On the determination of the potential from gravity disturbances along a fixed direction

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ABSTRACT

The problem of determining the potential from gravity disturbances along a fixed direction arises in the analysis of residuals of Doppler tracking of Lunar Orbiter satellites for features of the gravity field of the moon. This problem is solved in two ways, by an integral formula and by means of spherical harmonics.

1. Introduction

The component of the gravity disturbance vector along a fixed spatial direction is among the most recent observable data. In fact, Muller (1968) and Sjogren have obtained a set of lunar gravity disturbances by analysis of residuals of Doppler tracking of Lunar Orbiter satellites. These disturbances are in the direction to the earth, which varies with respect to a moon-fixed reference frame by about $\pm 7°$; they refer to altitudes of a few hundred kilometers and cover about 60 % of the moon's surface.

The anomalous gravity potential of the moon will be denoted by T. Then the gravity disturbance vector is the vector grad T. To simplify the problem is keeping with the accuracy of measurement, we replace the slightly variable direction from the lunar orbiter satellite to the earth by the constant direction from the center of the moon to the center of the earth, which, being fixed with respect to the moon, may be considered as the z-axis of a rectangular spatial coordinate system. Thus the observed gravity disturbances will be identified with $\partial T/\partial z$, the z-component of grad T.

To make the problem well-defined theoretically, we may assume that $\partial T/\partial z$ is given over the whole surface of a sphere concentric to and enclosing the moon.

Let R be the radius of the sphere on which $\partial T/\partial z$ is given, and R_0 be the radius of the moon, such that $R > R_0$. The computation of the anomalous potential T may then be split up into the following two steps:

1. Given $\partial T/\partial x$ on the sphere $r = R$, computation of T on the same sphere by means of an integral formula.
2. Downward continuation of T from the sphere $r = R$ to the lunar surface $r = R_0$.

Step 2 poses no new theoretical problems. Neither do so prediction and continuation techniques required for obtaining $\partial T/\partial x$ on the whole surface of the sphere from values measured along the orbit ($R - R_0$ should represent a mean elevation of the satellite above the moon). Thus there remains to find an integral formula for Step 1.

This will be done in sec. 2; in sec. 3 we shall outline an alternative solution by means of spherical harmonics.

2. An integral formula

We first remark that the function $\partial T/\partial z$ given on the sphere must satisfy the conditions

$$\left.\begin{aligned}\iint_\sigma \left(\frac{\partial T}{\partial z}\right)_\sigma P_{nn}(\cos\theta)\cos n\lambda d\sigma = 0,\\ \iint_\sigma \left(\frac{\partial T}{\partial z}\right)_\sigma P_{nn}(\cos\theta)\sin n\lambda d\sigma = 0,\end{aligned}\right\} \quad (n = 0, 1, 2, 3, \ldots) \tag{1}$$

where $d\sigma$ is the element of solid angle; θ (polar distance) and λ (longitude) are the spherical coordinates of $d\sigma$; $(\partial T/\partial z)_\sigma$ denotes the value of $\partial T/\partial z$ at the point (θ, λ) of the sphere; and $P_{nn}(\cos\theta)$ is the associated Legendre function of degree n and order n. The integration is extended over the whole sphere.

The conditions (1) express the fact that the spherical-harmonic expansion of $\partial T/\partial z$ contains no sectorial harmonics; they will be derived in sec. 3.

Since $\partial T/\partial z$ is harmonic together with T, we may use the Poisson integral formula

$$\frac{\partial T}{\partial z}(x, y, z) = \frac{R}{4\pi}\iint_\sigma \frac{r^2 - R^2}{l^3}\left(\frac{\partial T}{\partial z}\right)_\sigma d\sigma \tag{2}$$

to compute $\partial T/\partial z$ at a point (x, y, z) outside the sphere from its values on the spherical surface. We have

$$R^2 = \xi^2 + \eta^2 + \zeta^2, \tag{3a}$$

$$r^2 = x^2 + y^2 + z^2 > R^2, \tag{3b}$$

$$l^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2, \tag{3c}$$

where (ξ, η, ζ) are the coordinates of the surface element $R^2 d\sigma$. Since, according to (1) with $n = 0$, the function $\partial T/\partial z$ contains no zero-degree spherical harmonic, we may replace (2) by

$$\frac{\partial T}{\partial z}(x, y, z) = \frac{R}{4\pi}\iint_\sigma \left(\frac{r^2 - R^2}{l^3} - \frac{1}{r}\right)\left(\frac{\partial T}{\partial z}\right)_\sigma d\sigma, \tag{4}$$

in which the zero-degree harmonic is removed in exactly the same way as in equation (2–159) of (Heiskanen & Moritz, 1967, p. 90) the harmonics of degrees zero and one are removed.

Now T is found by integrating (4) with respect to z and interchanging the order of integrations:

$$T(x, y, z) = \int_{\infty}^{z} \frac{\partial T}{\partial z} dz = \frac{R}{4\pi} \iint_{\sigma} \left[\int_{\infty}^{z} \left(\frac{r^2 - R^2}{l^3} - \frac{1}{r} \right) dz \right] \left(\frac{\partial T}{\partial z} \right)_{\sigma} d\sigma.$$

We thus finally obtain

$$T(x, y, z) = \frac{R}{4\pi} \iint_{\sigma} F(x, y, z) \left(\frac{\partial T}{\partial z} \right)_{\sigma} d\sigma \tag{5}$$

with
$$F(x, y, z) = \int_{\infty}^{z} \left(\frac{r^2 - R^2}{l^3} - \frac{1}{r} \right) dz.$$

The evaluation of this integral, taking (3a, b, c) into account, by the standard methods of integration gives

$$F(x, y, z) = -\frac{2\xi(x-\xi) + 2\eta(y-\eta)}{l(l+z-\zeta)} - \frac{2\zeta}{l} + \ln \frac{l+z-\zeta}{r+z}. \tag{6}$$

These formulas are valid for $z \geqslant 0$ and for $x^2 + y^2 + z^2 \geqslant R^2$, that is, for a point outside or on the sphere of radius R. In order to get a regular result (5) in spite of the singularities of the function (6) together with its obvious counterpart for $z < 0$,[1]

$$F(x, y, z) = \int_{-\infty}^{z} \left(\frac{r^2 - R^2}{l^3} - \frac{1}{r} \right) dz = \frac{2\xi(x-\xi) + 2\eta(y-\eta)}{l(l-z+\zeta)} - \frac{2\zeta}{l} - \ln \frac{l-z+\zeta}{r-z}, \tag{6'}$$

the conditions (1) must be satisfied.

3. Use of spherical harmonics

An alternative solution uses spherical harmonics. The anomalous potential T may be expanded into a series of spatial spherical harmonics:

$$T(r, \theta, \lambda) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{R}{r} \right)^{n+1} P_{nm}(\cos\theta) \left(a_{nm} \cos m\lambda + b_{nm} \sin m\lambda \right). \tag{7}$$

From Maxwell's multipole representation of spherical harmonics (MacMillan, 1958, p. 403) it follows that

$$\frac{\partial}{\partial z} \left(\frac{P_{nm}(\cos\theta) \cos m\lambda}{r^{n+1}} \right) = -(n-m+1) \frac{P_{n+1,m}(\cos\theta) \cos m\lambda}{r^{n+2}},$$

$$\frac{\partial}{\partial z} \left(\frac{P_{nm}(\cos\theta) \sin m\lambda}{r^{n+1}} \right) = -(n-m+1) \frac{P_{n+1,m}(\cos\theta) \sin m\lambda}{r^{n+2}}. \tag{8}$$

Thus the differentiation of (7) with respect to z gives

$$\frac{\partial T}{\partial z} = -\sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{R}{r} \right)^{n+2} P_{n+1,m}(\cos\theta) \frac{n-m+1}{R} \left(a_{nm} \cos m\lambda + b_{nm} \sin m\lambda \right).$$

[1] In fact, if $x^2 + y^2 > R^2$, both (6) and (6′) may be used for $-\infty < z < \infty$.

We now replace the subscript n by $n-1$ to obtain

$$\frac{\partial T}{\partial z} = -\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\left(\frac{R}{r}\right)^{n+1} P_{nm}(\cos\theta)\frac{n-m}{R}(a_{n-1,m}\cos m\lambda + b_{n-1,m}\sin m\lambda). \qquad (9)$$

We see that (9) contains no tesseral harmonic $m=n$, which proves our conditions (1).

The function $\partial T/\partial x$ is supposed to be given on the sphere $r=R$. It may thus be expanded into a series of surface spherical harmonics:

$$\frac{\partial T}{\partial z} = \sum_{n=1}^{\infty}\sum_{m=0}^{n-1} P_{nm}(\cos\theta)(c_{nm}\cos m\lambda + d_{nm}\sin m\lambda) \qquad (10)$$

with $c_{nn}=d_{nn}=0$. Then the coefficients a_{nm} and b_{nm} are obtained by

$$a_{nm} = -\frac{R}{n-m+1}c_{n+1,m}, \quad b_{nm} = -\frac{R}{n-m+1}d_{n+1,m}, \qquad (11)$$

and (7) gives T in space down to the lunar surface $r=R_0$.

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References

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ОБ ОПРЕДЕЛЕНИИ ПОТЕНЦИАЛА ИЗ ВОЗМУЩЕНИЙ СИЛЫ ТЯЖЕСТИ ВДОЛЬ ФИКСИРОВАННОГО НАПРАВЛЕНИЯ

Задача определения потенциала из возмущений силы тяжести вдоль фиксированного направления возникает при анализе невязок для допплеровского слежения лунных орбитальных спутников с целью определения гравитационного поля Луны. Эта задача решается двумя способами: интегральной формулой и с помощью сферических гармоник.