A simple model of a certain type of Frontal Wave in the atmosphere

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(Manuscript received October 4, 1968, revised version January 24, 1969)

ABSTRACT

A simple model of small waves occurring on upper fronts in the rear of quasi-stationary surface cold fronts was derived. The latent heat release is partly included in the modelling assumptions. The analysis of the frequency equation shows that the most intense wave development is to be expected in the regions with the greatest slope of the upper front. During the development of an unstable wave the cusp point is formed on the front. Stable waves are possible in the model, too. The influence of orography may be incorporated in a simplified parametrical form.

Introduction

In this paper a certain type of progressive wave on the frontal surface of the Margules type is studied analytically to give a simple theoretical model of small waves occuring on upper fronts in the rear of quasi-stationary surface cold fronts. Such small waves are often connected with pronounced precipitation activity. Therefore, the influence of the latent heat release is partly included in the model. Typical synoptic conditions responsible for the development of these waves in middle Europe are described, e.g., in (Brádka, 1967). The following discussion focuses attention on the basic stability properties of the waves in question.

Basic equations

Let us consider the slanting frontal plane of the Margules type separating two basic currents $U_1 = \text{const}$, $U_2 = \text{const}$. Only zonal currents are discussed later for the sake of simplicity but the analysis remains essentially unchanged if meridional basic currents are considered. The air densities ϱ_1, ϱ_2 will be regarded as constants in each air mass while the colder air is assumed to lie below the frontal plane. The static stability of the system is thus insured. The angle between the frontal plane and the horizontal plane will be denoted as α. The co-ordinate system is chosen as follows: the x-axis is formed by the intersection of the two planes and is orientated eastward; the y-axis lies in the frontal plane and forms with the meridian the angle α; the z-axis which is normal to the frontal plane is pointed upward into the warmer air. We thus use a "quasi-standard" co-ordinate system connected with the frontal plane (Fig. 1).

Let us deal with wave motions which arose on the interface of the two currents. We shall assume that the geometrical co-ordinates of air particle x, y, z are related to the corresponding Lagrange parameters a, b, c as follows:

$$x = a + Ut + r(b, c)\sin k(a-\gamma t), \quad y = b + s(b, c) - r(b, c)\cos k(a-\gamma t), \quad z = c + \sigma(b, c) - Mr(b, c)\cos k(a-\gamma t), \; M = \text{const}. \tag{1}$$

This choice represents from the formal point of view a certain generalization of the assumptions used by Gerstner (Kotshin, 1963; Lamb, 1932) in his classical model of trochoidal waves. The relations (1) may be specified for the individual air masses by means of the indices 1, 2. For the velocity components we have

$$\left.\begin{aligned} v_x &= \frac{\partial x}{\partial t} = -k\gamma r(b, c)\cos k(a-\gamma t) + U, \\ v_y &= \frac{\partial y}{\partial t} = -k\gamma r(b, c)\sin k(a-\gamma t), \\ v_z &= \frac{\partial z}{\partial t} = -Mk\gamma r(b, c)\sin k(a-\gamma t) = Mv_y \end{aligned}\right\} \tag{2}$$

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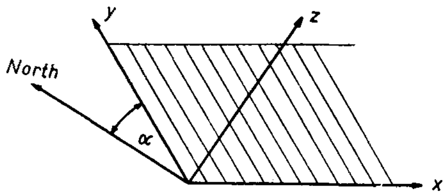

Fig. 1. The "quasi-standard" co-ordinate system.

where γ denotes the phase velocity, $k = 2\pi L^{-1}$ is the wave number, $r(b, c)$ characterizes the wave amplitude and $s(b, c)$ and $\sigma(b, c)$ are auxiliary functions the form of which will be discussed later. The meaning of the dimensionless parameter M which relates simply v_z with v_y, will also be illustrated later.

The above quantities a, b, c which serve to identify an individual air particle, are not the initial co-ordinates of a particle. Hence, the continuity equation in the Lagrangian form reads with respect to the assumption on incompressibility

$$\frac{\partial(x, y, z)}{\partial(a, b, c)} = \frac{\partial(x_0, y_0, z_0)}{\partial(a, b, c)} \tag{3}$$

where x_0, y_0, z_0 now denote the initial co-ordinates of the particle to which a, b, c refer, i.e., $x_0 = x(a, b, c, t_0)$ etc. Physically, a, b and c may be used to define the motion of air particles in the straight basic current U in the absence of waves.

If we neglect dissipative forces and confine ourselves to motions of synoptic character, in middle and higher geographical latitudes, we may write the components of forces per unit mass as $F_x = 2\Omega v_y \sin(\varphi - \alpha) - 2\Omega v_z \cos(\varphi - \alpha) \approx 2\Omega v_y \sin(\varphi - \alpha)$, $F_y = -g \sin\alpha - 2\Omega v_x \sin(\varphi - \alpha)$, $F_z = -g\cos\alpha + 2\Omega v_x \cos(\varphi - \alpha) \approx -g\cos\alpha$. Here g denotes the gravity acceleration, φ is the geographical latitude and Ω the angular velocity of the earth's rotation. The validity of the inequality $|\partial^2 z/\partial t^2| < g\cos\alpha$ is further assumed for the large-scale motions. By means of this assumption small-scale convective movements are excluded. If the notation $G = g\sin\alpha$, $g^* = g\cos\alpha$, $f^* = 2\Omega\sin(\varphi - \alpha)$ is introduced, the equations of motion may be written in the Lagrangian form as follows:

$$\left(f^*\frac{\partial y}{\partial t} - \frac{\partial^2 x}{\partial t^2}\right)\frac{\partial x}{\partial \lambda_i} - \left(f^*\frac{\partial x}{\partial t} + G + \frac{\partial^2 y}{\partial t^2}\right)\frac{\partial y}{\partial \lambda_i} - g^*\frac{\partial z}{\partial \lambda_i} = \frac{\partial}{\partial \lambda_i}\left(\frac{p}{\varrho}\right), \quad \lambda_i = a, b, c. \tag{4}$$

The symbol p stands here for pressure. The quantity f^* which corresponds to the Coriolis parameter $f = 2\Omega\sin\varphi$, will be regarded as constant due to the relatively small horizontal dimensions of the short frontal waves the wavelength of which usually lies in the interval $10^5 \leq L < 10^6$ m. By the scale of the waves the use of the cartesian co-ordinate system may also be substantiated. Eqs. (3) and (4) are the governing equations of the model.

With the aid of Eqs. (1) and (2) the equations of motion (4) can be written as:

$$\left.\begin{aligned}
&rk(k\gamma^2 - f^*\gamma - H - Mg^*)\sin k(a - \gamma t) = \frac{\partial}{\partial a}\left(\frac{p}{\varrho}\right),\\
&\left[rk\gamma(f^* - k\gamma) + (H + Mg^*)\frac{\partial r}{\partial b} + k\gamma r\frac{\partial s}{\partial b}(f^* - k\gamma)\right]\cos k(a - \gamma t)\\
&\quad + k\gamma(k\gamma - f^*)r\frac{\partial r}{\partial b} - g^*\frac{\partial \sigma}{\partial b} - H\left(1 + \frac{\partial s}{\partial b}\right) = \frac{\partial}{\partial b}\left(\frac{p}{\varrho}\right),\\
&\left[rk\gamma(f^* - k\gamma)\frac{\partial s}{\partial c} + (H + Mg^*)\frac{\partial r}{\partial c}\right]\cos k(a - \gamma t) + k\gamma(k\gamma - f^*)r\frac{\partial r}{\partial c}\\
&\quad - g^*\left(1 + \frac{\partial \sigma}{\partial c}\right) - H\frac{\partial s}{\partial c} = \frac{\partial}{\partial c}\left(\frac{p}{\varrho}\right)
\end{aligned}\right\} \tag{5}$$

where $H = f^*U + G$. We shall try to integrate the system (5) to obtain a relation for the pressure $p(a, b, c, t)$. For this purpose the auxiliary functions $s(b, c)$ and $\sigma(b, c)$ are to be specified conveniently. In order that Eqs. (5) may be integrable we put

$$\left.\begin{aligned}
&s(b, c) = -b + k^{-1}\ln r(b, c),\\
&\sigma(b, c) = \frac{1}{g^*}\left[\tfrac{1}{2}k(H + Mg^*)r^2(b, c) - k^{-1}(k\gamma^2 - f^*\gamma - Mg^*)\ln r(b, c)\right].
\end{aligned}\right\} \tag{6}$$

By means of this choice the system (5) may be rearranged to the form

$$\left.\begin{aligned}\frac{\partial}{\partial\lambda_i}\Bigg[& r(f^*\gamma - k\gamma^2 + H + Mg^*)\cos k(a-\gamma t)\\ & -\tfrac{1}{2}kr^2(f^*\gamma - k\gamma^2 + H + Mg^*)\\ & -k^{-1}(f^*\gamma - k\gamma^2 + H + Mg^*)\ln r\\ & -g^*c - p\varrho^{-1}\Bigg] = 0.\end{aligned}\right\} \quad (7)$$

By integration of Eqs. (7) we obtain (except for an additive constant)

$$p(a, b, c, t) = -g^*\varrho c + \varrho(f^*\gamma - k\gamma^2 + H + Mg^*) \times [r(b, c)\cos k(a-\gamma t) - \tfrac{1}{2}kr^2(b, c) - k^{-1}\ln r(b, c)]. \quad (8)$$

The equation of continuity (3) yields after rearrangement and with respect to Eqs. (1) and (6) the relation

$$\frac{\partial(x, y, z)}{\partial(a, b, c)} = \left(\frac{1}{kr} - kr\right)\frac{\partial r}{\partial b} = \frac{\partial(x_0, y_0, z_0)}{\partial(a, b, c)}. \quad (9)$$

Eq. (9) indicates that mass is conserved in the model and that the motion described by Eqs. (1) and (6) is dynamically possible. Obviously, Eqs. (6) serve as a sort of the condition of integrability of the governing equations (3) and (4).

The general vorticity equation of the model may be derived as follows. If t is regarded as a parameter, Eqs. (4) may be united and rewritten to a compact form

$$\left(f^*\frac{\partial y}{\partial t} - \frac{\partial^2 x}{\partial t^2}\right)dx - \left(f^*\frac{\partial x}{\partial t} + G + \frac{\partial^2 y}{\partial t^2}\right)dy = d\left(\frac{p}{\varrho} + g^*z\right).$$

Since the expression $d(p\varrho^{-1} + g^*z)$ is a total differential, the expression on the left-hand side of the last equation must be a total differential, too. Hence

$$\frac{\partial}{\partial y}\left(f^*v_y - \frac{\partial v_x}{\partial t}\right) + \frac{\partial}{\partial x}\left(f^*v_x + G + \frac{\partial v_y}{\partial t}\right) = 0.$$

With respect to the assumption $f^* = \text{const}$ we thus arrive at the vorticity equation in the form

$$\frac{\partial \xi}{\partial t} + f^*D = 0 \quad (10)$$

with

$$\xi = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y},\ D = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}.$$

In Eq. (10), $\partial\xi/\partial t$ corresponds to $d\xi/dt$ in Eulerian concept.

Conditions on the frontal surface

Let us consider the conditions on the frontal surface under the assumption that the frontal surface is physically identical with the inclined frontal plane separating the basic currents. The original frontal plane may be described without loss of generality by the relation $c = 0$. It follows from the condition of non-penetration of air particles across the frontal surface and from Eqs. (1) that the parametric equations of the frontal surface after the wave-like deformation read

$$x = a + Ut + r(b, 0)\sin k(a-\gamma t),\ y = b + s(b, 0) - r(b, 0)\cos k(a-\gamma t),\ z = \sigma(b, 0) - Mr(b, 0)\cos k(a-\gamma t). \quad (11)$$

The frontal surface may be constructed by means of Eqs. (11) if the function r and the quantity M are known. In co-ordinates a, b and c, the equation of the frontal surface assumes a simple form $c = 0$.

The formulation of the dynamic condition is generally not easy when using the Lagrange method because air particles adjacent to the frontal surface may generally have various values of the Lagrangian parameters a and b after the wave-like deformation. But we can consider the continuity of the frontal pressure field in an initial time $t_0 = 0$ when the wave-like deformation and wave amplitude are supposed to be very small. Now one can demand the dynamic condition to be fulfilled in the form $p_1(a_1 = a_2,\ b_1 = b_2,\ c = 0,\ t = 0) = p_2(a_2 = a_1,\ b_2 = b_1,\ c = 0,\ t = 0)$. The air particles adjacent to the frontal surface have the same geometrical position. Therefore the following condition may be prescribed: $x_1 = x_2$, $y_1 = y_2$, $z_1 = z_2$ for $a_1 = a_2$, $b_1 = b_2$, $c = 0$, $t = 0$. In addition, we put $M_1 = M_2 = M = \text{const}$ in order for the continuity of the vertical motions to be satisfied on the frontal surface.

The continuity of the frontal pressure then demands with respect to Eq. (8) that $r_1(b_1 = b_2, c = 0) = r_2(b_2 = b_1, \; c = 0)$, $\varrho_1(f^*\gamma - k\gamma^2 + f^*U_1 + G + Mg^*) = \varrho_2(f^*\gamma - k\gamma^2 + f^*U_2 + G + Mg^*)$. Hence the frequency equation of the model is obtained:

$$\gamma = \frac{1}{2k}\left\{f^* \pm \left[f^{*2} + 4kf^*\frac{U_1\varrho_1 - U_2\varrho_2}{\varrho_1 - \varrho_2} + 4k(G + Mg^*)\right]^{\frac{1}{2}}\right\}. \qquad (12)$$

It follows from Eqs. (1) and (6) that the fulfillment of the above mentioned condition $x_1 = x_2$, $y_1 = y_2$, $z_1 = z_2$ for $a_1 = a_2$, $b_1 = b_2$, $c = 0$, $t = 0$ is now guaranteed if the initial wave amplitude is supposed to be so small that, strictly speaking, the terms containing $r^2(b, 0)$ can be neglected with sufficient accuracy in the expression for z in Eqs. (1).

Eq. (12) may be rearranged and simplified by the requirement $\varrho_1(f^*U_1 + G) = \varrho_2(f^*U_2 + G)$, which results in the well-known Margules equation for the slope of the quasi-stationary frontal plane:

$$\operatorname{tg} \alpha = -f(\varrho_1 U_1 - \varrho_2 U_2)\,[g(\varrho_1 - \varrho_2) - 2(\varrho_1 U_1 - \varrho_2 U_2)\Omega \cos \varphi]^{-1}. \qquad (13)$$

The quantity $\operatorname{tg} \alpha$ is related directly to the frontal shear in ϱ and U because of the identity

$$\varrho_1 U_1 - \varrho_2 U_2 \equiv \frac{U_1 + U_2}{2}(\varrho_1 - \varrho_2) + \frac{\varrho_1 + \varrho_2}{2}(U_1 - U_2).$$

Let us denote the colder air mass lying below the frontal plane by the index 1, the warmer air by the index 2. Let us further assume in agreement with typical synoptic conditions the existence of a westerly current $U_2 > 0$ in the warmer air above the frontal plane while $U_1 \gtrless 0, \varrho_1 |U_1| < \varrho_2 U_2$. Hence $\operatorname{tg} \alpha > 0$. (With meridional basic currents, this corresponds to a warm southerly wind above the frontal plane, α being now the angle between the y-axis and the parallel.) By a combination of Eqs. (12) and (13) one finds

$$\gamma = k^{-1}\{\Omega \sin(\varphi - \alpha) \pm [\Omega^2 \sin^2(\varphi - \alpha) + kgM \cos \alpha]^{\frac{1}{2}}\}. \qquad (14)$$

The expression for the pressure (8) may now be rewritten to the form $p(a, b, c, t) = \varrho(f^*U + G)[r(b, c) \cos k(a - \gamma t) - \frac{1}{2}kr^2(b, c) - k^{-1} \ln r(b, c)] - g^*\varrho c + \text{const}$. Analogically the expression for $\sigma(b, c)$ in Eqs. (6) may be rearranged: $\sigma(b, c) = (1/2g^*)kr^2(b, c) \;\; (f^*U + G + Mg^*)$. Under typical conditions the difference $\sigma_1(b_1 = b_2, c = 0) - \sigma_2(b_2 = b_1, \; c = 0) = (1/2g^*)f^*kr^2(b_1 = b_2, \; c = 0) \times (U_1 - U_2)$ is a small number. Hence the relation $z_1(a_1 = a_2, \; b_1 = b_2, \; c = 0, \; t = 0) = z_2(a_2 = a_1, \; b_2 = b_1, \; c = 0, \; t = 0)$ is fulfilled approximately even for finite values of $r^2(b, 0)$, whereas analogical conditions for x and y are valid exactly. These requirements, together with Eqs. (1), prescribe the character of the initial wave-like deformation (the particles adjacent to the frontal surface practically adhere one to another during the initial wave-like impulse which even may be of finite amplitude type in the first approximation).

Since v_y and v_z are continuous on the frontal surface and since the equation of the frontal surface assumes a simple form $c = 0$, we may write the kinematic condition as follows:

$$\left[\left(\frac{\partial x_1}{\partial t} - \frac{\partial x_2}{\partial t}\right)\frac{\partial(c, y, z)}{\partial(a, b, c)}\right]_{\substack{a_1 = a_2,\, b_1 = b_2 \\ c = 0,\, t = 0}} = (U_1 - U_2)\left(\frac{\partial y}{\partial a}\frac{\partial z}{\partial b} - \frac{\partial y}{\partial b}\frac{\partial z}{\partial a}\right)_{\substack{a_1 = a_2,\, b_1 = b_2 \\ c = 0,\, t = 0}} = 0. \qquad (15)$$

Eqs. (15), (1) and (6) generally imply that $(k^2/3g^*)(f^*U + G + Mg^*)(\partial r^3/\partial b) - M(\partial r/\partial b = 0$ at $b_1 = b_2$, $c = 0$. Hence the function $r(b, c)$ may take the general form

$$r(b, c) = B(b)C(c) + \Gamma(c), \; C(0) = 0, \; \Gamma(0) > 0,$$
$$\Gamma_1(0) = \Gamma_2(0). \qquad (16)$$

By this choice we confine ourselves to a particular type of the solution fulfilling the kinematic condition. It follows from Eqs. (16) that $r(b, 0) = \Gamma(0) = \text{const}$. That is, the amplitude of the wave-like deformation of the frontal surface is independent of b. The condition $r_1(b_1 = b_2, \; c = 0) = r_2(b_2 = b_1, c = 0)$ mentioned earlier obviously is satisfied by Eqs. (16). One may demand that $\partial r/\partial c < 0$ since the real waves observed in the atmosphere are rather "shallow", as far as their vertical extent is considered. However, the exact form of the functions B, C and Γ will not be examined here.

Stability properties

Let us put in Eq. (14) formally $\alpha = \Omega = 0$, $M = 1$. We thus get the relation for the gravity waves derived by Gerstner. On the other hand, for $M = 0$ and for positive sign before the square root the frequency equation (14) degenerates to the expression for simple inertial oscillations which reflects the stabilizing effect of the earth's rotation. Since the model must be reducible to simpler inertial oscillations, only the choice of the positive sign before the square root has any physical meaning. For $M < 0$ unstable amplified waves may occur on the frontal surface; the case $M \geqslant 0$ leads only to stable (neutral) waves.

It is necessary, in order to complete the structure of the model, that the parameter M be estimated. To do so, we shall deal in a very simplified form with some thermodynamical aspects with special emphasis on the condensation effects in those parts of the wave where precipitation activity takes place. For the sake of simplicity, the air is assumed to be saturated. Let us consider a mass of $1 + W_s$ tons of air which consists of one ton of dry air and W_s tons of saturated water vapour. Here W_s is the saturation mixing ratio. Let us assume that the latent heat released per unit time by the condensation of $-(\partial W_s/\partial t)$ tons of water vapour is used exclusively to heat the dry air. We thus neglect the heating of the water vapour and assume that the condensation products fall out immediately. Because of the latent heat release, the first law of thermodynamics may be written in the Lagrangian variables as follows:

$$-l\frac{\partial W_s}{\partial t} = c_p\frac{\partial T}{\partial t} - \frac{1}{\varrho}\frac{\partial p}{\partial t}. \tag{17}$$

The symbol l denotes the latent heat of condensation. The specific heat at constant pressure c_p (as well as the specific gas constant R) are taken to be the values for dry air. The saturation mixing ratio is connected with the temperature T and the saturation water vapour pressure e_s by the relation $W_s = \varepsilon e_s(p - e_s)^{-1} \approx \varepsilon e_s p^{-1}$, where $\varepsilon = \text{const} = 0.622$. Hence $\partial e_s/\partial t = \varepsilon^{-1}\partial/\partial t\,(pW_s)$ so that the combination with the Clausius-Clapeyron equation in the approximate form $\partial T/\partial t = RT^2(\varepsilon l e_s)^{-1}\partial e_s/\partial t$ yields

$$\frac{\partial T}{\partial t} = \frac{T}{\varepsilon l \varrho W_s}\frac{\partial}{\partial t}(pW_s). \tag{18}$$

This is a relation valid for the phase change in question. From Eqs. (18) and (17) we get with the aid of the equation of state

$$\frac{\partial W_s}{\partial t} = \frac{(\varepsilon l - c_p T)\,W_s}{(\varepsilon l^2 W_s + c_p R T^2)\,\varrho}\frac{\partial p}{\partial t} \tag{19}$$

where $\varepsilon l - c_p T > 0$. Condensation thus takes place with $\partial p/\partial t < 0$. Eq. (19) is identical with Berkofsky's relation derived by means of Eulerian variables (Berkofsky, 1962).

Eq. (19) is based, inter alia, on the approximate relation $W_s \approx \varepsilon e_s p^{-1}$ which was used for the sake of comparison with Berkofsky's relation. If the complete relation $W_s = \varepsilon e_s(p - e_s)^{-1}$ is used we get

$$\frac{\partial W_s}{\partial t} = \frac{(\varepsilon l - c_p T)(\varepsilon + W_s)\,W_s}{[\varepsilon l^2 W_s(\varepsilon + W_s) + c_p R T^2 \varepsilon]\,\varrho}\frac{\partial p}{\partial t}$$

instead of Eq. (19). The numerical value of the fraction on the right-hand side of the last equation is practically the same as in Eq. (19).

If the difference between the numerical value of the specific humidity and the mixing ratio is ignored, the individual change in the saturation mixing ratio may be expressed in the absence of turbulent diffusion of humidity as follows (Marchuk 1967):

$$\begin{aligned}\frac{\partial Ws}{\partial t} &= -c_p l^{-1}(\delta_d - \delta_s)\,V = -c_p l^{-1}(\delta_d - \delta_s)\\ &\quad\times(v_z \cos\alpha + v_y \sin\alpha) = -c_p l^{-1}(\delta_d - \delta_s)v_y\\ &\quad\times(M\cos\alpha + \sin\alpha).\end{aligned} \tag{20}$$

In Eqs. (20) use was made of the relation for v_z in Eqs. (2). The symbol V denotes the vertical velocity related to the horizontal plane, δ_d is the dry-adiabatic temperature lapse rate and δ_s the saturation adiabatic one. By comparison of Eqs. (20) and (19) and after substituting for v_y and $\partial p/\partial t$ one arrives at the expression

$$M = \frac{(\varepsilon l - c_p T)\,W_s l(f^* U + G)}{(\varepsilon l^2 W_s + c_p R T^2)\,c_p(\delta_d - \delta_s)\cos\alpha} - \operatorname{tg}\alpha, \tag{21}$$

where $\operatorname{tg}\alpha$ is given by the Margules relation (13). The fraction

$$A = \frac{(\varepsilon l - c_p T)\,W_s l}{(\varepsilon l^2 W_s + c_p R T^2)\,c_p \varrho(\delta_d - \delta_s)}$$

must be regarded as a constant for the air both below and above the frontal surface due to the assumptions $M_1 = M_2 = M =$ const and $\varrho_1(f^*U_1 + G) = \varrho_2(f^*U_2 + G)$, so that the quantities W_s and T in Eq. (21) must be replaced by typical mean values which fulfill the condition $A_1 = A_2 = A$. The analysis of this condition shows that it is satisfied if $(T_2/T_1)^2 > W_{s2}/W_{s1}$. Naturally, the value of A and M may vary from case to case.

The parameter M is now expressed in terms of the slope of the original frontal plane and of representative mean values of certain thermodynamical quantities. The slope of the frontal plane is to be considered in the range of synoptically important values of α, thus approximately in the interval $0 < \mathrm{tg}\,\alpha < 1/50$. The assumption $M =$ const obviously does not permit adequate description of the development of the moisture field and serves, together with other simplifications, only to give an approximate solution to the problem. The model is likely to be useful as a first rough theoretical information on the three-dimensional behaviour of the waves under discussion.

From the relations (21) and (14) we get, using again the concise notation, the resultant frequency equation of the model:

$$\gamma = k^{-1}\left\{\tfrac{1}{2}f^* + \left[\tfrac{1}{4}f^{*2} - kg^*\,\mathrm{tg}\,\alpha + \frac{kg^*A\varrho(f^*U+G)}{\cos\alpha}\right]^{\frac{1}{2}}\right\}$$

$$= k^{-1}\left\{\tfrac{1}{2}f^* + \left[\tfrac{1}{4}f^{*2} - kg^*\,\mathrm{tg}\,\alpha + kg^*\frac{(\varepsilon l - c_pT)\,W_s l(G+f^*U)}{(\varepsilon l^2 W_s + c_pRT^2)\,c_p(\delta_d - \delta_s)\cos\alpha}\right]^{\frac{1}{2}}\right\}. \quad (22)$$

The foregoing insight into Eq. (14) indicates that Eq. (22) describes a certain type of modified inertial-shear-gravity waves. Let us deal more thoroughly with the inequality $M < 0$ which represents the necessary condition of instability. The practical form of this inequality reads with respect to Eq. (21)

$$M \approx \frac{(\varepsilon l - c_pT)\,W_s lG}{(\varepsilon l^2 W_s + c_pRT^2)\,c_p(\delta_d - \delta_s)\cos\alpha} - \mathrm{tg}\,\alpha < 0 \quad (23)$$

since $G \gg f^*|U|$ under typical conditions ($\mathrm{tg}\,\alpha > 1/500$, $|U| \leq 20$ m sec^{-1}). The physical interpretation of the inequality (23) seems to be rather simple. The parameter M reflects the influence of three factors controlling the stability of the wave: the destabilizing shear effect expressed by the term $-\mathrm{tg}\,\alpha$ and the combined stabilizing influence of the gravity and the vertical motions acting during the condensation and precipitation process in the saturated air. The stabilizing tendency of the vertical velocities is probably comparable with that described by Kuo (1953) and may be seen directly from Eq. (21). The maximum (and in the saturated air unreachable) tendency to instability would occur with $\tilde{M} = -\mathrm{tg}\,\alpha = \lim_{W_s \to 0} M$. But the vertical velocity related to the horizontal plane, i.e., the quantity $V = v_y(M\cos\alpha + \sin\alpha)$ would now equal zero. Hence unstable waves generally exhibit a quasi-horizontal motion regime (the angle β between the three-dimensional velocity vector and the horizontal plane fulfills the inequality $0 < \beta < \alpha$ for $0 > M > -\mathrm{tg}\,\alpha$). With instability the destabilizing shearing effect must prevail over the combined influence of gravity and precipitation release due to large-scale vertical motions. The fulfillment of the condition (23) depends strongly on the value of W_s while the probability of instability increases with decreasing W_s. Numerical estimates indicate that only negative temperatures (in °C) are "favoured" by unstable waves. Hence the wave development is much preferred (at least in this model) if condensation takes place in the mean at sub-zero temperatures resulting in the formation of supercooled liquid water in the cloud system of the upper wave.

The complete instability criterion based on Eq. (14) reads $k > -(f^{*2}/4Mg^*)$) and defines for $M < 0$ the wave numbers of unstable waves for which the shearing effect is greater than the stabilizing influence of gravity, vertical motion and Coriolis force. For given values of the thermodynamical quantities incorporated in M the amplification of the unstable waves is the greater, the greater the slope of the original frontal plane and the shorter the wave-length. The most intense wave development, vertical velocities and precipitation are thus to be expected in the regions with the greatest slope of the upper front. This conclusion is in agreement with the synoptic experience, at least in middle Europe. Since in the range of unstable waves the role of the phase velocity is taken over by its real part γ_{re}, the unstable waves propagate with a velocity $\gamma_{re} = k^{-1}\Omega\sin(\varphi - \alpha) = f^*/2k$, i.e., independently of the currents $U_{1,2}$, while the wave velocity equals half the phase velocity of inertial waves. From an order-of-magnitude estimate it is immediately obvious that the model allows only slowly moving waves in the unstable region. For $f^* = 10^{-4}$ sec^{-1} and $L = 5 \times 10^5$ m, for example, we obtain $\gamma_{re} \doteq 4$ m sec^{-1}. On the other hand, with $M < 0$, $k < -(f^{*2}/4Mg^*)$ we get stable, relatively fast waves ($\gamma_{re} = \gamma$) fulfilling also the condition $0 < \beta < \alpha$. The stability diagram of the model may be expressed in terms of k and $|M|$. The corresponding graph is a rectangular hyperbola $k = f^{*2}/(4|M|g^*)$. For the critical wave-length separating stable and unstable waves we have

$$L_{cr} = \frac{8\pi|M|g^*}{f^{*2}}, \quad 0 \geqslant M > -\mathrm{tg}\,\alpha. \tag{24}$$

The limiting situation $M = 0$, $v_z = 0$ corresponds to stable inertial waves as shown earlier with the aid of Eq. (14). Approximately, $L_{cr} \propto \sin\alpha$ because of $f^* \approx f$. The graphical representation of Eq. (24) is obvious. The case of stable waves with $M > 0$ seems to be unrealistic. In reality, new short waves occur upstream relative to the warm basic current U_2 which generally requires a negative or zero group velocity $(\partial/\partial k)(k\gamma)$. This is fulfilled with unstable waves or stable waves obeying the conditions $M < 0$, $k < (f^{*2}/(4|M|g^*))$. However, the group velocity is positive for $M > 0$. Therefore, this case is to be excluded.

The assumption $M = \text{const}$ seems to be the most restrictive and artificial simplification of this model and a source of certain logical inconsistencies since it leads to replacing actual values of W_s and T by mean values, as already stated earlier. On the other hand, this "parameterized" approach simplifies the mathematics of the model and permits an analytical solution of the problem. From the physical point of view, the assumption $M = \text{const}$ controls first of all the stability of the waves in such a way that the wave amplification takes place with condensation at sub-zero temperatures, as mentioned above. This property of the model may be compared with reality. Under typical conditions described in (Brádka 1967) the active part of the upper front where precipitation release is pronounced, is located in the height about 4 km and sub-zero temperatures occur there even during the summer period. We thus may conclude that the requirement $M = \text{const}$ is not the best but synoptically rather plausible.

One may ask how is M related to the rate of precipitation. The condensation (precipitation) rate per unit volume is $j = -\varrho(\partial W_s/\partial t)$ so that Eq. (20) yields

$$M = \frac{jl}{\varrho c_p(\delta_d - \delta_s) v_y \cos\alpha} - \mathrm{tg}\,\alpha,\ v_y > 0.$$

The assumption $M = \text{const}$ thus implies, l and $\delta_d - \delta_s$ being approximately constant, that $j(\varrho v_y)^{-1}$ is constant. From the last equation the stabilizing role of the condensation process may be seen again. Hence the model is "contradictory": The condensation due to vertical motions in the saturated air tends to stabilize the wave but, at the same time, pronounced instability connected with strong amplification is generally accompanied by more intense vertical velocities and precipitation than weak instability.

Wave geometry and development

On the plane surface of the earth the relation $V = v_y(M\cos\alpha + \sin\alpha) \equiv 0$ would be valid. Hence we may infer that it is impossible to introduce the wave motions on the earth's surface since in the saturated air $M \neq -\mathrm{tg}\,\alpha$ and a constant value of M is assumed everywhere. Therefore, the model is related theoretically to waves on the sloped interface separating two semi-infinite air masses, and, thus, in practice to waves on upper fronts. Orographic influences cannot be taken into account in a direct way, either. A possible parametric form of the incorporation of orography will be discussed later.

The geometry of the frontal wave follows from Eqs. (11) and (16) which describe the wave-like deformation of the frontal surface. Let us put $y' = y - k^{-1}\ln r(b, 0)$, $z' = z - \sigma(b, 0)$, $\Theta = ka$. In the definition of y' the relation for s of Eqs. (6) is used. From Eqs. (11) we now get, owing to $r(b, 0) = \Gamma(0) = \text{const}$, the parametric equation of a space curve representing, from the physical point of view, a wave-like deformation of any material straight line parallel to the x-axis which lay in the original frontal plane. These curves may be regarded as a sort of generating curves of the frontal surface. Their equation reads for $t = 0$ and for any value of b

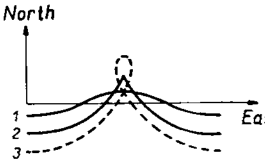

Fig. 2. The possible forms of the front. The curves 1, 2, 3 refer to $t<\tau$, $t=\tau$, $t>\tau$, respectively.

$$x = k^{-1}\Theta + r_0 \sin \Theta, \quad y' = -r_0 \cos \Theta$$

$$z' = -Mr_0 \cos \Theta \tag{25}$$

where $r_0 = r(b, 0) = \Gamma(0)$. Eqs. (25) represent a three-dimensional generalization of similar relations valid for Gerstner's trochoidal waves. The space curve (25) is now represented in projection to the xy'-plane by an abbreviated cycloid $x = k^{-1}\Theta + r_0 \sin \Theta$, $y' = -r_0 \cos \Theta$, $r_0 < k^{-1}$ the mean square slope S of which may be determined from the formula

$$S^2 = \frac{1}{L}\int_0^L \left(\frac{dy'}{dx}\right)^2 dx = 1 - \left[1 - \left(\frac{2\pi r_0}{L}\right)^2\right]^{\frac{1}{2}} < 1.$$

The lengthening of the generating curves due to the initial wave-like impulse may be characterized by means of the quantity P which determines the length of the projection to the xy'-plane over an interval equal to the wavelength. In the present case we get, by virtue of the foregoing relation,

$$P = \int_0^L \left[1 + \left(\frac{dy'}{dx}\right)^2\right]^{\frac{1}{2}} dx = \frac{2L}{\pi}\left[1 + (2S^2 - S^4)^{\frac{1}{2}}\right] \times E\left(\frac{2(2S^2 - S^4)^{\frac{1}{4}}}{1 + (2S^2 - S^4)^{\frac{1}{2}}}\right) < \frac{4L}{\pi},$$

where E denotes a complete elliptic integral of the second kind (Monin, 1967). The relative lengthening of the generating curves due to the initial wave-like deformation then is

$$\frac{P - L}{L} < \frac{4}{\pi} - 1.$$

The projection of the generating curves (25) to the xz'-plane yields

$$x = k^{-1}\Theta + r_0 \sin \Theta, \quad z' = -Mr_0 \cos \Theta. \tag{26}$$

Eqs. (26) describe for $r_0 < k^{-1}$ essentially a modified ("reduced") abbreviated cycloid, the "vertical" amplitude of which is $|M|$-times greater than the "horizontal" one of the curve $x = k^{-1}\Theta + r_0 \sin \Theta$, $y' = -r_0 \cos \Theta$. Thus, the vertical motion is controlled by the thermodynamical factors incorporated in M.

The front in the meteorological sense of the word (i.e., the intersection of the frontal surface and the horizontal plane) would be identical with one of generating curves (25) if $M = \tilde{M} = -\mathrm{tg}\,\alpha$, as indicated by the relation $z' = My'$. Of course, it is not the case due to the existence of vertical motions in the saturated air. However, it illustrates the geometrical meaning of the limiting and fictitious situation of a "dry" wave with a purely horizontal motion field ($W_s \to 0$, $V \to 0$). The possible forms of the front are shown schematically in Fig. 2.

The foregoing discussion describes in essence the geometry of stable waves or the incipient stage of unstable waves. Let us now deal with the geometry of an unstable amplified wave for $t > 0$. From Eqs. (11) we get

$$x' = k^{-1}\Theta + r_0 \exp(k\gamma_{\mathrm{im}} t) \sin(\Theta - k\gamma_{\mathrm{re}} t),$$

$$y' = -r_0 \exp(k\gamma_{\mathrm{im}} t) \cos(\Theta - k\gamma_{\mathrm{re}} t),$$

$$z' = -Mr_0 \exp(k\gamma_{\mathrm{im}} t) \cos(\Theta - k\gamma_{\mathrm{re}} t). \tag{27}$$

In Eqs. (27) the following abbreviations are used: $x' = x - Ut$, $\gamma_{\mathrm{im}} = k^{-1}[-(\frac{1}{4}f^{*2} + kMg^*)]^{\frac{1}{2}}$. During the eastward motion of the wave its amplitude increases. As soon as the wave is amplified (for $t = \tau$) to the value $r_0 \exp(k\gamma_{\mathrm{im}}\tau) = k^{-1}$, the original abbreviated cycloid obtains the shape of the simple cycloid in the $x'y'$-plane and the cusp point develops. A similar transformation applies in the $x'z'$-plane to the modified cycloid the "tip" of which is now pointed downward with $M < 0$. For prescribed r_0, τ may be estimated from the relation $\tau = (k\gamma_{\mathrm{im}})^{-1} \ln (kr_0)^{-1}$. If, for the sake of illustration, the limiting case $M = \tilde{M} = -\mathrm{tg}\,\alpha$ is mentioned again, the front is essentially a simple cycloid, too, with the "tip" pointed northward (cf. Fig. 2, curve 2). A certain geometrical similarity with the beginning of the occlusion process may be seen here for $M < 0$. One thus may conclude that the case $M < 0$, i.e., relatively short, slow, unstable waves ($k > -(f^{*2}/(4Mg^*))$ and relatively long, relatively fast, stable waves fulfilling the condition $k < -(f^{*2}/(4Mg^*))$, might have some similarity with reality. We note that the discussion of the geometry of the waves leads again to the conclusion that only the case $M < 0$ is to be considered. From the more general point of view, the formation of the cusp point during the development of the unstable waves may be regarded as the most important feature of the internal dynamics of the waves under consideration. The model indicates that it is possible to describe to a certain degree the qualitative change in the shape of unstable frontal waves, even within the framework of some "naive" and crude assumptions.

The path of an individual frontal particle ($\Theta = \text{const}$, $c = 0$) may be characterized with respect to Eqs. (27) by the equation of an ellipsoid

$$\frac{(x' - k^{-1}\Theta)^2}{r_0^2 \exp(2k\gamma_{\text{im}} t)} + \frac{y'^2}{2r_0^2 \exp(2k\gamma_{\text{im}} t)} + \frac{z'^2}{2M^2 r_0^2 \exp(2k\gamma_{\text{im}} t)} = 1. \qquad (28)$$

We recognize here again a certain generalization of Gerstner's model. The length of the half-axes depends on time since the ellipsoid "expands" with unstable amplified waves. Hence the cusp point formation may be described by means of the corresponding lengths of the half-axes. For $t = \tau$ we find from Eq. (28) that the half-axes in x', y' and z'-direction are equal to k^{-1}, $\sqrt{2}k^{-1}$ and $\sqrt{2}|M|k^{-1}$, respectively.

Let us return to the geometry of the wave the further development of which ($t > \tau$) is again indicated by Eqs. (27). For $r_0 \exp(k\gamma_{\text{im}} t) > k^{-1}$ an elongated cycloid develops in the $x'y'$-plane characterized by a loop in which the warm air is trapped. An analogical development takes place in the $x'z'$-plane. The front in a fictitious "dry" wave ($M = \tilde{M} = -\text{tg}\,\alpha$, $V = 0$) is represented by an elongated cycloid with the loop of the warm air on its northern side (cf. Fig. 2, curve 3). The loop gradually spreads northward during the further amplification of the wave. Of course, the physical validity of this model ends, with unstable waves, probably in the stage $t = \tau$ when the "tip" develops. With the formation of the cusp point on the front the dissipative effects, which are not considered in this model, begin to play a substantial role due to the enormous curvature of the front in the vicinity of the "tip". The above solution also does not represent the typical asymmetry of the wave which in reality we usually observe, and is in this respect less general than the numerical solution of a similar problem studied in (Kasahara, 1965). On the other hand, the present paper corrects, improves and extends the author's earlier, preliminary considerations (Vítek, 1965, 1967).

Let us estimate, for the sake of orientation, some typical values describing the development of an unstable wave in this model. We put $T = -5°\text{C}$, $W_s = = 0.004$, $\delta_s/\delta_d = 0.6$, $\text{tg}\,\alpha = 1/130$. Now, $M \doteq -10^{-3}$ and practically all waves are unstable. For $f^* = 10^{-4}$ sec^{-1} and a very short wave $L = 10^5$ m with a finite initial amplitude $r_0 = L(10^3\pi)^{-1}$ we have $\tau \doteq 7940$ sec. For $L = 8 \times 10^5$ m, τ equals, ceteris paribus, approximately 22 720 sec. These values probably correspond to a moderate degree of instability since even lower temperatures may occur in the vicinity of the upper front. The vertical velocity related to the horizontal plane is $V = v_y(M \cos\alpha + \sin\alpha) \approx v_y(M + \text{tg}\,\alpha)$. Obviously, V has the same sign as v_y. For the present values of M and $\text{tg}\,\alpha$, $V \doteq 0.0067\ v_y$, where the numerical coefficient is independent of L. The critical wave-length separating stable and unstable waves may lie, according to Eq. (24), within the interval $10^5 \leqslant L < 10^6$ m if $-M < 4 \times 10^{-5}$. Hence both stable and unstable waves are allowed in the model, the stable ones being possible for very small values of $|M|$.

The wave motion studied is rotational. To prove this a slight modification of the procedure described by Lamb (1932) may be used. The expression $v_x \delta x + v_y \delta_y = \delta[(U - \gamma) r(b, c) \sin k(a - \gamma t)] + (U - \gamma k^2 r^2(b, c)]\delta a$ is not a total differential so that the circulation equals $2\gamma k^2 r(\partial r/\partial b)\delta a\,\delta b$ and the vorticity of the element (a, b) is

$$\xi = \frac{2\gamma k^2 r}{[\partial(x, y)]/[\partial(a, b)]}\frac{\partial r}{\partial b}.$$

Substituting for x and y from Eqs. (1) and using the first relation of Eqs. (6) we get

$$\xi(b, c) = 2\gamma k^3 r^2(b, c)\ [1 - k^2 r^2(b, c)]^{-1}. \qquad (29)$$

Eq. (29) is a relation analogous to the one for Gerstner's trochoidal (rotational) waves. Vorticity is conserved, i.e., air particles having the same b and c have the same vorticity for each t. On the frontal surface the vorticity conservation also takes place, as may be seen from Eqs.

(16). Obviously, $\xi(b, 0) = \text{const}$. The vorticity equation (10) states that vorticity conservation is connected with the non-divergent flow. The formula (29) makes the relative vorticity infinite for $r = k^{-1}$ but this case is practically meaningless. For any realistic value of $r(b, c)$ fulfilling the inequality $r < k^{-1}$ we have $\xi(b, c) > 0$. Hence anti-cyclonic vorticity cannot be generated in the area of the wave. Eq. (29) indicates that the waves discussed are generally connected with weak cyclonic vorticity. This conclusion seems to be in accordance with synoptic experience.

Concluding remarks

The whole "dynamic" part of the model, i.e., Eqs. (1)–(14), was constructed on the basis of some generalization of Gerstner's assumptions. But the model may alternatively start out from the vorticity equation (10). It is sufficient, in order to satisfy Eq. (10), that the velocity components be chosen so that $D \equiv 0$, $\partial\xi/\partial t \equiv 0$. This, within the general physical concept of the model, leads back to Eqs. (1), to the condition of integrability (6) and to the continuity equation (9). Finally, the integration of Eqs. (4) produces Eq. (8). The further discussion is the same as previously. The model represents a special case of the integration of the primitive equations in the Lagrangian form independently of the starting point but the procedure based directly on Eqs. (1) seems to be more natural and corresponds basically to the standard explanation of Gerstner's rotational waves.

The model does not account for orographic influences which may modify significantly the vertical velocity pattern. In spite of it, the orographic or other "local" phenomena may be introduced indirectly in a simplified parametrical form. Let us put $z(a, b, c, t) = c + \sigma(b, c) - Mr(b, c) \cos k(a - \gamma t) + h(t)$. Hence

$$v_z = -Mk\gamma r(b, c) \sin k(a - \gamma t) + \frac{\partial h(t)}{\partial t} = Mv_y + \frac{\partial h}{\partial t}. \tag{30}$$

The integration of Eqs. (4) is unaffected by Eq. (30) if $h(t)$ is a linear function of t or if

$$\left|\frac{\partial^2 h}{\partial t^2}\right| = \left|\frac{d^2 h}{dt^2}\right| < g^*.$$

Equation of continuity (3) is not violated, either. But Eqs. (17) and (18) now indicate that the integration constant in the relation for $p(a, b, c, t)$ is to be replaced by an integration function $q(t)$ so that

$$\frac{(\varepsilon l - c_p T)\, W_s}{(\varepsilon l^2 W_s + c_p R T^2)\varrho} \frac{\partial q}{\partial t} = -\frac{c_p}{l}(\delta_d - \delta_s) \cos\alpha \frac{\partial h}{\partial t}. \tag{31}$$

The stability properties of the model are unchanged by the condition (31). Now the additional vertical velocity $v_{za} = \partial h/\partial t$ and other additional quantities $\partial p_a/\partial t = \partial q/\partial t$ and $\partial W_{sa}/\partial t$ may be defined as follows:

$$v_{za} = -\frac{A}{\cos\alpha}\frac{\partial p_a}{\partial t} = -\frac{l}{c_p(\delta_d - \delta_s)\cos\alpha}\frac{\partial W_{sa}}{\partial t}. \tag{32}$$

As a consequence of Eq. (32) and the modelling assumptions used, the additional vertical velocity may be interpreted either as an individual pressure change or as an additional sink for water vapour. This is due to the fact that the condensation rate per unit volume generally is $-\varrho(\partial W_s/\partial t)$. The "local" component v_{za} which represents a uniform contribution to the "dynamic" vertical velocity, is to be specified according to need. Alternatively, the additional water vapour sink may be prescribed which is to be regarded as homogeneous in space. One assumes tacitly a positive sink $v_{za} > 0$ as long as saturated conditions are considered. For sufficiently large values of v_{za}, i.e., for $v_{za} > |M| k\gamma r$, the whole body of the wave may be regarded as saturated and the thermodynamical relations (17)–(20) are valid everywhere.

It is well known from the synoptic analysis that the short waves often pass along the upper front without any pronounced increase in amplitude. This property is not in contradiction with the model since both stable and unstable waves are allowed. But in the formative period of the wave development a certain degree of instability probably does exist in any case. Unfortunately, it is difficult to formulate the transition from the unstable stage to the stable one of an individual wave in a quantitative way since the above solution fails to give any information as to what takes place during the development of the cloud system of the wave. One process which necessarily tends to stabilize the wave, may be evaporation from falling precipitation, provided the cloud system is only partly saturated. This results in additional cooling of the ascendent air mass. The stabilizing effect may be seen in terms of M in the very first approximation as follows. Eq. (20) is now to be replaced by the following relation (Marchuk 1967):

$$\frac{\partial W_s}{\partial t} = -c_p l^{-1}(\delta_d - \delta_s)V + n(\bar{m} - m)u. \quad (33)$$

In the last equation, n is a positive coefficient, u is the "specific humidity" of water and m the water vapour density. The bar denotes the maximum value of m. Let us put schematically and for the sake of simplicity $\bar{m} - m = \text{const}$, $u \propto V$. We then have from Eq. (33)

$$\frac{\partial W_s}{\partial t} = [-c_p l^{-1}(\delta_d - \delta_s) + N]V, \quad N > 0. \quad (34)$$

Eq. (34) is based on a very simplified concept of superposition and cannot be exactly valid. But it shows at least that the probability of relatively large negative values of M is reduced. Hence, one may expect that the cooling of the ascending air due to evaporation from falling precipitation may contribute, in a somewhat later stage of development, to the stabilization of the wave. The complete discussion of these questions is complicated since, inter alia, a change in wave-length may simultaneously take place in real situations. Some qualitative arguments indicate that the stabilization of the wave may result in the acceleration of the wave propagation, provided a train of simple, originally unstable waves was generated earlier. Let us assume that the first wave induced was transformed into a stable one. Accordingly, the first wave is moving faster. The consequence is that all the following unstable waves are continually being drawn out in length so that their velocities of propagation gradually increase as they advance. At the same time the probability of the transformation into stable waves increases due to the drawing out in length. This effect has the same sense as the stabilizing effect of evaporation. A similar type of acceleration due to the difference in wave-length is known from the theory of surface waves.

Summing up we may say that some basic characteristics of the short waves on upper fronts may be interpreted with the aid of a certain generalization of Gerstner's rotational waves. The modelling assumptions used involve the vorticity conservation in the non-divergent flow but the existence of unstable amplified waves is allowed. During the development of the wave the cusp point is formed on the front. The model cannot describe the non-linear interactions between the thermodynamical and dynamical effects in spite of the fact that the governing equations (3) and (4) are essentially nonlinear. This is unsatisfactory, but perhaps the best that one can expect from the assumption $M = \text{const}$ and other simplifications used. It is felt that, in order to obtain a more consistent model, the thermodynamics of the waves in question should be considered more thoroughly. We hope that the present model provides a framework to do so. From the more general point of view, we believe that the modified Gerstner's concept could be used even for more detailed numerical investigations of the frontal systems which are among the major factors in weather forecasting.

Acknowledgement

The present paper was elaborated during a year's stay at the Atmospheric Research Section, Meteorological Branch, Department of Transport, in Toronto, Canada. The author is grateful to the National Research Council of Canada for its award of a Postdoctorate Fellowship which gave an opportunity to work in a stimulating and generous milieu.

The author wishes to express his gratitude to Dr. M. Shabbar for reading the manuscript and offering valuable suggestions.

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ПРОСТАЯ МОДЕЛЬ ОДНОГО ТИПА ФРОНТАЛЬНЫХ ВОЛН В АТМОСФЕРЕ

Выводится простая модель для волн малой амплитуды на верхних фронтах позади квазистационарных поверхностей холодных фронтов. В модели частично учитывается выделение скрытой теплоты. Анализ уравнения для частот показывает, что наибольшего развития волн следует ожидать в областях с наибольшим наклоном верхнего фронта. Во время развития неустойчивой волны на фронте формируется остроугольная точка. В модели возможны и устойчивые волны. Влияние орографии также может быть учтено в модели в некоторой упрощенной параметрической форме.