frac{2\alpha^2}{F_0'^2}\int_{\alpha-1}^{\alpha}\frac{\delta_0}{\varrho}d\varrho + \frac{\alpha^2\mu^2 t_2^2}{F_0'^2}\delta_0(\alpha)\right. \\ &\left.- \int_{\alpha-1}^{\alpha}\left(1 - \frac{\alpha^2}{\varrho^2}\right)\frac{t_4}{\varrho}\left\{1 + \frac{F_0'^2}{\varrho^2}\frac{d\delta_0}{d\pi_0}(\gamma - 2)\right\}d\varrho\right]. \end{aligned}$$

The cnoidal wave profile is

$$\gamma = \alpha + q\left[1 - lk^2 + lk^2 C_2^n\left(\frac{\theta}{2}\sqrt{\frac{ql|\lambda|}{h}}, k\right)\right] + 0(q), \quad (5.2)$$

whose period is

$$T = 4\sqrt{\frac{h}{ql|\lambda|}}\int_0^{\pi/2}\frac{dt}{\sqrt{1 - k^2\sin^2 t}}, \quad (5.3)$$

and the speed of the cnoidal wave is

$$\bar{u}_1 = -\frac{2\sqrt{gh}}{(2a - h)}\frac{\alpha^2}{F_0'}\left\{\frac{F_0'^2}{\alpha^2} + \frac{q}{h}K[3 - l(k^2 + 1)]\right\} + 0(q), \quad (5.4)$$

where $\lambda = m_1 | 3m_0$.

Conclusions

We have established through equation (3.21), that due to the compressibility, there exist two critical speeds. Thus there exist two cnoidal waves, in contrast to one in incompressible medium. The critical speed is also seen to depend on the topography of the channel in addition to the density of the fluid. The profile and the period of the cnoidal wave at the free surface of a compressible fluid over a toroidal channel of arbitrary cross section are given by equation (3.42) and (3.43) respectively. These when the compressibility of the medium tends to zero, reduce to the results of Peters (1966). By making the radius of the torodial channel infinite, we have obtained, as a limiting case, the results for the propagation of a cnoidal wave in a compressible fluid in a straight channel of arbitrary cross section. In particular, (4.1) and (4.3) give the profile and speed of the cnoidal wave. Further, we note that equation (4.1) and (4.4) giving the cnoidal wave profile and the critical speed are same as those of Shen (1968). Finally, we observe that by making the wave length of the cnoidal wave infinite the equations (4.1) and (4.3) define the profile and speed of a solitary wave in a compressible fluid in a straight channel of arbitrary cross section, and these (equations (4.6) and (4.7)) are in agreement with those of Raman (1967).

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КНОИДАЛЬНЫЕ ВОЛНЫ В СЖИМАЕМОЙ ЖИДКОСТИ В ТОРОИДАЛЬНОМ КАНАЛЕ

В этой статье мы обсуждаем задачу о кноидальных волнах на свободной поверхности сжимаемой жидкости в тороидальном канале. Для анализа задачи мы пользуемся техникой разложения зависимых переменных в ряд по степеням параметра. Получено выражение для профиля кноидальной волны; в конце обсуждаются некоторые частные случай. Точность полученных результатов устанавливается путем сравнения с результатами Шена.