Some transformation properties for the coefficients in a spherical harmonics expansion of the earth's external gravitational potential

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ABSTRACT

The earth's external gravitational potential may be mathematically expressed as an infinite series of spherical harmonics. The coefficients in such series depend numerically on the location and orientation of the adopted system of reference with respect to the body of the earth. The present subject is to derive general formulae for the numerical changes of the harmonic coefficients due to both a re-orientation and a re-location of the reference frame. Such formulae could be of value in critical studies of the impact of artificial satellites on dynamical geodesy, especially when combining the results with those from terrestrial geodesy. Use has been made of a transformation property of spherical harmonics which seems to be not a part of standard mathematics. Some simple numerical examples are given.

Introduction

The earth's external gravitational potential $U(r, \varphi, \lambda)$ may be expressed as an expansion in terms of solid spherical harmonics

$$U(r, \varphi, \lambda) = \frac{GM}{r} \sum_{n=0}^{\infty} \sum_{s=0}^{n} \left(\frac{a}{r}\right)^n (J_n^s \cos s\lambda + K_n^s \sin s\lambda) \cdot P_n^s(\sin \varphi) \tag{1}$$

coefficients J_n^s and K_n^s being defined as

$$\begin{aligned} J_n^s &= \varepsilon_s \frac{(n-s)!}{(n+s)!} \frac{1}{Ma^n} \int_M r^n \cos s\lambda \cdot P_n^s(\sin \varphi)\, dm. \\ K_n^s &= \varepsilon_s \frac{(n-s)!}{(n+s)!} \frac{1}{Ma^n} \int_M r^n \sin s\lambda \cdot P_n^s(\sin \varphi)\, dm \end{aligned} \tag{2}$$

Here GM stands for the product of the universal constant of gravitation G and the total mass M of the earth; the mass of the atmosphere will be neglected. a is an arbitrary scaling factor which conveniently can be taken as to coincide with the equatorial radius of the earth. r, λ and φ (=latitude) are the conventional spherical coordinates and the notation of the associated Legendre functions of the first kind could follow Hobson (1931):

$$P_n^s(x) = \frac{(-1)^m}{2^n n!} (1-x^2)^{m/2} \frac{d^{n+m}}{dx^{n+m}} (x^2-1)^n \qquad \begin{matrix} |x| \leq 1 \\ |s| \leq n \end{matrix}$$

$$\varepsilon_s = \begin{cases} 1 & \text{if} \quad s = 0 \\ 2 & \text{if} \quad s > 0. \end{cases}$$

The element of mass dm stands for $\sigma(r, \varphi, \lambda) r^2 \cos\varphi dr d\varphi d\lambda$ where $\sigma(r, \varphi, \lambda)$ is the specific mass of the earth's interior.

The integrations are to be carried out throughout the space occupied by the masses and (1) is uniformly convergent outside any origin-centred sphere which contains all masses in its interior (Kellogg, 1929).

Nowadays $\sigma(r, \varphi, \lambda)$ is an unknown function of r, φ and λ and consequently the integrals (2) cannot be evaluated in a straightforward manner. However Stokes' constants J_n^s and K_n^s—as they are sometimes referred to—can be determined phenomenologically from measurements in the external field, for which purpose the analysis of orbits of artificial satellites proved to be a powerful tool.

But the numerical values of J_n^s and K_n^s depend on the location and the orientation of the adopted reference system of coordinates with respect to the earth's body. It is geodetic practice to disregard coefficients J_1^0, J_1^1, K_1^1, J_2^1 and K_2^1 as being zero by definition. This amounts to saying that the origin of the reference system coincides with the centre of mass of the earth and that the axis $\varphi = \pm\pi/2$ coincides with one of the earth's principal axes of inertia. The latter is then identified with its mean axis of rotation, the direction of which with respect to the body of the earth can be determined from surface measurements of astronomical latitude. Surface measurements of the magnitude of gravity on the other hand fail to define the centre of mass unambiguously (Molodenskii *et al.*, 1962). In connection with such problems of definition it seems to be of at least some theoretical interest to investigate the numerical relationship between the coefficients J_n^s and K_n^s in two systems of reference mutually related to each other by a rotation or a translation.

As a matter of fact the potential is numerically invariant to such transformations and only the mathematical expression of the potential is affected. The transformation of this expression amounts to a transformation of the J_n^s and K_n^s coefficients. Such is effected in a straightforward manner by formal transformations of the integrals in (2).

Here it is convenient to write (1) in the more symmetrical form

$$U(r, \varphi, \lambda) = \frac{GM}{a} \sum_{n=0}^{\infty} \sum_{s=-n}^{n} \left(\frac{a}{r}\right)^{n+1} \operatorname{Re}\{(J_n^s + iK_n^s) e^{-is\lambda}\} \cdot P_n^s(\sin\varphi). \tag{3}$$

with $i^2 = -1$, Re operating on a complex expression as to take its real part. Associated with (3) we have the complex coefficients

$$J_n^s + iK_n^s = \frac{(n-s)!}{(n+s)!} \frac{1}{Ma^n} \int_M r^n e^{is\lambda} P_n^s(\sin\varphi) dm \tag{4}$$

It should be noted that in both (3) and (4) s can assume integer negative values. Hence these formulae involve first kind associated Legendre functions of negative order. This should not disturb us, because these are readily expressible in terms of those of positive order

$$P_n^{-s}(\sin\varphi) = (-1)^s \frac{(n-s)!}{(n+s)!} P_n^s(\sin\varphi)$$

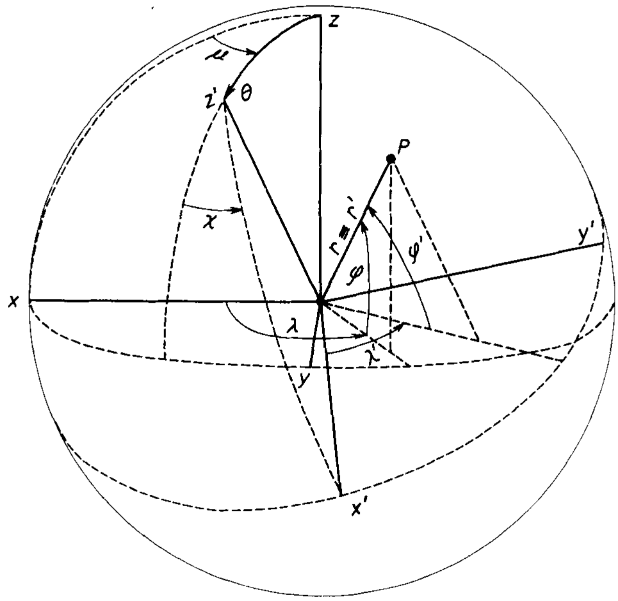

Fig. 1. Rotation; geometry and notation.

This leads to the relation

$$J_n^{-s} + iK_n^{-s} = (-1)^s \frac{(n+s)!}{(n-s)!} \overline{(J_n^s + iK_n^s)}$$

and (3) reduces to (1). This establishes the relationship between representations (1) and (3).

The cases of rotation and translation will now be treated subsequently.

The rotation

Let a mass distribution be given together with two right-handed cartesian systems of reference x, y, z and x', y', z', mutually related by a rotation. Let the spherical coordinates of point P be r, φ, λ and r', φ', λ' when referred to systems x, y, z and x', y', z' respectively (Fig. 1).

Then the external potential referred to systems x, y, z and x', y', z' will read subsequently

$$U(r, \varphi, \lambda) = \frac{GM}{a} \sum_{p=0}^{\infty} \sum_{l=-p}^{p} \left(\frac{a}{r}\right)^{p+1} \operatorname{Re}\{(A_p^l + iB_p^l) e^{-il\lambda}\} P_p^l(\sin\varphi) \tag{5}$$

and $$U'(r', \varphi', \lambda') = \frac{GM}{a} \sum_{n=0}^{\infty} \sum_{s=-n}^{n} \left(\frac{a}{r'}\right)^{n+1} \operatorname{Re}\{(J_n^s + iK_n^s) e^{-is\lambda'}\} P_n^s(\sin\varphi') \tag{6}$$

where $$A_p^l + iB_p^l = \frac{(p-l)!}{(p+l)!} \frac{1}{Ma^p} \int_M r^p e^{il\lambda} P_p^l(\sin\varphi)\, dm \tag{7}$$

and
$$J_n^s + iK_n^s = \frac{(n-s)!}{(n+s)!}\frac{1}{Ma^n}\int_M (r')^n e^{is\lambda'} P_n^s(\sin\varphi')\,dm' \tag{8}$$

Here
$$dm = \sigma(r,\varphi,\lambda)\,r^2\cos\varphi\,dr\,d\varphi\,d\lambda$$
$$dm' = \sigma'(r',\varphi',\lambda')\,(r')^2\cos\varphi'\,dr'\,d\varphi'\,d\lambda'$$

where obviously $\sigma(r,\varphi,\lambda)\equiv\sigma'(r',\varphi',\lambda')$ for all values of r,φ,λ and r',φ',λ'. In the present case $r\equiv r'$, but in the case of a translation this identity will not hold.

If the rotation from x, y, z to x', y', z' is specified by the Eulerian angles μ, θ, χ as indicated in Fig. 1, we have the relations

$$e^{-is\lambda'}P_n^s(\sin\varphi') = \sum_{l=-n}^{n}\frac{(n-l)!}{(n-s)!}\bar{S}_{2n}^{n+s,\,n+l}(\mu,\theta,\chi)\,e^{-il\lambda}P_n^l(\sin\varphi) \tag{9}$$

and
$$e^{is\lambda'}P_n^s(\sin\varphi') = \sum_{l=-n}^{n}\frac{(n-l)!}{(n-s)!}S_{2n}^{n+s,\,n+l}(\mu,\theta,\chi)\,e^{il\lambda}P_n^l(\sin\varphi) \tag{10}$$

with

$$S_{2n}^{n+s,\,n+l}(\mu,\theta,\chi) = \begin{cases} (-1)^{n-l}\binom{n-s}{n+l} e^{i(s\chi+l\mu)}(\sin\tfrac{1}{2}\theta)^{s-l}(\cos\tfrac{1}{2}\theta)^{-s-l} \\ \qquad \times\, {}_2F_1(-n-l,\, n-l+1;\; -s-l+1;\, \cos^2\tfrac{1}{2}\theta) \quad \text{if} \quad s+l\leqslant 0 \\ (-1)^{n+s}\binom{n+s}{n-l} e^{i(s\chi+l\mu)}(\sin\tfrac{1}{2}\theta)^{-s+l}(\cos\tfrac{1}{2}\theta)^{s+l} \\ \qquad \times\, {}_2F_1(-n+l,\, n+l+1;\, s+l+1;\, \cos^2\tfrac{1}{2}\theta) \quad \text{if} \quad s+l\geqslant 0 \end{cases} \tag{11}$$

where conventionally

$${}_2F_1(\alpha,\beta;\gamma\colon x) = 1 + \frac{\alpha\cdot\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x^2 + \dots$$

$\bar{S}_{2n}^{n+s,\,n+l}(\mu,\theta,\chi)$ is conjugate complex to $S_{2n}^{n+s,\,n+l}(\mu,\theta,\chi)$.

(9) and (10) together with (11) are transformation formulae for the surface spherical harmonics (Courant & Hilbert, 1953; Erdélyi *et al.*, 1953). Equivalent formulae have been derived by Schmidt (1899), Hönl (1934), Satô (1950) and more recently by Jeffreys (1965).

Obviously (9) and (10) are generalizations of the well-known addition-theorem for Legendre-polynomials ($s=0$) and can be derived on observing a generating function for spherical harmonics, quoted in section 3.

Inserting (10) into (8) and taking (7) into account we obtain the required result

$$J_n^s + iK_n^s = \sum_{k=-n}^{n}\frac{(n+k)!}{(n+s)!}\bar{S}_{2n}^{n+s,\,n+k}(\mu,\theta,\chi)\cdot(A_n^k + iB_n^k) \tag{12}$$

with

$$S_{2n}^{n+s,\,n+k}(\mu,\theta,\chi)=\begin{cases}(-1)^{n-k}\binom{n-s}{n+k}e^{-i(s\chi+k\mu)}(\sin\tfrac{1}{2}\theta)^{s-k}(\cos\tfrac{1}{2}\theta)^{-s-k}\\ \qquad\times{}_2F_1(-n-k,n-k+1;\,-s-k+1;\cos^2\tfrac{1}{2}\theta)\quad\text{if}\quad k\leqslant -s\\ (-1)^{n+s}\binom{n+s}{n-k}e^{-i(s\chi+k\mu)}(\sin\tfrac{1}{2}\theta)^{-s+k}(\cos\tfrac{1}{2}\theta)^{s+k}\\ \qquad\times{}_2F_1(-n+k,n+k+1;s+k+1;\cos^2\tfrac{1}{2}\theta)\ \text{if}\ k\geqslant -s\end{cases}\tag{13}$$

Apparently the same result can be obtained by applying (9) to (5) and (6) but the derivation is then more involved and has been omitted here.

A translation

We proceed as before, trying to transform the integral in (4). Doing so however we face the problem of finding an appropriate transformation formula for the solid spherical harmonics of type $r^n e^{is\lambda} P_n^s(\sin\varphi)$. Although such formulae are known for special translations (Hobson, 1931; Robin, 1957), a general formula doesnot seem to be available, although the matter can hardly have escaped attention and the present result might not be new. The derivation of such a formula is not difficult if we observe a generating function for the surface spherical harmonics given by Erdélyi *et al.* (1953) and referred to already in the previous section; see also Courant & Hilbert (1953).

Define vectors

$$\mathbf{u}=(1-t^2,\,i+it^2,\,-2t)\quad\text{and}\quad \mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}$$

with again $i^2=-1$.

Define the functions $H_n^m(\mathbf{r})$ as to be generated by

$$(\mathbf{u}\cdot\mathbf{r})^n=t^n\sum_{m=-n}^{n}H_n^m(\mathbf{r})\cdot t^m\tag{14}$$

where $\mathbf{u}\cdot\mathbf{r}$ stands for the scalar product of $\mathbf{u}$ and $\mathbf{r}$.

Interpret x, y, z as the right-handed cartesian coordinates of a point P referred to system x, y, z.

Let cartesian system x', y', z' be related to x, y, z by

$$\mathbf{r}'=\mathbf{r}-\boldsymbol{\rho}\tag{15}$$

where

$$r'=\begin{pmatrix}x'\\y'\\z'\end{pmatrix}\quad\text{and}\quad \varrho=\begin{pmatrix}\delta x\\\delta y\\\delta z\end{pmatrix}.$$

(15) specifies a translation by $\boldsymbol{\rho}$, the effect of which we want to investigate (Fig. 2).

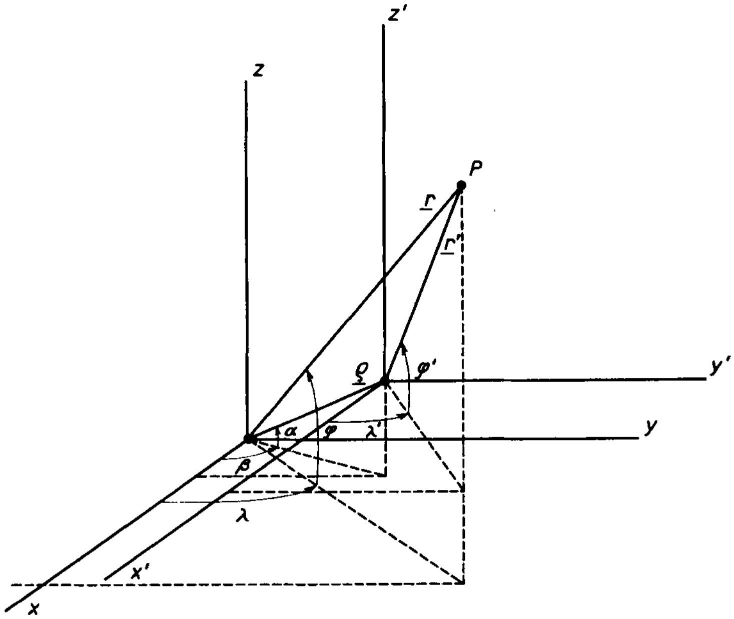

Fig. 2. Translation; geometry and notation.

In consequence of (14) we also have

$$(\mathbf{u}\cdot\mathbf{r}')^n = t^n \sum_{m=-n}^{n} H_n^m(\mathbf{r}')\cdot t^m. \tag{16}$$

On the other hand

$$(\mathbf{u}\cdot\mathbf{r}')^n = [\mathbf{u}\cdot(\mathbf{r}-\boldsymbol{\varrho})]^n = (\mathbf{u}\cdot\mathbf{r}-\mathbf{u}\cdot\boldsymbol{\varrho})^n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}(\mathbf{u}\cdot\mathbf{r})^{n-k}(\mathbf{u}\cdot\boldsymbol{\rho})^k$$

$$= t^n \sum_{k=0}^{n}(-1)^k\binom{n}{k}\left\{\sum_{h=-n+k}^{n-k} H_{n-k}^h(\mathbf{r})\cdot t^h\right\}\left\{\sum_{j=-k}^{k} H_k^j(\boldsymbol{\varrho})t^j\right\}, \tag{17}$$

since again

$$(\mathbf{u}\cdot\boldsymbol{\varrho})^n = t^n \sum_{m=-n}^{n} H_n^m(\boldsymbol{\varrho})\cdot t^m.$$

Combining (16) and (17) we find after some changes of indices:

$$H_n^m(\mathbf{r}') = \sum_{p=0}^{n} \sum_{l>\frac{-p}{m-n+p}}^{<\frac{p}{m+n-p}} (-1)^{n-p}\binom{n}{p} H_{n-p}^{m-l}(\boldsymbol{\varrho})\cdot H_p^l(\mathbf{r}) \tag{18}$$

where symbol $l > \dfrac{a}{b}$ means:

$$l - \begin{cases} a = \text{if} & a \geqslant b \\ b = \text{if} & a \leqslant b \end{cases}$$

and similarly

$$l \lessgtr{}^{a}_{b} \quad \text{means} \quad l = \begin{cases} a & \text{if} \quad a \leqslant b \\ b & \text{if} \quad a \geqslant b. \end{cases}$$

Now it can be proved that the $H_n^m(r)$ as defined by (14) form a complete system of linearly independent harmonic polynomials of degree n (see Erdélyi *et al.*, 1953 and Courant & Hilbert, 1953) and making the substitution:

$$\mathbf{r} = r \begin{pmatrix} \cos\varphi \cos\lambda \\ \cos\varphi \sin\lambda \\ \sin\varphi \end{pmatrix},$$

where r stands for the absolute value of $\mathbf{r}$, we find

$$H_n^m(\mathbf{r}) = (-1)^{n+m} \frac{2^h n!}{(n+m)!} r^n e^{-\imath m\lambda} P_n^m(\sin\varphi). \tag{19}$$

Making similar substitutions

$$\mathbf{r}' = r' \begin{pmatrix} \cos\varphi' \cos\lambda' \\ \cos\varphi' \sin\lambda' \\ \sin\varphi' \end{pmatrix}; \quad \boldsymbol{\rho} = \varrho \begin{pmatrix} \cos\alpha \cos\beta \\ \cos\alpha \sin\beta \\ \sin\alpha \end{pmatrix}$$

and substituting the result together with (19) in (18) we obtain

$$(r')^n e^{-\imath m\lambda'} P_n^m(\sin\varphi') =$$

$$= \sum_{p=0}^{n} \sum_{l > {}^{-p}_{m-n+p}}^{< {}^{p}_{m+n-p}} (-1)^{n-p} \binom{n+m}{p+l} \varrho^{n-p} e^{-\imath(m-l)\beta} P_{n-p}^{m-l}(\sin\alpha)\, r^p e^{-\imath l\lambda} P_p^l(\sin\varphi): \tag{20}$$

It follows immediately that also

$$(r')^n e^{\imath m\lambda'} P_n^m(\sin\varphi') =$$

$$= \sum_{p=0}^{n} \sum_{l > {}^{-p}_{m-n+p}}^{< {}^{p}_{m+n-p}} (-1)^{n-p} \binom{n+m}{p+l} \varrho^{n-p} e^{\imath(m-l)\beta} P_{n-p}^{m-l}(\sin\alpha)\, r^p e^{\imath l\lambda} P_p^l(\sin\varphi) \tag{21}$$

(21) is the required transformation formula for solid spherical harmonics of type

$$r^n e^{\imath s\lambda} P_n^s(\sin\varphi).$$

If we abbreviate (21) as

$$(r')^n e^{\imath m\lambda'} P_n^m(\sin\varphi') = \sum_{p=0}^{n} \sum_{l > {}^{-p}_{m-n+p}}^{< {}^{p}_{m+n-p}} K_{n,m}^{p,l}(\varrho, \alpha, \beta) \cdot r^p e^{\imath l\lambda} P_p^l(\sin\varphi) \tag{22}$$

where

$$K_{n,m}^{p,l}(\varrho, \alpha, \beta) = (-1)^{n-p} \binom{n+m}{p+l} \varrho^{n-p} e^{\imath(m-l)\beta} P_{n-p}^{m-l}(\sin\alpha) \tag{23}$$

then coefficients $K^{p,l}_{n,m}(\varrho, \alpha, \beta)$ are functions of the elements ϱ, α, β which define the translation.

If, in particular, $\alpha = \pi/2$ (20) reduces to

$$(r')^n e^{-im\lambda'} P_n^m(\sin\varphi') = \sum_{p=m}^{n} (-1)^{n-p} \binom{n+m}{p+m} \varrho^{n-p} r^p e^{-im\lambda} P_p^m(\sin\varphi). \tag{24}$$

An equivalent formula for this case is given by Robin (1957).

Inserting (21) into (8) and observing (7) we obtain:

$$J_n^s + iK_n^s = \sum_{p=0}^{n} \sum_{l > {-p \atop s-n+p}}^{< {p \atop s+n-p}} (-1)^{n-p} \frac{(n-s)!}{(p-l)!\,(n+s-p-l)!}$$
$$\times \left(\frac{\varrho}{a}\right)^{n-p} P_{n-p}^{s-l}(\sin\alpha)\, e^{i(s-l)\beta} (A_p^l + iB_p^l) \tag{25}$$

which is the required transformation formula. It relates the coefficients in two spherical harmonics expansions of the gravitational potential when these expansions are referred to mutually translated systems of reference.

Introducing finally the coefficients of type (23) we find the alternative expression:

$$J_n^s + iK_n^s = \sum_{p=0}^{n} \sum_{l > {-p \atop s-n+p}}^{l < {p \atop s+n-p}} a^{-n+p} \frac{(n-s)!\,(p+l)!}{(n+s)!\,(p-l)!} K^{p,l}_{n,s}(\varrho, \alpha, \beta) \cdot (A_p^l + iB_p^l). \tag{26}$$

Formulae (25) or (26) bear some similarity to Steiner's wellknown theorem on the moments of inertia of a body with respect to parallel axes. See f.i. Joos (1967).

It is not difficult to derive corresponding expressions in terms of cartesian coordinates x, y, z; x', y', z' and translation components $\delta x, \delta y, \delta z$ but such expressions turn out to be somewhat more complicated and the present formulae in terms of spherical coordinates are more readily applicable.

Discussion

The complex harmonic coefficients $J_n^s + iK_n^s$ as defined by (4) proved to be convenient in the mathematical process of deriving (12) and (25). On the other hand representation (3) of the potential which goes with them does not seem to agree with current practice, because it involves the first kind Legendre functions of negative order s. But like pointed out in section 1 the interrelationship between expressions (1) and (3) is a simple one.

Given a representation of type (1) the complex coefficients $J_n^s + iK_n^s$ can immediately be written down for $s \geqslant 0$. For $s < 0$ they are obtained from these by

$$J_n^{-s} + iK_n^{-s} = (-1)^s \frac{(n+s)!}{(n-s)!} \overline{(J_n^s + iK_n^s)}$$

as demonstrated in section 1. In this way the corresponding representation of type (3) can be easily constructed. To obtain explicit expressions of J_n^s and K_n^s in terms of A_p^1 and B_p^1, in (12) and (25) real and imaginary parts have to be separated as usual.

Simplified formulae can be derived for a zonal earth, i.e. a hypothetical earth the external gravitational field of which is symmetric with respect to the z-axis. In this simplified case, which however is not too far from reality, we have $A_p^1 = 0$ if $l \neq 0$ and $B_p^1 = 0$ for all p and l.

Consequently (12) reduces to:

$$J_n^s + iK_n^s = \frac{n!}{(n+s)!} \bar{S}_{2n}^{n+s,\,n} \cdot A_n^0$$

with

$$\bar{S}_{2n}^{n+s,\,n} = (-1)^{n+s} \binom{n+s}{n} e^{-is\chi} (\sin \tfrac{1}{2}\theta)^{-s} (\cos \tfrac{1}{2}\theta)^s \, {}_2F_1(-n, n+1; s+1; \cos^2 \tfrac{1}{2}\theta)$$

$$= (-1)^n \frac{(n-s)!}{n!} e^{-is\chi} P_n^s(-\cos\theta) = (-1)^s \frac{(n-s)!}{n!} e^{-is\chi} P_n^s(\cos\theta).$$

Hence for a zonal earth

$$J_n^s + iK_n^s = (-1)^s \frac{(n-s)!}{(n+s)!} e^{-is\chi} P_n^s(\cos\theta) \cdot A_n^0 \tag{27}$$

μ does not appear, like it should not. Separating real and imaginary parts we find:

$$\begin{pmatrix} J_n^s \\ K_n^s \end{pmatrix} = (-1)^s \frac{(n-s)!}{(n+s)!} \begin{pmatrix} \cos s\chi \\ -\sin s\chi \end{pmatrix} \cdot P_n^s(\cos\theta) \cdot A_n^0. \tag{27 a}$$

Interpreting this equation it should be kept in mind that

$$P_n^s(1) = \begin{cases} 1 & \text{if} \quad s = 0 \\ 0 & \text{if} \quad s > 0 \end{cases}$$

Doing so, we find, as we should, $J_n^0 = A_n^0$, $K_n^s = 0$ if $\theta = 0$.
For $\chi = 0$:

$$J_n^s = (-1)^s \frac{(n-s)!}{(n+s)!} P_n^s(\cos\theta) \cdot A_n^0 \tag{27 b}$$
$$K_n^s = 0 \text{ for all } n, s \text{ and } \theta.$$

(25) reduces for a zonal earth to

$$J_n^s + iK_n^s = \sum_{p=0}^{n-s} (-1)^{n-p} \frac{(n-s)!}{p!\,(n+s-p)!} \left(\frac{\varrho}{a}\right)^{n-p} P_{n-p}^s(\sin\alpha) e^{is\beta} \cdot A_p^0. \tag{28}$$

This expression is substantially simpler than (25) and may be further specialized to some interesting cases:

(a): $\alpha = \pi/2$; this represents a translation along the z-axis.

Since
$$P^s_{n-p}(1) = \begin{cases} 1 & \text{if} \quad s=0 \\ 0 & \text{if} \quad s>0 \end{cases}$$

we obtain

$$J^0_n = \sum_{p=0}^{n} (-1)^{n-p} \binom{n}{p} \left(\frac{\delta z}{a}\right)^{n-p} A^0_p \tag{28a}$$

$K^s_n = 0$ for all n and s.

(b): $\alpha = \beta = 0$, i.e. a translation along the x-axis. We obtain immediately

$$J^s_n = \sum_{p=0}^{n-s} (-1)^{n-p} \frac{(n-s)!}{p!\,(n+s-p)!} P^s_{n-p}(0) \left(\frac{\delta x}{a}\right)^{n-p} A^0_p \tag{28b}$$

$K^s_n = 0$ for all n and s.

(c): $\alpha = 0$, $\beta = \pi/2$; a translation in y-direction. Comparing real and imaginary parts we find

$$\text{for} \quad s \text{ even} \begin{cases} J^s_n = \displaystyle\sum_{p=0}^{n-s} (-1)^{n-p+s/2} \frac{(n-s)!}{p!\,(n+s-p)!} P^s_{n-p}(0) \left(\frac{\delta y}{a}\right)^{n-p} A^0_p \\ K^s_n = 0 \end{cases}$$
$$s \text{ odd} \begin{cases} J^s_n = 0 \\ K^s_n = \displaystyle\sum_{p=0}^{n-s} (-1)^{n-p+(s-1)\frac{1}{2}} \frac{(n-s)!}{p!\,(n+s-p)!} P^s_{n-p}(0) \left(\frac{\delta y}{a}\right)^{n-p} A^0_p \end{cases} \tag{28c}$$

Numerical estimates and a conclusion

To be able to appreciate the order of magnitude involved in practical cases let us generalize our zonal earth still further, only retaining the predominant terms $A^0_0 = 1$ and $A^0_2 \approx -\frac{1}{2} \times 10^{-3}$.

Then (27a) gives as non-trivial result

$$J^0_2 = (\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}) A^0_2$$
$$K^0_2 = 0$$
$$J^1_2 = \tfrac{1}{2}\cos\chi \sin\theta \cos\theta \cdot A^0_2$$
$$K^1_2 = -\tfrac{1}{2}\sin\chi \sin\theta \cos\theta \cdot A^0_2$$
$$J^2_2 = \tfrac{1}{8}\cos 2\chi \sin^2\theta \cdot A^0_2$$
$$K^2_2 = -\tfrac{1}{8}\sin 2\chi \sin^2\theta \cdot A^0_2.$$

If we assume θ and χ to be small quantities, say of order 10^{-5}, the only non-zero results would be:

$$J_2^0 = A_2^0;\ |J_2^1| = \left|\frac{\theta}{2} A_2^0\right| \approx 0.002 \times 10^{-6}.$$

With the same generalizations of the gravitational field we find similar estimates when using (28a) with $|\delta z/a| = 10^{-5}$, corresponding to a translation of about 60 m along the rotation-axis:

$$|J_2^0 - A_2^0| = 0.01 \times 10^{-8}$$

$$|J_3^0| = 0.15 \times 10^{-7}$$

$$|J_4^0| = 1.5 \times 10^{-13} \text{ etc.}$$

These figures are very small. It is interesting to compare them with the accuracies attained in the determination of J_2^0 and J_3^0 by the analysis of the orbits of artificial satellites. We take Kozai's contribution of zonal harmonics to the Smithsonian Standard Earth 1966 (Lundquist & Veis, 1966). Here we are especially interested in the standard errors and we adopt their least optimistic estimates:

$$0.003 \times 10^{-6} \quad \text{for} \quad J_2^0$$

and

$$0.010 \times 10^{-6} \quad \text{for} \quad J_3^0.$$

The corrections to A_2^0 and A_3^0 due to a shift of origin, some 60 m along the rotation-axis, turned out to be very small, but as regards A_3^0 of the order of the standard error quoted. This means that such a shift would hardly introduce a bias in the derived value of J_3^0. The other harmonic coefficients are only influenced to a smaller extend, as is easily seen from (28a), where the lower degree harmonics go with higher powers of $\delta z/a$. J_3^0 proves to be the critical case. If we take a 6 m shift instead of 60 m, we find that this would generate a J_3^0-term of magnitude 0.0015×10^{-6}, which is well within to-day's standard error.

This is an example and similar, perhaps more important conclusions might be drawn for some of the other harmonic coefficients. The general picture is clear from (28a) through (28c). Adopt 10^{-5} (about 60 m again) as the maximum conceivable value for $|\delta \varrho/a|$, then noting that the factorials are at most of order 10 for the terms which are not obviously negligible, we easily pick out the relevant terms, limiting ourselves to those $> 10^{-9}$

$$|J_3^0| = 3\left|\frac{\delta z}{a} A_2^0\right| = 0.015 \times 10^{-6} \text{ from (28a)}$$

$$|K_3^1| = \tfrac{1}{2}\left|\frac{\delta y}{a} A_2^0\right| = 0.0025 \times 10^{-6} \text{ from (28c)}.$$

Hence in addition we find, generated by a y-shift, a K_3^1-term the magnitude of which might not be entirely negligible in future.

A conclusion could be that within the contemporary accuracies of satellite geodesy a definition of the reference system with respect to the earth's centre of mass to within say 50 m comes up to all practical requirements as regards the determination of the external field. The subject might become relevant as soon as apparent standard errors of the order of 10^{-10} are attained.

This conclusion will appear almost trivial to many, but it seems worth while to have some numerical estimates available.

Perhaps it is different with the moon, where the centre of mass relative to the centre of figure is not so well defined; an excentricity of some 2 km is not out of the question (Michael, 1967). Moreover the moon's radius is roughly four times smaller than the earth's, so that in all in selenodetic investigations $|\delta\varrho/a|$ could be of order 10^{-3}, some hundred times as big as the value we thought conceivable in geodesy.

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НЕКОТОРЫЕ ТРАНСФОРМАЦИОННЫЕ СВОЙСТВА ДЛЯ КОЭФФИЦИЕНТОВ РАЗЛОЖЕНИЯ ВНЕШНЕГО ГРАВИТАЦИОННОГО ПОТЕНЦИАЛА ЗЕМЛИ ПО СФЕРИЧЕСКИМ ГАРМОНИКАМ

Внешний гравитационный потенциал Земли может быть математически представлен в виде бесконечного ряда по сферическим гармоникам. Коэффициенты этого ряда численно зависят от положения и ориентации по отношению к фигуре Земли принятой системы отсчета. Целью настоящей статьи является вывод общих формул для изменений коэффициентов разложения при перемещении и переориентации системы отсчета. Такие формулы могут быть полезны при критическом анализе динамической геодезии искусственных спутников, особенно при анализе их результатов совместно с данными земной геодезии. Было использовано трансформационное свойство сферических гармоник, которое, по-видимому, не встречалось ранее в стандартной матема тике. Приводятся некоторые простые численные примеры.