An attempt to solve the Geodetic Boundary Value Problem, using a double layer approach for Malkin's function

Stokes' accuracy accepted

By ERIK TENGSTRÖM, *Geodetic Institute, Uppsala University, Sweden*

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ABSTRACT

The Boundary Value Problem of Physical Geodesy is usually treated as the problem of finding reliable corrections to be applied to an accepted ellipsoidal model of the Earth in order to get the true values of the gravitational force and the shape of its level surfaces in external space. The accuracy in determining these corrections corresponds, as a rule, to neglecting errors of the order of the flattening of the Earth (3 ‰ of obtained values). This first approximation identifies the Earth's topography with the so-called *Telluroid* (see Fig. 1 and below), and uses as only boundary values s.c. *true free air anomalies* (see not:s).

To study the solvability of this problem, integral equations are well suited. For numerical work, they are now-a-days not difficult to handle, because of the availability of powerful high-speed computers. As to my opinion, there is not yet any fully satisfactory solution obtained, which fulfills all boundary conditions. If it should turn out, that no available harmonic models can satisfy the surface-condition *and* the infinity-conditions (see below) with sufficient accuracy (see above), we have to admit, that we need additional information along the Earth's surface to be able to solve this important problem, by reducing it to the level case of Stokes (see Appendix 1).

The paper describes a direct use of a double-layer approach for the s.c. *Malkin's function* H_y (see below), from which all wanted corrections in space may be deduced. This approach is suggested, because the solvability condition of the resulting *singular* Fredholm-equation can be used for defining, beforehand, the correct model gravity γ. The solution, by definition of the double layer potential, satisfies the weaker infinity condition

$$\lim_{r\to\infty} rH_y = 0$$

If it also satisfies the stronger condition

$$\lim_{r\to\infty} r^2H_y = 0$$

accurately enough, ought to be investigated further.

Appendix 1 deals with the simpler problem of a purely oceanic Earth. It was mainly written to *explain* theoretically the procedure, when solving the general case (arbitrary topography) with double layer approach.

Appendix 2 develops several formulas, which could be used in numerical computations, when dealing with the general case.

Appendix 3 gives an additional explanation of the relationship between model gravity and the measured gravity along the Earth's surface.

Notations

Σ	Mean Earth sphere with radius R
S	Telluroid
σ	Unit sphere
P, Q	Points on telluroid with radii r_P, r_Q
$P_\Sigma = P', Q'$	Orthogonal projections from P, Q to Σ
P_y	External point with radius r
n_P, ν_Q	Normals to telluroid at P and Q
β	Slope of telluroid
$\operatorname{tg} \beta_\psi$	Tangent of slopecomponent in direction of ψ, angular radius of r_Q from r_P. Observe its sign! $\operatorname{tg} \beta_\psi = (\partial H_Q / R \partial \psi)$
Δ_y	Distance $P_y - Q$,
Δ_s	Distance $P - Q$; $D = \Delta_s / r_P = \sqrt{\mu^2 - 2\mu \cos \varphi + 1}$ where $\mu = r_Q / r_P$, or generally in D_y, $\mu = r_Q / r$
Δ_Σ	Distance $P' - Q'$
Δ_σ	$\Delta_\Sigma / R = 2 \sin \psi / 2$
H_P, H_Q	Normal heights of P and Q
A, α	Azimuth of ψ (α also flattening of Earth Ellipsoid)
Φ, ν	Momentum of double layer (φ also density of single layer)
$\Delta g_F(P)$	True free air anomaly $g(P) - \gamma(P_e)$, P_e point on spherop with same potential as P has on its geop.
γ_e	Equator value of γ_0, the normal gravity on Ellipsoid
T_y, H_y	Disturbing potential, resp. Malkin's function at P_y
Y_N, P_N	General spherical harmonic, resp. Legendrian of order N
k^2	Gravitational constant $6.67 \cdot 10^{-8}$ cgs; $k^2 M$: Mass constant of Earth
$(U_n^{(\nu)}, \lambda_n)$	Characteristic system for a normal kernel $K(P, Q)$
N'_P	Height anomaly at P
ξ_P, η_P	Deflections of the vertical at P

The problem in Geodesy of finding the s.c. disturbing potential T, which is equal to the difference between the Newtonian potential for the Earth and the Newtonian potential of the ellipsoidal model, at one and the same point in space, has been treated in different ways during the last 30–40 years, using as boundary values the true free air anomalies Δg_F along the Earth's surface without making reductions, containing unknown densities and curvatures.[1]

Molodenskij *et al.* tries to determine T directly as a single layer potential, the density of which satisfies a *regular* integral equation of Fredholm's type (1962). Here the infinity condition

[1] *Important remark:* The effect on Δg_F of now-a-days errors in geocentric positions of P and Q (see figure) can always be neglected when deriving T and its directional derivatives to within the accuracy of Stokes. Cf. also Aardoom's paper in this number of Tellus.

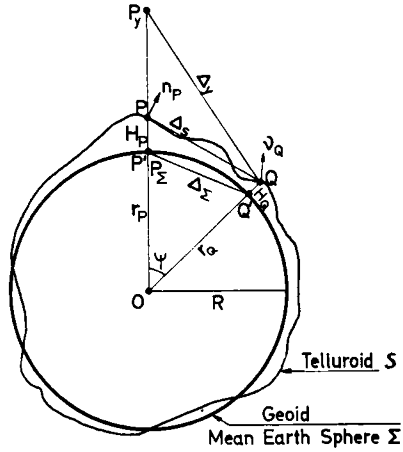

Fig. 1. In the first approximation (accuracy of Stokes), the *Geoid* and *Ellipsoid* are identified with same sphere Σ, the surface of the Earth with the *Telluroid*. O is the gravity centre of the Earth $\equiv$ geometrical centre of the Ellipsoid. The angular positions of P' and Q' on Σ are equal to ellipsoidal latitude and longitude of P and Q, the normal heights H_P and H_Q of which define the Telluroid.

$$\lim_{r\to\infty} rT = 0 \tag{1}$$

cannot be ascertained in advance, because Φ in

$$T = \int_S \frac{\Phi}{\Delta_y} dS \tag{2}$$

must be solved, before the equator gravity of the model γ_e can be computed, S being the s.c. Telluroid, or, the surface created by points at distance H (normal height) from the mean Earth sphere.

$$\Phi(\Delta g_F, S) \quad \text{must fulfil} \quad \int_S \Phi dS = 0 \tag{1 a}$$

But it must also fulfil the stronger condition

$$\lim_{r\to\infty} r^2 T = 0 \tag{3}$$

$$\int_S \Phi Y_1 dS = 0 \tag{3 a}$$

where Y_1 is a general spherical harmonic of first order. It has not yet been proved, that the Δg_F also satisfy (3a), even if Δg_F is correct, according to (1a).

Bjerhammar (1960–63) (cf. Aranov & Arnold) makes use of a formal "downward continuation" to define from Δg_F and topography a new anomaly Δg^* along the mean Earth sphere, satisfying the (simplified) integralequation of the first kind

$$\Delta g_F = \frac{H}{2\pi} \int_\Sigma \frac{\Delta g^*}{\Delta^3_{PQ'}} d\Sigma \tag{4}$$

If this equation can be solved, or, rather the equation

$$\Delta g_F = \frac{1}{4\pi} \int_\sigma \sum_{n=2}^{\infty} (2n+1)\mu^{n+2} P_n \Delta g^* d\sigma, \tag{4 a}$$

Δg^* gives by means of Stokes' generalized formula

$$T = \frac{r}{4\pi} \int_\sigma \left(\sum_{n=2}^{\infty} \frac{2n+1}{n-1} P_n \mu^{n+2} \right) \Delta g^* \, d\sigma$$

the correct disturbing potential in empty space. It is, even in the simplified case of a telluroid with Lyapunow-properties, still unclear, under which circumstances (4a) may be solved for an analytical Δg_F-model. Starting with (4) and defining an analytical Δg^*-function, which, through (4), gives rise to an analytical Δg_F, coinciding with measured values in a finite number of discrete points, the problem is solvable, but the result depends on the form of the function, defined by a finite number of coefficients. In this case, the infinity conditions

$$\int_\sigma \Delta g^* \, d\sigma = 0 \quad \text{and} \quad \int_\sigma \Delta g^* Y_1 \, d\sigma = 0$$

must also be proved to hold.

Brovar (1964) uses *Malkin's function*

$$H = \frac{1}{r} \frac{\partial}{\partial r} (r^2 T),$$

and forms a Fredholm equation from the approach ($n_s = \nu_Q$ in not:s and Fig. 1)

$$H(P_y) = \frac{1}{2\pi} \int_S \nu(Q) \frac{\partial}{\partial n_S} \left(\frac{1}{\Delta_y} \right) dS_Q + \frac{1}{2\pi} \int_S \frac{\mu(Q)}{\Delta_y} dS_Q \tag{5}$$

the sum of a double layer and a single layer.

With
$$\mu = \frac{\nu \cos (n_S r_S)}{2 r_S}$$

the total kernel disappears, if $S \to \sum$ (Stokes' spherical case).

The resulting integral equation in

$$\delta g = -\frac{\nu r_S}{R^2} \tag{5a}$$

is regular, and gives for any Δg_F distribution a unique solution, as in Molodenskij's case. δg must, however, satisfy the infinity conditions

$$\int_\sigma \delta g \, d\sigma = 0 \tag{5b}$$

$$\int_\sigma \delta g \left(\tfrac{3}{2} \cos \psi + \frac{1}{R} \sin \psi \frac{\partial H}{\partial \psi} \right) d\sigma = 0 \tag{5c}$$

γ_e cannot be defined (cf. Molodenskij) before the integral equation has been solved for the whole Earth. The solution

$$\delta g(\Delta g_F, S)$$

is naturally very elaborous to obtain. The validity of (5c) is still questionable. T is, of course, derived from the definition of H, integrating with respect to r and watching the infinity conditions for T.

Because of the difficulty to define γ_e before starting to work on H (or T), which characterises the previous approaches, I have tried to utilize, for H, the double layer directly:

$$H_{Py} = \int_S \Phi(Q) \frac{\partial}{\partial \nu}\left(\frac{1}{\Delta_y}\right) dS \tag{6}$$

with the infinity condition

$$\lim_{r\to\infty} rH = 0 \tag{1'}$$

automatically fulfilled.

It would seem logical to try to find a harmonic function, which a priori also fulfils the stronger condition

$$\lim_{r\to\infty} r^2 H = 0 \tag{3'}$$

It can be shown, however, that no multipole potential of second order, which should obey both the necessary infinity conditions, can approach the limits along the surface, given by

$$\lim_{\substack{r\to r_P \\ P_y\to P}} H_{Py} = -r_P \Delta g_F(P)$$

The form of such a potential would be

$$\int_S \mu \frac{\partial^2}{\partial \nu_Q^2}\left(\frac{1}{\Delta_y}\right) dS_Q$$

It neither defines a Fredholm equation, nor a meaningful equation of the first kind. The kernel

$$\frac{\partial^2}{\partial \nu^2}\left(\frac{1}{\Delta_S}\right) = \frac{3\cos^2(\Delta_S \nu_Q) - 1}{\Delta_S^3}$$

does not satisfy the demands of kernels with improper singularity!

The integral equation of the *double layer* is from (6)

$$2\pi\Phi(P) - \int_S \frac{\cos(\Delta_S \nu_S)}{\Delta_S^2} \Phi(Q)\, dS_Q = -r_P \Delta g_F(P) \tag{7}$$

The equation has an *improper* singularity for $\Delta = 0$; S is assumed to be a Lyapunow-surface model with exponent α (positive), and we see, that the kernel always satisfies

$$\left|\frac{\cos(\Delta_\nu \nu_S)}{\Delta_S^2}\right| \leqslant \frac{|C|\,\Delta_S^{\alpha(1-k)}}{\Delta_S^{2-\alpha k}}$$

($k < (2/\alpha)$; $k < 1$), which means, that it is absolutely less than the quotient between a continuous function, going to zero with Δ_S, and a power of Δ_S with an exponent *smaller* than 2. (7) may therefore be treated, using all theorems of Fredholm.

The homogeneous equation to (7) is

$$2\pi\Phi(P) - \int_S \frac{\cos(\Delta_S \nu_S)}{\Delta_S^2} \Phi(Q)\, dS_Q = 0. \tag{8}$$

It has only one linearly independent solution, a constant $\varkappa$, because

$$\int_S \varkappa \frac{\cos(\Delta_S \nu_S)}{\Delta_S^2} dS_Q = 2\pi\varkappa.$$

Remark: it can be shown, that the kernel of (7) is a *normal* one, wherefore the solution, if it exists, can be expressed by means of the resolvent of Schmidt in known way, using the welldefined system of characteristic functions and eigenvalues, obtainable by successive approximations (same theorems as for a Hermitian kernel are valid). A simplified procedure of getting such a solution may be illustrated, using the spherical case, see appendix 1!

In order to obtain a solution of the *singular* integral equation (7), $-r_Q g_F(Q)$ must be orthogonal to the one only existing, linearly independent solution $\mu(\varphi, \lambda)$ of the *transposed* homogeneous equation

$$2\pi\mu(P) + \int_S \frac{\cos(\Delta_S n_S)}{\Delta_S^2} \mu(Q)\, dS_Q = 0. \tag{9}$$

that is

$$\int_S \mu(Q)\, r_Q \Delta g_F(Q)\, dS_Q = 0. \tag{10}$$

With Stokes' accuracy, (10) may be written as

$$\int_\sigma \chi(Q)\, \Delta g_F(Q)\, d\sigma_Q = 0 \tag{10 a}$$

and (9)—after some algebraic computations—as

$$2\pi\chi(P) - \int_\sigma \frac{1}{\Delta_{PQ}^3} r_Q^3 \left\{\frac{r_P}{r_Q} - \cos\psi_{PQ} + \frac{1}{r_P}\frac{\partial H_P}{\partial \psi} \sin\psi_{PQ}\right\} \chi(Q)\, d\sigma_Q = 0 \tag{9 a}$$

where now $$\chi = \mu \sec\beta.$$

χ is proportional to a characteristic function, and the simple successive approximation method by *Schmidt* can by conveniently applied. In the spherical case, (8) and (9) are identical (real Hermitian = symmetrical kernel), and therefore the first approximation $\chi_1 = 1$ is practical. The second approximation is then

$$\chi_2 = +\frac{1}{2\pi}\int_\sigma \frac{1}{\Delta_{PQ}^3} r_Q^3 \left\{\frac{r_P}{r_Q} - \cos\psi_{PQ} + \frac{1}{r_P}\frac{\partial H_P}{\partial \psi} \sin\psi_{PQ}\right\} d\sigma_Q. \tag{9 b}$$

The right number might be written

$$1 + \frac{1}{2\pi}\int_\sigma \left\{\tfrac{1}{2} r_Q \left(\frac{\Delta_{\Sigma Q}^2}{\Delta_{PQ}^3} - \frac{1}{\Delta_{\Sigma Q}}\right) + \frac{r_Q^3}{\Delta_{PQ}^3}\left(\frac{H_P - H_Q}{r_Q} + \frac{\partial H_P}{r_P \partial \psi} \sin\psi_{PQ}\right)\right\} d\sigma_Q,$$

$$\chi_2 = 1 - \frac{3}{8\pi}\int_\sigma r_Q \frac{(H_P - H_Q)^2}{\Delta_{PQ}^3}\, d\sigma + \frac{1}{2\pi}\int_\sigma \frac{r_Q^3}{\Delta_{PQ}^3}\left(\frac{H_P - H_Q}{r_Q} + \frac{\partial H_P}{r_P \partial\psi}\sin\psi_{PQ}\right) d\sigma_Q$$

$$= 1 + S_1(\varphi, \lambda) + S_2(\varphi, \lambda) \tag{9c}$$

Here S_1 is a local correction term, varying with P but always negative, S_2 is also local and varying with position, but it also varies in sign. As known, the s.c. terra-incorrection is a positive local correction, the expression of which (constant density ϱ assumed) is

$$C_t = \tfrac{1}{2} k^2 \varrho R^2 \int_\sigma \frac{(H_P - H_Q)^2}{\Delta_{PQ}^3} d\sigma_Q + \dots \tag{11}$$

Therefore, in such a case, S_1 may be written

$$S_1 \cong -\frac{3}{4\pi}\frac{C_t}{k^2 \varrho R} \tag{11a}$$

which shows the role of C_t in deriving γ_e [2]. Probably the function $S_1 + S_2$ and its further approximations for the real Earth is very small and will not change appreciably the value of γ_e, obtained by putting

$$\int_\sigma \Delta g_F \, d\sigma = 0 \quad \text{(cf. spherical case)}$$

where now
$$\Delta g_F = g - \gamma_e f_0 + f_H, \tag{12}$$

with
$$f_0 = 1 - 0.0052884 \sin^2\varphi_e - 0.0000059 \sin^2\varphi_e$$

$$f_H = (0.30878 - 0.00045 \sin^2\varphi_e)\, H - 0.0000000725\, \mathrm{H}^2$$

corresponding to a model with $\alpha = 1/297{,}0$ and $a = 6{,}378{,}400$ m. For the correct treatment of the B.V.-problem it is, however, necessary to take it into account, if its effect on γ_e is bigger than some tenths of a mgal.

Using anomalies, defined by g, S and correct γ_e according to (10a), we may obtain solutions Φ of (7), and these solutions give H_y-functions, which satisfy the weaker infinity condition (1). But a correct solution must also satisfy (3′), which means, that the following property of $\Phi(\Delta g_F, S)$ should be true. It has been derived in a similar manner as (9a), letting r in $r^2 H_y$ go to infinity. We obtained

$$\int_\sigma \Phi(Q)\left\{\cos\psi_{PQ} + \frac{\partial H_Q}{r_Q \partial\psi}\sin\psi_{PQ}\right\} r_Q^2 \, d\sigma_Q = 0. \tag{13}$$

Equation (7) can be written

$$\Phi(P) - \frac{1}{4\pi}\int_\sigma 2\frac{r_P r_Q^2}{\Delta_S^3}\left\{\frac{r_Q}{r_P} - \cos\psi_{PQ} - \frac{1}{r_Q}\frac{\partial H_Q}{\partial\psi}\sin\psi_{PQ}\right\}\Phi(Q)\, d\sigma_Q = -\frac{r_P}{2\pi}\Delta g_F(P). \tag{7a}$$

According to the properties of the kernel in (7), the solution of (7a) may be expressed as a Schmidt-series, from a derivable system of characteristic functions and eigenvalues (page 006). In practical work, the solution is obtained directly by a convenient successive approximation-procedure.

As we have accepted Stokes' accuracy, (13) and (7a) may be simplified to

$$\int_{\psi} \Phi(Q)\left\{\cos\psi_{PQ} + \frac{\partial H_Q}{R\partial\psi}\sin\psi_{PQ}\right\} d\sigma_Q = 0 \tag{13a}$$

and

$$\Phi(P) - \frac{1}{4\pi}\int_{\sigma}\frac{2R^3}{\Delta_S^3}\left\{1-\cos\psi_{PQ} - \frac{1}{R}\left(H_P - H_Q + \frac{\partial H_Q}{\partial\psi}\sin\psi_{PQ}\right)\right\}\Phi(Q)\,d\sigma_Q = -\frac{R}{2\pi}\Delta g_F(P). \tag{7b}$$

If we could reduce Δg_F from the telleroid to the mean Earth sphere, according to the solution of the integral equation of the first kind (Bjerhammar, 1963; cf. also Tengström, 1965)

$$\Delta g_F(P) = \frac{1}{4\pi}\int_{\sigma}\sum_{n=2}^{\infty}(2n+1)\mu^{n+2}P_n(\Delta g_F(Q) + G(Q))\,d\sigma_Q, \tag{14}$$

μ being equal to $R/(R+H_P)$, the result $\Delta g_F + G$ could be used to construct the correct T_y above the topography by means of Stokes' generalized formula (see appendix 1). In this case, the infinity-conditions

$$\int_{\sigma}(\Delta g_F + G)\,d\sigma = 0$$

and
$$\int_{\sigma}(\Delta g_F + G)\,Y_1\,d\sigma = 0$$

would be valid.

From the anomaly-distribution $\Delta g_F + G$ along the mean Earth sphere, the correct H_y along the telluriod and everywhere in space might be computed as a double layer potential of momentum φ, satisfying

$$\varphi(P) - \frac{1}{4\pi}\int_{\sigma}\varphi(Q)\frac{1}{\Delta_\sigma}\,d\sigma = -\frac{R}{2\pi}\{\Delta g_F(P) + G(P)\}. \tag{15}$$

Here, obviously, φ/R is an angular function, distributed along the unit sphere. Now, (7b) can be transformed to

$$\Phi(P) - \frac{1}{4\pi}\int_{\sigma}\Phi(Q)\frac{1}{\Delta_\sigma}\,d\sigma = -\frac{R}{2\pi}\Bigg[\Delta g_F(P) + \int_{\sigma}\frac{R}{\Delta_S^3}\left\{H_P - H_Q + \frac{\partial H_Q}{\partial\psi}\sin\psi_{PQ}\right\}\Phi(Q)\,d\sigma + \int_{\sigma}\frac{3}{4}\frac{(H_P-H_Q)^2}{\Delta_S^3}\Phi(Q)\,d\sigma\Bigg]. \tag{16}$$

The sum of the integrals ($g(P)$) inside [ ] cannot be equal to $G(P)$, because $\Phi \neq \varphi$, Φ being distributed along the telluroid, φ along the mean sphere and producing the same H_y. So, forming anomalies

$$\Delta g_F^{(n)}(P) = \Delta g_F(P) + g^{(n-1)}(P),$$

starting with

$$\Delta g_F^{(1)}(P) = \Delta g_F(P) - \frac{R^2}{2\pi}\left\{\int_\sigma \frac{\Delta g_F(Q)}{\Delta_S^3}\left(H_P - H_Q + \frac{\partial H_Q}{\partial \psi}\sin\psi_{PQ}\right)d\sigma \right.$$

$$\left. + \frac{3}{4R}\int_\sigma \frac{\Delta g_F(Q)}{\Delta_S^3}(H_P - H_Q)^2 d\sigma\right\} \tag{18}$$

the equation and its successors

$$\Phi(P) - \frac{1}{4\pi}\int_\sigma \Phi(Q)\frac{1}{\Delta_\sigma}d\sigma = -\frac{R}{2\pi}\Delta g_F^{(1)}(P) \tag{19}$$

cannot be solved by means of the Schmidt-series for the spherical case (see appendix 1), using the characteristic functions and eigenvalues of the spherical kernel $1/\Delta_\sigma$, because the condition of solvability

$$\int_\sigma \Delta g_F^{(n)} d\sigma = 0$$

is contradictory to the condition, already fulfilled

$$\int_\sigma \Delta g_F \chi d\sigma = 0.$$

However, as a solution Φ exists, (7 b) can be solved, using e.g. an appropriate approximation method. The result is a moment-function, which should be applied to the surface of the telluroid, giving

$$H_y = \int_S \Phi(Q)\frac{\partial}{\partial \nu_S}\left(\frac{1}{\Delta_y}\right)dS.$$

When (14) is solvable, we have no difficulties. Using $\Delta g_F + G$ in Stokes' generalized formula, our problem is unravelled. If discrete measurements are used to secure a solution of (14), this means an indefinite interpolation-process, and so an indefinite solution is obtained. I have tried to show (Tengström, 1965), that (14) is not generally solvable. Therefore I myself regard this approach for very steep topography and irregular Bouguer field as being impossible to apply.

The difficulties with (7) and (16) is, that H_y, resulting from the *existent* solution Φ, which we arrive at by our successive approximations, and integrated to give T_y, does not in general satisfy the stronger infinity-condition (3′) respectively (13a). Φ, expanded in general spherical harmonics, contains in this case, obviously, all orders or all, except order zero (see Appendix 2), and so does $\partial H/\partial\varphi \sin\psi$ for a given arbitrary topography, and therefore (13a) does not hold. It has only to be shown, that this condition is satisfactorily well fulfilled in the case of the real Earth and its topography. Then, if the error falls below the one, accepted by Stokes, the solution of (16) is the answer to the main question of Physical Geodesy. If not, we have to admit, that additional information (vertical derivative-measurements of gravity) is necessary to be able to obtain the external Newtonian gravitational field correctly, also to within the accuracy of Stokes.

APPENDIX 1

Spherical case of double-layer

We have here

$$H_{Py} = \frac{1}{r}\frac{\partial}{\partial r}(r^2 T_y) = \int_\Sigma \Phi(Q)\frac{\partial}{\partial \nu}\left(\frac{1}{\Delta_y}\right) d\Sigma \tag{1*}$$

$$\lim_{\substack{r\to r_P\\ Py\to P}} H_{Py} = -r_P \Delta g_F(P); \quad \lim_{r\to\infty} r^2 H_{Py} = 0. \tag{2* a b}$$

Φ *shall be determined.* If this is possible, the disturbing potential $T = W - U =$ the difference between actual Newtonian potential of Earth and Newtonian potential of Ellipsoid at the same point in space, is obtained by integrating (1*) with respect to r, putting the constant of integration = zero.

The integral equation of the double-layer is

$$2\pi\Phi(P) - \int_\Sigma \frac{\cos(\Delta\nu)_\Sigma}{\Delta_\Sigma^2}\Phi(Q)\,d\Sigma = -r_P\Delta g_F(P). \tag{3*}$$

The equation has an *improper* singularity for $\Delta_\Sigma = 0$. Σ is a Lyapunow-surface with $\alpha = 1$, so

$$\Delta_\Sigma^{-\frac{1}{2}} \cdot C \cdot \Delta_\Sigma = C\Delta_\Sigma^{\frac{1}{2}},$$

which is continuous, and goes to zero for $\Delta_\Sigma \to 0$. The kernel

$$K(PQ) = \frac{\cos(\Delta\nu)_\Sigma}{\Delta_\Sigma^2} \leqslant \frac{C\Delta_\Sigma^{\frac{1}{2}}}{\Delta_\Sigma^{\frac{3}{2}}}, \quad \text{where } \tfrac{3}{2} < 3.$$

(3*) may therefore be treated, using all theorems of Fredholm (and also of Schmidt in this symmetrical case).

The homogeneous equation

$$2\pi\Phi(P) - \int_\Sigma \frac{\cos(\Delta\nu)_\Sigma}{\Delta_\Sigma^2}\Phi(P)\,d\Sigma = 0 \tag{4*}$$

has the only linearly independent solution $\varkappa$ (a constant), because

$$\int_\Sigma \varkappa\frac{\cos(\Delta\nu)_\Sigma}{\Delta_\Sigma^2}\,d\Sigma = 2\pi\varkappa.$$

In order to obtain a solution of the singular equation (3*), $-r_Q\Delta g_F(Q)$ must be orthogonal to the only existing linearly independent solution μ of the transposed homogeneous equation

$$2\pi\mu(P)+\int_{\Sigma}\frac{\cos(\Delta n)_{\Sigma}}{\Delta_{\Sigma}^{2}}\mu(Q)\,d\Sigma=0, \tag{6*}$$

that is,

$$0=\int_{\Sigma}\mu(Q)\,r_Q\,\Delta g_F(Q)\,d\Sigma,$$

or, as here $r_Q=r_P=R$

$$\int_{\Sigma}\mu(Q)\,\Delta g_F(Q)\,d\Sigma=0. \tag{5*}$$

As, for the sphere, $-\cos(\Delta\nu)_{\Sigma}=\cos(\Delta n)_{\Sigma}$, (4*) gives the solution of (6*), namely

$$\mu=\text{constant}.$$

Therefore (5*) indicates, that Δg_F must not contain any spherical harmonics of zero order. We now have to define the properties of the field on the ellipsoidal model, so that

$$\int_{\sigma}\Delta g_F\,d\sigma=0, \tag{7*}$$

σ being the unit sphere.

As

$$\Delta g_F=g_{\text{geoid}}-\gamma_{\text{ellipsoid}},$$

(7*) means, that $\bar{g}=\bar{\gamma}$, mean value over the unit sphere of measured g should be equal to mean value of γ over the ellipsoid. From this condition γ_e, the normal equatorvalue of γ, can be computed, using measurements g all over the world, and the Δg_F, securing a solution of (3*), are known.

(3*) is in our spherical case

$$2\pi\Phi(P)-\int_{\Sigma}\Phi(Q)\,\frac{\sin\dfrac{\psi}{2}}{4R^2\sin^2\dfrac{\psi}{2}}\,d\Sigma=-R\Delta g_F(P),$$

or

$$2\pi\Phi(P)-\int_{\Sigma}\Phi(Q)\,\frac{d\Sigma}{4R^2\sin\dfrac{\psi}{2}}=-R\Delta g_F(P),$$

which gives

$$\Phi(P)-\frac{1}{8\pi}\int_{\sigma}\Phi(Q)\,\frac{d\sigma}{\sin\dfrac{\psi}{2}}=-R\Delta g_F(P)\,\frac{1}{2\pi}, \tag{8*}$$

or

$$\boxed{\Phi(P)-\frac{1}{4\pi}\int_{\sigma}\Phi(Q)\,\frac{1}{\Delta_{\sigma}}\,d\sigma=-R\Delta g_F(P)\frac{1}{2\pi}}$$

Here,

$$\frac{1}{\Delta_{\sigma}}=1+P_1(\cos\psi)+P_2(\cos\psi)+\dots. \tag{9*}$$

The general solution of

$$\boxed{\Phi(P)-\frac{1}{4\pi}\int_\sigma \Phi(Q)\frac{1}{\Delta_\sigma}\,d\sigma=0} \tag{8 a*}$$

is $\Phi=\varkappa$ with $\lambda=1/4\pi$ being an eigenvalue.

The permissible Schmidt-expansion therefore gives the particular solution:

$$\Phi(P)_{\text{part}}=-R\Delta g_F(P)\frac{1}{2\pi}+\frac{1}{4\pi}\sum_{n,\nu}^{\infty}\lambda_n^{(\nu)}\frac{U_n^{(\nu)}(P)}{\lambda_n-\dfrac{1}{4\pi}}$$

with $$\lambda_n^{(\nu)}=\frac{1}{2\pi}\int_\sigma -R\Delta g_F(Q)\,U_n^{(\nu)}(Q)\,d\sigma,$$

and with $\lambda_1=1/4\pi$ missing: The $U_n^{(\nu)}$ are normalized, which means, that $\int(U_n^{(\nu)})^2 d\sigma=1$. The bilinear series for the kernel $1/\Delta_\sigma$ is

$$\frac{1}{\Delta_\sigma(P,Q)}=\frac{U_1^{(1)}(P)\,U_1^{(1)}(Q)}{\lambda_1}+\frac{U_2^{(1)}(P)\,U_2^{(1)}(Q)+U_2^{(2)}(P)\,U_2^{(2)}(Q)+U_2^{(3)}(P)\,U_2^{(3)}(Q)}{\lambda_2}+\dots \tag{10*}$$

with $$\lambda_1=\frac{1}{4\pi};\quad U_1^{(1)}(P)=\sqrt{\frac{1}{4\pi}}P_0(P)=\sqrt{\frac{1}{4\pi}};\quad \lambda_2=\frac{3}{4\pi};$$

$$U_2^{(1)}(P)=\sqrt{\frac{3}{4\pi}}P_1^{(0)}(P);\quad U_2^{(2)}(P)=\sqrt{\frac{3}{4\pi}}P_1^{(1)}(P)\cos\lambda(P);$$

$$U_2^{(3)}(P)=\sqrt{\frac{3}{4\pi}}P_1^{(1)}(P)\sin\lambda(P)\ \text{etc., and}\quad \lambda_n=\frac{2n-1}{4\pi}.$$

so, $$\Phi(P)_{\text{part}}=-R\Delta g_F(P)\frac{1}{2\pi}+\frac{1}{4\pi}\sum^{\infty}\left(\int_\sigma\frac{-R\Delta g_F(Q)}{2\pi}U_n^{(\nu)}(Q)\,d\sigma\right)\frac{U_n^{(\nu)}(P)}{\dfrac{2n-1}{4\pi}-\dfrac{1}{4\pi}},$$

or $$\boxed{\Phi(P)_{\text{part}}=-R\Delta g_F(P)\frac{1}{2\pi}+\tfrac{1}{2}\sum^{\infty}\left(\int_\sigma\frac{-R\Delta g_F(Q)}{2\pi}U_n^{(\nu)}(Q)\,d\sigma\right)\frac{U_n^{(\nu)}(P)}{n-1}} \tag{11*}$$

with $n=1$ missing.

Check on (10*) gives

$$\frac{1}{\Delta_\sigma}=1+[\cos\vartheta(P)\cos\vartheta(Q)+\sin\vartheta(P)\sin\vartheta(Q)\cos\lambda(P)\cos\lambda(Q)$$
$$+\sin\vartheta(P)\sin\vartheta(Q)\sin\lambda(P)\sin\lambda(Q)]+\dots$$
$$=1+\cos\vartheta(P)\cos\vartheta(Q)+\sin\vartheta(P)\sin\vartheta(Q)\cos(\lambda(Q)-\lambda(P))$$
$$=1+\cos\psi_{PQ}+\dots=1+P_1(\cos\psi)+P_2(\cos\psi)+\dots$$

From this we get the partial sum over ν for a certain n, e.g. $n=2$

$$\tfrac{1}{2}\sum_{\nu=1}^{3}\left(\int_\sigma \frac{-R\Delta g_F(Q)}{2\pi} U_2^{(\nu)}(Q)\,d\sigma\right)\frac{U_2^{(\nu)}(P)}{1} = \tfrac{1}{2}\left[\left(\int_\sigma \frac{-R\Delta g_F(Q)}{2\pi}\sqrt{\frac{3}{4\pi}}\cos\vartheta(Q)\,d\sigma\right)\right.$$

$$\times\sqrt{\frac{3}{4\pi}}\cos\vartheta(P) + \left(\int_\sigma \frac{-R\Delta g_F(Q)}{2\pi}\sqrt{\frac{3}{4\pi}}\sin\vartheta(Q)\cos\lambda(Q)\,d\sigma\right)\sqrt{\frac{3}{4\pi}}\sin\vartheta(P)\cos\lambda(P)$$

$$\left.+\left(\int \frac{-R\Delta g_F(Q)}{2\pi}\sqrt{\frac{3}{4\pi}}\sin\vartheta(Q)\sin\lambda(Q)\,d\sigma\right)\sqrt{\frac{3}{4\pi}}\sin\vartheta(P)\sin\lambda(P)\right]$$

$$= -\tfrac{1}{2}\frac{3R}{4\pi}\int\frac{\Delta g_F}{2\pi}[\cos\vartheta(P)\cos\vartheta(P) + \sin\vartheta(P)\sin\vartheta(Q)\cos\lambda(P)\cos\lambda(Q)$$

$$+\sin\vartheta(P)\sin\vartheta(Q)\sin\lambda(P)\sin\lambda(Q)]\,d\sigma = -\frac{3}{(4\pi)^2}\int_\sigma P_1(\cos\psi_{PQ})\Delta g_F\,d\sigma,$$

and generally

$$-\frac{2n-1}{(4\pi)^2}\frac{R}{n-1}\int_\sigma P_{n-1}(\cos\psi_{PQ})\,\Delta g_F(Q)\,d\sigma,$$

so we have

$$\boxed{\Phi(P) = -R\frac{1}{(4\pi)^2}\int_\sigma\left[\sum_2^\infty \frac{2n-1}{n-1}P_{n-1}(\cos\psi_{PQ})\,\Delta g_F(Q)\right]d\sigma - R\Delta g_F(P)\frac{1}{2\pi}} \qquad (12^*)$$

If now the anomaly is

$$\Delta g_F = Y_1 + Y_2 + Y_3 + \dots,$$

which is already secured, (12*) may be written

$$\Phi(P) = -\frac{R}{2\pi}\left[Y_1(P) + Y_2(P) + \dots + \frac{1}{8\pi}\left(3\frac{4\pi}{1}\frac{Y_1(P)}{3} + \frac{4\pi Y_2(P)}{2} + \dots\right)\right]$$

$$= -\frac{R}{2\pi}\left[Y_1(P)(1+\tfrac{1}{2}) + Y_2(P)(1+\tfrac{1}{4}) + \dots + Y_N(P)\left(1+\frac{1}{2N}\right) + \dots\right]$$

$$\boxed{\Phi(P) = -\frac{R}{2\pi}\sum_{N=1}^{\infty}\frac{2N+1}{2N}Y_N(P) = -\frac{R}{4\pi}\sum_{N=1}^{\infty}\frac{2N+1}{N}Y_N(P)} \qquad (13^*)$$

Now, to make H_y approach zero, according to the stronger condition

$$\lim_{r\to\infty} r^2 H_y = 0,$$

it follows (see general case), that

$$\boxed{\int_\sigma \Phi(Q)\cos\psi_{PQ}\,d\sigma = 0} \qquad (14^*)$$

Therefore $$-\frac{R}{4\pi}\int_{\sigma}\sum_{N=1}^{\infty}\frac{2N+1}{N}\,Y_N(Q)P_1(PQ)\,d\sigma=0.$$

This demands that,

$$\boxed{Y_1(P)=0}, \tag{15*}$$

or, that Δg_F *should not contain any spherical harmonics of 1st order.*

Therefore, if Δg_F satisfies this condition (*which it does, because the positions of gravity stations are always accurate enough, and because these stations are assumed to be located on a level surface, very near to the ellipsoidal level surface*),

$$\Phi(P)=-\frac{R}{4\pi}\sum_{N=2}^{\infty}\frac{2N+1}{N}\,Y_N(P). \tag{13a*}$$

From (1*), we now obtain

$$H_y=-\frac{R}{4\pi}\int_{\Sigma}\left[\sum_{N=2}^{\infty}\frac{2N+1}{N}\,Y_N(Q)\frac{\partial}{\partial\nu}\left(\frac{1}{\Delta_{P_yQ}}\right)\right]d\Sigma$$

$$=-\frac{R^3}{4\pi}\int_{\sigma}\left[\sum_{N=2}^{\infty}\frac{2N+1}{N}\,Y_N(Q)\frac{\partial}{\partial\nu}\left(\frac{1}{\Delta_{P_yQ}}\right)\right]d\sigma=\frac{1}{r}\frac{\partial}{\partial r}(r^2T_y), \tag{16*}$$

or $$\frac{\partial}{\partial r}(r^2T_y)=r\,H_y=-\frac{R^3r}{4\pi}\int_{\sigma}\sum_{N=2}^{\infty}\frac{2N+1}{N}\,Y_N(Q)\frac{\partial}{\partial\nu}\left(\frac{1}{\Delta_{P_yQ}}\right)d\sigma$$

$$r^2T_y=-\frac{R^3}{4\pi}\int_{\sigma}\left[\sum_{N=2}^{\infty}\frac{2N+1}{N}\,Y_N(Q)\left(\int r\frac{\partial}{\partial\nu}\left(\frac{1}{\Delta_{P_yQ}}\right)dr+C\right)\right]d\sigma,$$

or $$T_y=-\frac{R^3}{4\pi r^2}\int_{\sigma}\left[\sum_{N=2}^{\infty}\frac{2N+1}{N}\,Y_N(Q)\left(\int r\frac{\partial}{\partial\nu}\left(\frac{1}{\Delta_{P_yQ}}\right)dr+C\right)\right]d\sigma.$$

Here, $$\frac{\partial}{\partial\nu}\left(\frac{1}{\Delta_{P_yQ}}\right)=-\frac{1}{\Delta^2_{P_yQ}}\cos(\Delta_{P_yQ}\nu_Q)$$

with $$r^2=\Delta^2_{P_yQ}+R^2-2R\Delta_{P_yQ}\cos(\Delta_{P_yQ}\nu_Q),$$

or $$\cos(\Delta_{P_yQ}\nu_Q)=\frac{\Delta^2_{P_yQ}+R^2-r^2}{2R\Delta^3_{P_yQ}}$$

and $$r\frac{\partial}{\partial\nu}\left(\frac{1}{\Delta_{P_yQ}}\right)=-\frac{r}{\Delta^2_{P_yQ}}\cos(\Delta_{P_yQ}\nu_Q)=-\frac{r(\Delta^2_{P_yQ}+R^2-r^2)}{2R\Delta^3_{P_yQ}}.$$

Also, $$\Delta^2_{P_yQ}=r^2+R^2-2Rr\cos\psi_{PQ}.$$

$$\therefore\ r\frac{\partial}{\partial\nu}\left(\frac{1}{\Delta_{P_yQ}}\right)=-\frac{r}{2R}\cdot\frac{r^2+R^2-2Rr\cos\psi_{PQ}+R^2-r^2}{(r^2+R^2-2Rr\cos\psi_{PQ})^{\frac{3}{2}}}$$

$$= -\frac{r}{2R}\frac{2r^2\left[\left(\frac{R}{r}\right)^2 - \frac{R}{r}\cos\psi_{PQ}\right]}{r^3\left[1+\left(\frac{R}{r}\right)^2 - 2\left(\frac{R}{r}\right)\cos\psi_{PQ}\right]^{\frac{3}{2}}} = -\frac{1}{R}\mu\frac{\mu-\cos\psi_{PQ}}{(1+\mu^2-2\mu\cos\psi_{PQ})^{\frac{3}{2}}},$$

using $\mu = R/r$. With $d\mu = -(R/r^2)dr$; $dr = -(r^2/R)d\mu = -(R^2/(\mu^2R))d\mu$ or $dr = -(R/\mu^2)d\mu$, we obtain for T_y

$$T_y = -\frac{R^3}{2\pi r^2}\int_\sigma\left[\sum_{N=2}^{\infty}\frac{2N+1}{N}Y_N(Q)\int\left(-\frac{1}{2R}\mu\frac{\mu-\cos\psi_{PQ}}{(1+\mu^2-2\mu\cos\psi_{PQ})^{\frac{3}{2}}}\cdot-\frac{R}{\mu^2}\right)d\mu + C\right]d\sigma.$$

This may be written

$$T_y = -\frac{R^3}{4\pi r^2}\int_\sigma\left[\sum_{N=2}^{\infty}\frac{2N+1}{N}Y_N(Q)\int\left(\frac{1}{\mu}\frac{\mu-\cos\psi_{PQ}}{(1+\mu^2-2\mu\cos\psi_{PQ})^{\frac{3}{2}}}+C'\right)d\mu\right]d\sigma$$

$$= \frac{R^3}{4\pi r^2}\int_\sigma\left[\sum_{N=2}^{\infty}\frac{2N+1}{N}Y_N(Q)\int\left(\frac{1}{\mu}\frac{\cos\psi_{PQ}-\mu}{(1+\mu^2-2\mu\cos\psi_{PQ})^{\frac{3}{2}}}+C''\right)d\mu\right]d\sigma.$$

Now,
$$\frac{1}{\mu}\frac{\cos\psi_{PQ}-\mu}{(1+\mu^2-2\mu\cos\psi_{PQ})^{\frac{3}{2}}} = P_1(\cos\psi_{PQ})\frac{1}{\mu}+2P_2(\cos\psi_{PQ})+\ldots,$$

and
$$T_y = \frac{R^3}{4\pi r^2}\int_\sigma\left[\sum_{N=2}^{\infty}\frac{2N+1}{N}Y_N(Q)\left(P_1(\cos\psi_{PQ})\ln\mu + 2P_2(\cos\psi_{PQ})\mu\right.\right.$$
$$\left.\left.+\ldots+\frac{N}{N-1}P_N(\cos\psi_{PQ})\mu^{N-1}+\ldots+C''\right)\right]d\sigma$$
$$= \frac{r}{4\pi}\int_\sigma\sum_{N=2}^{\infty}\frac{2N+1}{N-1}P_N(\cos\psi_{PQ})\mu^{N+2}\Sigma Y_N\,d\sigma,$$

or
$$\boxed{T_y = \frac{r}{4\pi}\int_\sigma\left[\sum_{N=2}^{\infty}\frac{2N+1}{N-1}P_N\mu^{N+2}\right]\Delta g_F d\sigma,} \qquad (17^*)$$

which is the generalized formula of Stokes.

APPENDIX 2

The general solution Φ of (7 a) can be obtained as the sum of a Schmidt-expansion and the general solution of (8), that is,

$$\Phi(P) = -\frac{r_P}{2\pi}\Delta g_F(P) + \frac{1}{2\pi}\sum_{\nu,\,n=2}^{\infty}\lambda_n^{(\nu)}\frac{U_n^{(\nu)}(P)}{\lambda_n - \frac{1}{4\pi}} + \varkappa \qquad (20)$$

where $(U^{(\nu)}, \lambda)$ is the characteristic system of the kernel

$$K_\sigma = \frac{2r_Pr_Q^2}{\Delta_S^3}\left\{1-\cos\psi_{PQ} - \frac{1}{r_P}(H_P-H_Q) - \frac{1}{r_Q}\frac{\partial H_Q}{\partial\psi}\sin\psi_{PQ}\right\},$$

or $$\frac{1}{\Delta_\sigma}-\frac{2}{\Delta_S^3}\left\{r_Q^2(H_P-H_Q)+r_P r_Q\frac{\partial H_Q}{\partial\psi}\sin\psi_{PQ}\right\}+\text{higher orders}=\frac{1}{\Delta_\sigma}+f(PQ).$$

As the equation

$$\Phi(P)-\frac{1}{4\pi}\int_\sigma K_\sigma\Phi(Q)\,d\sigma_Q=0$$

is satisfied by any constant, $\sqrt{1/(4\pi)}$ is an eigenfunction (normalized) with eigenvalue $1/(4\pi)$ (see (20)). The other eigenfunctions and eigenvalues $U_n^{(\nu)}, \lambda_n$ may be obtained by successive approximations, e.g. putting $U_2^{(1)}=P_1\sqrt{3/(4\pi)}$ we get $\lambda_2=3/(4\pi)$, $U_2^{(1)}=P_1\sqrt{3/(4\pi)}$ as a first approximation. The next is

$$U_2^{(1)'}=\lambda_2'\int_\sigma K_\sigma P_1\sqrt{\frac{3}{4\pi}}\,d\sigma_Q=\lambda_2'\left\{P_1\sqrt{\frac{4\pi}{3}}+\int_\sigma f(PQ)P_1\sqrt{\frac{3}{4\pi}}\,d\sigma_Q\right\}$$
$$=\lambda_2'\left\{P_1\sqrt{\frac{4\pi}{3}}+\int_\sigma F_1(P)P_1^2(Q)\sqrt{\frac{3}{4\pi}}\,d\sigma_Q=\lambda_2'\{P_1+F_1\}\sqrt{\frac{4\pi}{3}}.\right.$$

Normalize:

$$\int_\sigma \lambda_2'^2(P_1+F_1)^2\frac{4\pi}{3}\,d\sigma=1$$

$$\lambda_2'^2=\frac{3}{4\pi}\frac{1}{\int_\sigma(P_1+F_1)^2d\sigma};\quad \lambda_2'=\sqrt{\frac{3/(4\pi)}{\int_\sigma(P_1+F_1)^2d\sigma}}$$

$$U_2^{(1)'}=\sqrt{\frac{3/(4\pi)}{\int_\sigma(P_1+F_1)^2d\sigma}}\,(P_1+F_1)\sqrt{\frac{4\pi}{3}}=\frac{P_1+F_1}{\sqrt{\int_\sigma(P_1+F_1)^2d\sigma}}\quad\text{etc.}$$

We know, that $(r_P/2\pi)\Delta g_F(P)$ contains all orders of spherical harmonics:

$$\frac{r_P}{2\pi}\Delta g_F(P)=Y_0+Y_1+Y_2+\dots \tag{21}$$

The infinite sum in (20) does *not* contain any constant (harmonics of zero order). Therefore, it is possible to determine $\varkappa$ in such a way, that Φ does not contain any zero order terms. Just put $\varkappa=Y_0$, Y_0 being known beforehand from the distribution of anomalies Δg_F and heights.

A solution, obtained by successive approximations, must not contain any constant, that is, it must be expressible as

$$\Phi=Y_1'+Y_2'+\dots \tag{22}$$

Note, that any constant $\varkappa$ in (20) gives the solution of Malkin's function (in space and near the surface S). The selection of (22) is only made for further developments of T_y, the disturbing potential and its directional derivatives. We obtain, at first, Malkin's function (here $\Phi \equiv \Phi_Q d\sigma \equiv d\sigma_Q$):

$$H_y = -\int_\sigma \Phi \frac{r_Q^2}{\Delta_y^3}\left\{r_Q - r\cos\psi - r\frac{\partial H_Q}{r_Q\partial\psi}\sin\psi\right\}d\sigma = \int_\sigma \Phi\frac{r_Q^2}{\Delta_y^3}\cdot r\left\{\cos\psi + \frac{\partial H_Q}{r_Q\partial\psi}\sin\psi - \frac{r_Q}{r}\right\}d\sigma$$

$$\int_\sigma \Phi r_Q^2\cdot\frac{1}{r}\frac{\cos\psi - \dfrac{r_Q}{r} + \dfrac{\partial H_Q}{r_Q\partial\psi}\cdot\sin\psi}{\left(1+\left(\dfrac{r_Q}{r}\right)^2 - 2\dfrac{r_Q}{r}\cos\psi\right)^{\frac{3}{2}}}d\sigma. \tag{23}$$

By virtue of (22), Φ, here, does not contain Y_0'.

We now go in for the determination of T_y etc., and obtain

$$r^2 T_y = \int_\sigma \Phi r_Q^2\left\{\int\frac{1}{r}\frac{\cos\psi - \dfrac{r_Q}{r} + \dfrac{\partial H_Q}{r_Q\partial\psi}\cdot\sin\psi}{\left(1+\left(\dfrac{r_Q}{r}\right)^2 - 2\dfrac{r_Q}{r}\cos\psi\right)^{\frac{3}{2}}}dr + C\right\}d\sigma \tag{24}$$

$$T_y = \frac{R}{r^2}\int_\sigma\left(\frac{r_Q}{R}\right)\Phi\left\{\int\mu\frac{\cos\psi - \mu + \dfrac{\partial H_Q}{r_Q\partial\psi}\sin\psi}{(1+\mu^2 - 2\mu\cos\psi)^{\frac{3}{2}}}dr + C\right\}d\sigma \tag{25}$$

where
$$\frac{r_Q}{r} = \mu;\quad dr = -\frac{r^2}{r_Q}d\mu = -\frac{1}{\mu^2}r_Q d\mu.$$

We, therefore, have

$$T_y = -\frac{R^2}{r^2}\int_\sigma\left(\frac{r_Q}{R}\right)^2\Phi\left\{\int\frac{1}{\mu}\frac{\cos\psi - \mu + (\partial H_Q)/(r_Q\partial\psi)\sin\psi}{(1+\mu^2-2\mu\cos\psi)^{\frac{3}{2}}}dr + C'\right\}d\sigma. \tag{26}$$

In this case, we cannot rely upon μ being <1. Therefore, we have to evaluate the μ-integral generally.

This integral is

$$\int\frac{1}{\mu}\frac{\cos\psi + (\partial H_Q)/r_Q\partial\psi)\cdot\sin\psi}{(1+\mu^2-2\mu\cos\psi)^{\frac{3}{2}}}d\mu - \int\frac{d\mu}{(1+\mu^2-2\mu\cos\psi)^{\frac{3}{2}}}$$

$$= \left(\cos\psi + \frac{\partial H_Q}{r_Q\partial\psi}\sin\psi\right)\underbrace{\int\frac{(1/\mu)\,d\mu}{(1+\mu^2-2\mu\cos\psi)^{\frac{3}{2}}}}_{\text{(I)}} - \underbrace{\int\frac{d\mu}{(1+\mu^2-2\mu\cos\psi)^{\frac{3}{2}}}}_{\text{(II)}}$$

$$T_y = -\frac{R^2}{r^2}\int_\sigma\left(\frac{r_Q}{R}\right)^2\Phi\left\{\left(\cos\psi + \frac{\partial H_Q}{r_Q\partial\psi}\sin\psi\right)\int\frac{d\mu/\mu}{(1+\mu^2-2\mu\cos\psi)^{\frac{3}{2}}}\right.$$
$$\left. - \int\frac{d\mu}{(1+\mu^2-2\mu\cos\psi)^{\frac{3}{2}}} + C'\right\}d\sigma$$
$$= -\frac{R^2}{r^2}\int_\sigma\left(\frac{r_Q}{R}\right)^2\Phi\left\{\left(\cos\psi + \frac{\partial H_Q}{r_Q\partial\psi}\sin\psi\right)(\text{I}) - (\text{II}) + C'\right\}d\sigma.$$

Integral (I): $$\int \frac{1}{\mu} \frac{d\mu}{(1+\mu^2-2\mu\cos\psi)^{\frac{3}{2}}} = \int \frac{d\mu}{\mu \sqrt{\bar{R}^3}}$$

$$\bar{R} = \mu^2 + 1 - 2\mu\cos\psi; \quad a = 1,\ b = -2\cos\psi,\ C = 1$$

$$\Delta = 4 - 4\cos^2\psi = 4\sin^2\psi$$

$$\int \frac{d\mu}{\mu\sqrt{\bar{R}^3}} = -\frac{2(-2\cos\psi\cdot\mu - 2 + 4\cos^2\psi)}{4\sin^2\psi\sqrt{\bar{R}}} + \int\frac{d\mu}{\mu\sqrt{\bar{R}}}$$

$$= \frac{\mu\cos\psi + 1 - 2\cos^2\psi}{\sin^2\psi\sqrt{\bar{R}}} - \ln\frac{2 - 2\mu\cos\psi + 2\sqrt{\bar{R}}}{\mu} = (\mathrm{I})\cdot$$

Integral (II): $$\int \frac{d\mu}{(1+\mu^2-2\mu\cos\psi)^{\frac{3}{2}}} = \int\frac{d\mu}{\sqrt{\bar{R}^3}} = \frac{2(2\mu - 2\cos\psi)}{4\sin^2\psi\sqrt{\bar{R}}}$$

$$T_y = -\frac{R^2}{r^2}\int_\sigma \left(\frac{r_Q}{R}\right)^2 \Phi\left[\left(\cos\psi + \frac{\partial H_Q}{r_Q\partial\psi}\sin\psi\right)\cdot\right.$$

$$\left.\times\left\{\frac{\mu\cos\psi - \cos 2\psi}{\sin^2\psi\sqrt{\bar{R}}} + \ln\frac{\mu}{2(1-\mu\cos\psi - \sqrt{\bar{R}})}\right\} - \frac{2(2\mu - 2\cos\psi)}{4\sin^2\psi\sqrt{\bar{R}}} + C'\right] d\sigma. \quad (27)$$

Insert $$D^2 = \mu^2 + 1 - 2\mu\cos\psi = \bar{R}.$$

As $(r_Q/R)^2\Phi(Q)$ with Stokes' accuracy might be written $1 + 2(H_Q/R) = 1$ we have for any C' the solution for T_y

$$T_y = -\frac{R^2}{r^2}\int_\sigma \Phi\left[\frac{(\cos\psi + (\partial H_Q/(r_Q\partial\psi))\cdot\sin\psi)(\mu\cos\psi - \cos 2\psi) - \mu + \cos\psi}{D\sin^2\psi}\right.$$

$$\left. + \left(\cos\psi + \frac{\partial H_Q}{r_Q\partial\psi}\sin\psi\right)\ln\frac{\mu}{2(1-\mu\cos\psi + D)} + C'\right] d\sigma. \quad (27\,\mathrm{a})$$

Now, let $r \to r_P$, we have *the heightanomaly* N'_P from

$$\mu \to \frac{r_Q}{r_P} = 1 - \frac{H_P - H_Q}{r_P},$$

at least to Stokes' accuracy

$$N'_P = -\frac{R^2}{\gamma_P r_P^2}$$

$$\times \int_\sigma \Phi\left[\frac{\left\{\cos\psi + \dfrac{\partial H_Q}{r_Q\partial\psi}\sin\psi\right\}\left\{\left(1 - \dfrac{H_P - H_Q}{r_P}\right)\cos\psi - \cos 2\psi\right\} - 1 + \dfrac{H_P - H_Q}{r_P} + \cos\psi}{\sin^2\psi\,(\Delta_{PQ}/r_P)}\right.$$

$$\left. + \left(\cos\psi + \frac{\partial H_Q}{r_Q\partial\psi}\sin\psi\right)\ln\frac{1 - ((H_P - H_Q)/r_P)}{2(1 - [1 - ((H_P - H_Q)/r_P)]\cos\psi + (\Delta_{PQ}/r_P))} + C'\right] d\sigma, \text{ or}$$

$$\boxed{\begin{aligned} N'_P = -\frac{R^2}{\gamma_P r_P^2}\int_\sigma \Phi\Bigg[&\frac{2\cos\psi - 1}{D} - \left(\cos\psi + \frac{\partial H_Q}{r_Q\partial\psi}\sin\psi\right)\ln 2\left(2\sin^2\frac{\psi}{2}\right. \\ &\left. + D + D\frac{H_P - H_Q}{r_P D}\cos\psi\right) + \frac{\partial H_Q}{r_Q\partial\psi}\frac{\cos\psi - \cos 2\psi}{D\sin\psi} + \frac{H_P - H_Q}{r_P D} \\ &\times\left(1 - \frac{\partial H_Q}{r_Q\partial\psi}\cot\psi\right) - \frac{H_P - H_Q}{r_P D}\left\{\cos\psi + \frac{\partial H_Q}{r_Q\partial\psi}\sin\psi\right\}D\Bigg]d\sigma \end{aligned}} \tag{28}$$

In the spherical case we should therefore have

$$N_\Sigma = -\frac{1}{G}\int_\sigma \Phi_\Sigma\left[\frac{1}{2\sin(\psi/2)} - 2\sin(\psi/2) - \cos\psi\ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right) - \ln 4\cos\psi\right]d\sigma,$$

and as Φ now does not contain first order

$$N_\Sigma = -\frac{1}{G}\int_\sigma \Phi_\Sigma\left[\frac{1}{2\sin(\psi/2)} - 2\sin\frac{\psi}{2} - \cos\ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right)\right]d\sigma. \tag{29}$$

The Schmidt-series for Φ_Σ is now:

$$\Phi_\Sigma(P) = -R\Delta g(P)\frac{1}{2\pi} + \frac{1}{4\pi}\sum_{\nu,\, n=1}^{\infty}\lambda_n^{(\nu)}\frac{U_n^{(\nu)}(P)}{\lambda_n - (1/(4\pi))} + \varkappa = 0.$$

where $$\lambda_n^{(\nu)} = -\frac{R}{2\pi}\int_\sigma \Delta g\, U_n^{(\nu)} d\sigma, \text{ which gives}$$

$$\Phi_\Sigma = -\frac{R}{2\pi}\left[\Delta g + \frac{1}{8\pi}\sum_{N=2}^{\infty}\frac{2N+1}{N}\int_\sigma P_N \Delta g\, d\sigma\right],$$

as here Φ_Σ also lacks (because of known properties for Δg), first order terms. We obtain

$$T_y = -\frac{R^2}{r^2}\int_\sigma \Phi_\Sigma\frac{1}{R}\left[\int \mu\frac{\mu - \cos\psi}{(\mu^2 + 1 - 2\mu\cos\psi)^{\frac{3}{2}}}dr + C\right]d\sigma.$$

Here, $\mu = R/r < 1$; $dr = -(R/\mu^2)d\mu$, and we have

$$T_y = -\frac{R^2}{r^2}\int_\sigma \Phi_\Sigma\left[P_1\ln\mu + 2P_2\mu + \tfrac{3}{2}P_3\mu^2 + \dots + \frac{N}{N-1}P_N\mu^{N-1} + \dots + C\right]d\sigma.$$

Φ_Σ does not contain zero and and first order. Therefore the expression may be written ($C = 0$, $N \neq 1$)

$$T_y = -\frac{R^2}{r^2}\int_\sigma \Phi_\Sigma\left[\sum_{N=2}^{\infty}\frac{N}{N-1}P_N\mu^{N-1}\right]d\sigma.$$

$$\sum_{N=2}^{\infty}\frac{N}{N-1}P_N=\sum_{N=2}^{\infty}P_N+\sum_{N=2}^{\infty}\frac{P_N}{N-1}$$

$$=\left\{\frac{1}{2\sin(\psi/2)}-1-\cos\psi\right\}+\left\{1-\cos\psi-2\sin\frac{\psi}{2}-\cos\psi\ln\left(\sin\frac{\psi}{2}+\sin^2\frac{\psi}{2}\right)\right\}$$

$$=\frac{1}{2\sin(\psi/2)}-2\sin\frac{\psi}{2}-\cos\psi\ln\left(\sin\frac{\psi}{2}+\sin^2\frac{\psi}{2}\right)-2\cos\psi.$$

The last term is unnecessary, because Φ_Σ does not contain first order. So we have

$$N_\Sigma=-\frac{1}{G}\int_\sigma\Phi_\Sigma\left[\frac{1}{2\sin(\psi/2)}-2\sin\frac{\psi}{2}-\cos\psi\ln\left(\sin\frac{\psi}{2}+\sin^2\frac{\psi}{2}\right)\right]d\sigma$$

which coincides with (29).

From (27) we may also derive the deflections of the vertical

$$\xi_y''=+\frac{\partial N'}{\partial\psi}\frac{\partial\psi}{\partial x}\varrho''=+\frac{\partial(T_y/\gamma_y)}{\partial\psi}\frac{\partial\psi}{\partial x}\varrho''$$

$$\eta_y''=+\frac{\partial N'}{\partial\psi}\frac{\partial\psi}{\partial y}\varrho''=+\frac{\partial(T_y/\gamma_y)}{\partial\psi}\frac{\partial\psi}{\partial y}\varrho'' \tag{30}$$

$$\frac{\partial(T_y/\gamma_y)}{\partial\psi}=-\frac{T_y}{\gamma_y^2}\frac{\partial\gamma_y}{\partial\psi}+\frac{1}{\gamma_y}\frac{\partial T_y}{\partial\psi}=-\frac{N_y}{\gamma_y}\frac{\partial\gamma_y}{\partial\psi}+\frac{1}{\gamma_y}\frac{\partial T_y}{\partial\psi}\cong\frac{1}{\gamma_y}\frac{\partial T_y}{\partial\psi}.$$

According to (27 a) is

$$\frac{\partial T_y}{\partial\psi}=-\frac{R^2}{r^2}\int_\sigma\Phi\frac{\partial}{\partial\psi}\left[\qquad\qquad\right]d\sigma,$$

which after some computation gives

$$\frac{\partial T_y}{\partial\psi}=-\frac{R^2}{r^2}\int_\sigma\Phi\left[2\frac{\cos2\psi-\cos\psi(\mu-\cos\psi)}{D\sin\psi}+\frac{(\mu-2\cos\psi)(\mu\sin^2\psi+2D^2\cos\psi)}{D^3\sin\psi}\right.$$

$$\left.-\frac{\sin2\psi(1-D)}{2D(1-\mu\cos\psi+D)}+\sin\psi\ln\frac{2(1-\mu\cos\psi+D)}{\mu}\right]d\sigma \tag{31 I}$$

$$-\frac{R^2}{r^2}\int_\sigma\Phi\frac{\partial H_Q}{r_Q\partial\psi}\left[\frac{2}{D\sin^2\psi}\left(\mu\cos2\psi\sin^2\frac{\psi}{2}-\sin\psi\sin2\psi\right)\right.$$

$$+\frac{1}{D^3\sin^2\psi}(\cos2\psi-\mu\cos\psi)(\mu\sin^2\psi+2D^2\cos\psi) \tag{31 II}$$

$$\left.-\frac{\sin^2\psi}{D}\frac{(D+1)\mu}{1-\mu\cos\psi+D}+\cos\psi\ln\frac{\mu}{2(1-\mu\cos\psi+D)}\right]d\sigma-$$

$$-\frac{R^2}{r^2}\int_\sigma\Phi\frac{\partial}{\partial\psi}\left(\frac{\partial H_Q}{r_Q\partial\psi}\right)\left[\frac{(\mu\cos\psi-\cos2\psi)}{D\sin\psi}-\sin\psi\ln\frac{2(1-\mu\cos\psi+D)}{\mu}\right]d\sigma \tag{31 III}$$

that is
$$\frac{\partial T_y}{\partial \psi} = -\frac{R^2}{r^2}\int_\sigma [(\mathrm{I}) + (\mathrm{II}) + (\mathrm{III})]d\sigma$$

$$\xi_y'' = +\frac{R}{\gamma_y r^2}\varrho'' \int_\sigma [(\mathrm{I}) + (\mathrm{II}) + (\mathrm{III})]\cos\alpha\, d\sigma$$

$$\eta_y'' = +\frac{R}{\gamma_y r^2}\varrho'' \int_\sigma [(\mathrm{I}) + (\mathrm{II}) + (\mathrm{III})]\sin\alpha\, d\sigma. \tag{31 a}$$

Now, let $r \to r_p$,

$$\mu \to \frac{r_Q}{r_P} = 1 - \frac{H_P - H_Q}{r_P},$$

we get to the accuracy (at least) of Stokes

$$\xi_P'' = +\frac{R}{\gamma_P r_P^2}\varrho'' \int_\sigma [(\mathrm{I})_P + (\mathrm{II})_P + (\mathrm{III})_P]\cos\alpha\, d\sigma$$

$$\eta_P'' = +\frac{R}{\gamma_P r_P^2}\varrho'' \int_\sigma [(\mathrm{I})_P + (\mathrm{II})_P + (\mathrm{III})_P)\sin\alpha\, d\sigma. \tag{32}$$

Starting with the expression in the spherical case using Δg_F, we obtain a first approximation Φ_1, which is not a solution of an integral equation for the spherical double layer, but the value of which is certainly nearer to the correct value than e.g.

$$\Phi = -\frac{R}{2\pi}\Delta g_F.$$

We then have

$$\Phi_1 = -\frac{R}{2\pi}\left[\Delta g_F + \frac{1}{8\pi}\int_\sigma \left(\sum_{N=2}^{\infty}\frac{2N+1}{N}P_N\right)\Delta g_F d\sigma\right]. \tag{33}$$

Now
$$\sum_{N=2}^{\infty}\frac{2N+1}{N}P_N = \sum_{N=2}^{\infty}2P_N + \sum_{N=2}^{\infty}\frac{P_N}{N}.$$

Both sums can be derived from

$$\sum_{N=2}^{\infty}P_N\mu^{N-1} = \frac{1}{\sqrt{\mu^2 - 2\mu\cos\psi + 1}} - P_1 - \frac{1}{\mu}, \tag{34}$$

where
$$\mu \leqslant 1.$$

The first sum is obtained by putting $\mu = 1$ (letting μ go to 1) and multiplying by 2

$$2\sum_{N=2}^{\infty}P_N = \frac{2}{2\sin(\psi/2)} - 2\cos\psi - 2. \tag{35}$$

The second term is obtained by integrating (34) with respect to μ

$$\sum_{N=2}^{\infty} \frac{P_N}{N} \mu^N = \int \frac{d\mu}{\sqrt{\mu^2 - 2\mu\cos\psi + 1}} - P_1\mu - \ln\mu + C$$

$$= \ln(\mu - \cos\psi + \sqrt{\mu^2 - 2\mu\cos\psi + 1}) - P_1\mu - \ln\mu + C.$$

Let $$\mu \to 1$$

$$\sum_{N=2}^{\infty} \frac{P_N}{N} = \ln\left(2\sin^2\frac{\psi}{2} + \sqrt{2 - 2\cos\psi}\right) - \cos\psi + C$$

$$= \ln\left(\sin^2\frac{\psi}{2} + \sin\frac{\psi}{2}\right) - \cos\psi + C, \tag{36}$$

and $$\sum_{N=2}^{\infty} \frac{2N+1}{N} P_N = \frac{1}{\sin(\psi/2)} - 2\cos\psi - 2 - \cos\psi + \ln\left(\sin^2\frac{\psi}{2} + \sin\frac{\psi}{2}\right) + C$$

$$\Phi_1 = -\frac{R}{2\pi}\left(\Delta g_F + \frac{1}{8\pi}\int_\sigma \left[\frac{1}{\sin(\psi/2)} + \ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right)\right]\Delta g_F d\sigma \tag{37}$$

under the assumption—which is realistic for the real Earth—that Δg_F contains only small terms of zero and first order. This approximation corresponds to Molodenskij's use of Stokes' formula as the first step to solve his integral equation.

Generally, we have for the correct function

$$\Phi(P) - \frac{1}{4\pi}\int_\sigma K_\sigma \Phi(Q)\, d\sigma = -\frac{R}{2\pi}\Delta g_F(P) = \Phi_0(P).$$

The *first* approximation was

$$\Phi_1(P) = \Phi_0(P) + \frac{1}{4\pi}\int_\sigma \left[\frac{1}{2\sin(\psi/2)} + \tfrac{1}{2}\ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right)\right]\Phi_0(Q)\, d\sigma. \tag{38}$$

The *second* approximation is obtained from

$$\Phi_2(P) - \Phi_0(P) = \frac{1}{4\pi}\int_\sigma K_\sigma \Phi_1(Q)\, d\sigma,$$

and we get

$$\Phi_2(P) - \Phi_1(P) = d_1\Phi(P) = \frac{1}{4\pi}\int_\sigma \left[K_\sigma\Phi_1(Q) - \frac{1}{\Delta_\sigma}\Phi_0(Q)\right.$$

$$\left. - \tfrac{1}{2}\ln\left(\sin\frac{\psi}{2} + \sin^2\right)\frac{\psi}{2}\Phi(Q)\right] d\sigma = \frac{1}{4\pi}\int_\sigma \left(K_\sigma - \frac{1}{\Delta_\sigma}\right)\Phi_1(Q)\, d\sigma$$

$$+ \frac{1}{4\pi}\int_\sigma \frac{1}{\Delta_\sigma}[\Phi_1(Q) - \Phi_0(Q)]\, d\sigma - \frac{1}{8\pi}\int_\sigma \ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right)\Phi_0(Q)\, d\sigma. \tag{39}$$

For the *third* approximation we put

$$\Phi_3(P) - \Phi_0(P) = \frac{1}{4\pi}\int_\sigma K_\sigma \Phi_2(Q)\, d\sigma,$$

which combined with

$$\Phi_2(P) - \Phi_0(P) = \frac{1}{4\pi}\int_\sigma K_\sigma \Phi_1(Q)\,d\sigma$$

gives
$$d_2\Phi(P) = \frac{1}{4\pi}\int_\sigma K_\sigma[\Phi_2(Q) - \Phi_1(Q)]\,d\sigma = \frac{1}{4\pi}\int_\sigma K_\sigma d_1\Phi(Q)\,d\sigma. \tag{39 a}$$

Additional correction to get *4th approximation* is then

$$d_3\Phi(P) = \frac{1}{4\pi}\int K_\sigma d_2\Phi(Q)\,d\sigma, \quad \text{etc.} \tag{39 b}$$

$$\Phi = \Phi_1 + d_1\Phi + d_2\Phi + \dots \tag{40}$$

and this series is only convergent, if a solution exists, that is, if Δg_F is obeying

$$\int_\sigma \Delta g_F \chi\,d\sigma = 0$$

If not, $\Phi_0 = -(R/2\pi)\Delta g_F$ contains a *false* constant, which makes (40) diverge. Such a divergence is, however, not very dangerous—at least near the surface—because the false constant for nowadays γ_e is probably small, and the number of terms in the series (40) to attain Stokes' accuracy, are limited to two or three, even in the case of very steep topography and irregular Bouguer-field. The different propagation of Φ-errors, when computing heightanomalies or deflections, should be noted, however!

(39) can be written

$$d_1\Phi(P) = \frac{1}{4\pi}\int_\sigma\left[k_\sigma - \frac{1}{\Delta_\sigma}\right][\Phi_1(Q) - \Phi_1(P)]\,d\sigma + \frac{1}{4\pi}\int_\sigma \frac{1}{\Delta_\sigma} d_0\Phi(Q)\,d\sigma$$
$$- \frac{1}{8\pi}\int \ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right)\Phi_0(Q)\,d\sigma, \quad d_0\Phi = \Phi_1 - \Phi_0.$$

Now, $$K_\sigma - \frac{1}{\Delta_\sigma} = -\frac{R}{\Delta_s}\left[2\frac{R}{\Delta_s}\left(\frac{H_P - H_Q}{\Delta_s}\right) + \frac{R}{\Delta_s}\left[2 - \left(\frac{H_P - H_Q}{\Delta_s}\right)^2 + \dots\right]\operatorname{tg}\beta_\psi \cos\frac{\psi}{2}\right.$$
$$\left. = \frac{3}{2}\left(\frac{H_P - H_Q}{\Delta_s}\right)^2 + \frac{3}{8}\left(\frac{H_P - H_Q}{\Delta_s}\right)^4 + \dots\right],$$

so

$$\boxed{\begin{aligned} d_1\Phi(P) = &-\frac{R^2}{2\pi}\int_\sigma \frac{[H_P - H_Q][\Phi_1(Q) - \Phi_1(P)]}{\Delta_s^3}\,d\sigma - \frac{R^2}{4\pi}\int_\sigma \frac{\operatorname{tg}\beta_\psi \cos(\psi/2)}{\Delta_s^2}\left[2 - \left(\frac{H_P - H_Q}{\Delta_s}\right)^2\right] \\ &\times (\Phi_1(Q) - \Phi_1(P)) - \frac{3R^2}{4\pi}\int_\sigma \frac{1}{R\Delta_s}\left[\left(\frac{H_P - H_Q}{\Delta_s}\right)^2 + \tfrac{1}{4}\left(\frac{H_P - H_Q}{\Delta_s}\right)^4 + \dots\right] \\ &\times (\Phi_1(Q) - \Phi_2(P))\,d\sigma + \frac{R}{4\pi}\int \frac{1}{\Delta_\Sigma} d_0\Phi(Q)\,d\sigma \\ &- \frac{1}{8\pi}\int_\sigma \ln\left(\sin\frac{\psi}{2} + \sin^2\frac{\psi}{2}\right)\Phi_0(Q)\,d\sigma \end{aligned}}$$

(41)

The first term is approximately equal to twice the terraincorrection [5], the second takes the distribution of slopes into account (observe $\operatorname{tg}\beta_\psi = \partial H_Q/(R\partial\psi)$). The third term is usually small but takes observable values in very steep topography. The last two terms are smooth and small. In the remaining approximations

$$\underset{n>1}{d_n}\Phi(P) = \frac{1}{4\pi}\int_\sigma K_\sigma d_{n-1}\Phi(Q)\,d\sigma$$

$$K_\sigma = \frac{2R^3}{\Delta_s^3}\left[2\sin^2\frac{\psi}{2} - \operatorname{tg}\beta_\psi \sin\psi\right] - \frac{2R^2}{\Delta_s^3}(H_P - H_Q)$$

$$= \frac{4R^3}{\Delta_s^3}\left[\sin\frac{\psi}{2} - \operatorname{tg}\beta_\psi\cos\frac{\psi}{2}\right]\sin\frac{\psi}{2} - \frac{2R^2}{\Delta_s^3}(H_P - H_Q), \tag{42}$$

with sufficient accuracy.

Further investigations of practical computations of N'_P, ξ_P and η_P, according to (28) and (32) are going on! Also estimates of the errors in N'_y, ξ_y and η_y at high elevations (satellite-heights) because of Φ disobeying (13a), are being made!

Note: It is important, when dealing with the problems above, to be aware, always, of the difference between Δ_Σ and Δ_s, Δ_Σ being by definition smaller than Δ_s for all topography. Cf. [5] p. 7.

APPENDIX 3

Remark to condition:

$$\int_\sigma \Delta g_F \chi\, d\sigma = 0,$$

to be used to determine γ_e (for simplicity put $\bar{\gamma} = \gamma = \text{constant}$).

$$\Delta g_F = g + 0{,}3086\,H - \gamma = Y_0 + Y_1 + \ldots - \gamma.$$

If the condition were $$\int_\sigma \Delta g_F\, d\sigma = 0,$$

we would obtain $$\gamma_e = \gamma_1 \text{ (wrong)}$$

$$\int_\sigma (Y_0 + Y_1 + \ldots - \gamma_1)\,d\sigma = 0;$$

that is $$\boxed{\gamma_1 = Y_0}$$

But the condition is for correct $\gamma_e = \gamma_2$

$$\int_\sigma \Delta g_F \chi\, d\sigma = \int_\sigma (Y_0 + Y_1 + \ldots - \gamma_2)\left(1 + \varkappa + \frac{\bar{c}_t}{400\,000} + \ldots\right)d\sigma = 0,$$

where $\bar{c}_t$ does not contain zero order (this is $\varkappa$)

$$\chi = Y_0' + Y_1' + \dots \quad \text{with} \quad Y_1' = 1 + \varkappa$$

$$\therefore \int_\sigma (Y_0 + Y_1 + \dots - \gamma_2)(1 + \varkappa + Y_1' + Y_2' + \dots)\, d\sigma = 0,$$

$$\int_\sigma Y_0 Y_0' d\sigma + \int_\sigma Y_1 Y_1' d\sigma + \dots = \gamma_2(1+\varkappa)4\pi$$

$$\gamma_2 = \frac{4\pi\gamma_1(1+\varkappa)}{4\pi(1+\varkappa)} + \frac{\int_\sigma Y_1 Y_1' d\sigma}{4\pi(1+\varkappa)} + \frac{\int_\sigma Y_2 Y_2' d\sigma}{4\pi(1+\varkappa)} + \dots$$

$$\gamma_2 = \gamma_1 + \frac{\int_\sigma Y_1 Y_1' d\sigma + \dots}{4\pi(1+\varkappa)}.$$

Now

$$Y_1 = A_1 \cos\vartheta + B_1 \sin\vartheta \cos\lambda + C_1 \sin\vartheta \sin\lambda$$

$$Y_1' = A_1' \cos\vartheta + B_1' \sin\vartheta \cos\lambda + C_1' \sin\vartheta \sin\lambda \quad \text{etc.}$$

$$\int_\sigma Y_1 Y_1' d\sigma = A_1 A_1' \int_\sigma P_1^2 d\sigma = \frac{4\pi}{3} A_1 A_1'$$

$$\int_\sigma Y_2 Y_2' d\sigma = A_2 A_2' \int_\sigma P_2^2 d\sigma = \frac{4\pi}{5} A_1 A_2' \quad \text{etc.}$$

$$\gamma_2 - \gamma_1 = (1-\varkappa)\left[\tfrac{1}{3} A_1 A_1' + \tfrac{1}{5} A_2 A_2' + \dots\right].$$

Therefore: expand $g + 0.3086\,H$ in spherical harmonics $Y_0 + Y_1 + \dots$, and also $\chi = 1 + \varkappa + Y_1' + Y_2' + \dots$ and list the coefficients of P_1, P_2 etc. of $g + 0.3086\,H$, and in λ the coefficients of P_1', P_2', …, that is A and A$'$. Also determine $Y_0 = \gamma_1$ and $Y_0' = 1 + \varkappa$; $\varkappa = Y_0' - 1$. We then have

$$\boxed{\gamma_2 = Y_0 + (2 - Y_0')\left(\tfrac{1}{3} A_1 A_1' + \tfrac{1}{5} A_2 A_2' + \dots\right)}$$

which differs from γ_1 by $(2 - Y_0')(\qquad)$.

Observe, that very high orders in g, H and χ may influence this difference!

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ПОПЫТКА РЕШЕНИЯ ГРАНИЧНОЙ ЗАДАЧИ ГЕОДЕЗИИ ПРИ ИСПОЛЬЗОВАНИИ ПРИБЛИЖЕНИЯ ДВОЙНОГО СЛОЯ ДЛЯ ФУНКЦИИ МАЛКИНА И ПРИ ВЫПОЛНЕНИИ СТОКСОВОЙ ТОЧНОСТИ

Граничная задача физической геодезии обычно рассматривается как задача нахождения надёжных поправок к принимаемой эллипсоидальной модели Земли для получения истинных значений силы тяжести и формы её уровенных поверхностей во внешнем пространстве. Точность в определении этих поправок, как правило, соответствует пренебрегаемым ошибкам порядка сжатия Земли рис. I и Выше (3 ‰ от получаемых величин). В этом первом приближении земная топография идентифицируется с так наз. теллуроидом (см. го) и при этом в качестве единственных граничных величин используются так наз. истинные аномалии свободно воздуха (см. обозначения).

Для изучения разрешимости этой задачи хорошо подходят интегральные уравнения. Благодаря доступности мощных вычислительных машин они не представляют трудности для расчётов. По моему мнению, до сих пор ещё не получено полностью удовлетворительных решений, удовлетворяющих всем граничным условиям. Если окажется, что нет таких гармонических моделей, которые могут удовлетворить всем условиям на поверхности и на бесконечности (см. ниже) с достаточной точностью (см. выше), то мы должны признать, что необходима дополнительная информация на поверхности Земли для решения этой важной задачи, при сведении её к случаю Стокса (см. приложение 1).

В статье описывается прямое использование приближения двойного слоя для так наз. функции Малкина Ку (см. ниже), из которой все искомые поправки в пространстве могут быть выведены. Это приближение предлагается потому, что условие разрешимости результирующего сингулярного уравнения Фредгольма может быть использовано для предварительного определения корректной модели силы тяжести. Решение, потенциал двойного слоя по определению, удовлетворяет слабому условию на бесконечности

$$\lim_{r\to\infty} rH_y = 0.$$

Удовлетворяет ли оно с достаточной точностью более сильному условию

$$\lim_{r\to\infty} r^2H_y = 0,$$

должно быть исследовано в дальнейшем.

В Приложении 1 рассматривается более простая задача о Земле, сплошь покрытой океаном. Оно написано, в основном, с целью теоретического обънснения процедуры, применяемой при рассмотрении общего случая (произвольной топографии) в приближении двойного слоя.

В Приложении 2 выводятся различные формулы, которые могут быть использованы в численных расчётах при анализе общего случая.

В Приложении 3 даётся дополнительное объяснение связи между модельной силой тяжести и силой тяжести, измеряемой на поверхности Земли.